Solutions to Exercises in

Game Theory
Chapter 15

1. Consider the game $(\{1,2,3\},V)$ with $V(\{i\}) = \mathbb{R}_-$ for $i = 1,2,3$,

$$V(\{i,j\}) = \left\{(z_i,z_j) \,\middle|\, \exists (x_i,x_j) \in \mathbb{R}^2_+ : \min\left\{x_i + 4x_j, 4x_i + x_j\right\} = 5\right\} \text{ for } i,j \in \{1,2,3\}, i \neq j,$$
$$V(N) = \left\{z \in \mathbb{R}^3 \,\middle|\, \exists x \in \mathbb{R}^3_+ : \textstyle\sum_{i=1}^3 x_i \leq \frac{27}{8}, i = 1,2,3\right\}.$$

Then the core of $(\{1,2,3\},V)$ is the set of all payoffs (x_1,x_2,x_3) satisfying $x_1 + x_2 + x_2 = \frac{27}{8}$ and the inequalities

$$\min\left\{x_i + 4x_j, 2x_i + x_j\right\} = 5, i,j = 1,2,3, i \neq j,$$

which contains the point $\left(\frac{9}{8},\frac{9}{8},\frac{9}{8}\right)$, so it is nonempty. There is only one possible λ which can be used in the Shapley transfer principle, namely $\lambda = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$, so $v_\lambda(N) = \frac{9}{8}$ and $v_\lambda(\{i,j\}) = \frac{5}{3}$ for each i,j with $i \neq j$. Clearly, Core$(\{1,2,3\},v_\lambda)$ is empty, and so is the NTU core.

2. The reasonableness of the payoff $\left(\frac{1}{2},\frac{1}{2},0,\right)$ can be argued with reference to the fact that any other payoff vector in $V(N)$ could be improved by $\{1,2\}$ whereas this payoff vector cannot be improved by any coalition (it belongs to the core).

To show that $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ is a Shapley NTU value of (N,V), we first notice, that $\lambda = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ is normal to bd $V(N)$ at $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$, and that (N,v_λ) is given by

$$v_\lambda(\{i\}) = 0, i = 1,2,3, v(\{i,j\}) = \frac{1}{3}, i,j = 1,2,3, i \neq j, v_\lambda(N) = \frac{1}{3},$$

and the Shapley value of v_λ is

$$\phi_i(v_\lambda) = \frac{1}{6}\left(v_\lambda(\{i\}) + \sum_{j\neq i}\left[v(\{i,j\}) - v(\{i\})\right] + \left[v(N) - v(N\backslash\{i\})\right]\right) = \frac{1}{9}$$

for each i. Using that units have been changed by $\frac{1}{3}$ when moving to v_λ, we obtain that $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ is an NTU Shapley value.

It may be argued that the equal division between players reflects the power of coalitions in a better way than the core for which the principal importance is the possibilities of coalitional improvements.

3. Since $v(p,\cdot)$ assigns a number to every coalition (the minimum is well-defined under the given assumptions), we have that $(N, v(p,\cdot))$ is a TU game.

To show that the payoff $M(u_i, p, x_i)_{i\in N}$ for an equilibrium $(x_1,\dots,x_n,p)$ is the Shapley value of $(N, v(p,\cdot))$, we use the axiomatic approach to the Shapley value. First of all we notice that $v(p,\{i\}) = M(u,p,x_i) = p\cdot x_i$, each $i\in N$ and $v(p,N) = \sum_{i\in N} p\cdot x_i$ in this situation, and that the payoff vector $(p\cdot x_1,\dots,p\cdot x_n)$ is an imputation in $v(p,\cdot S)$. Now, the solution for games $(N, v(p,\cdot))$, where p is an equilibrium price vector, which gives the payoff vector $(p\cdot x_1,\dots,p\cdot x_n)$, clearly satisfies Pareto optimality, symmetry and the dummy axiom. Suppose that the economy is chosen such that x, x' and $x'+x''$ are equilibria with the same price vector p, giving rise to games $v(p,\cdot), v'(p,\cdot)$ and $v(p,\cdot)+v'(p,\c{d}ot)$, then the assignment of payoff vectors $(p\cdot x_1,\dots,p\cdot x_n)$, and $(p\cdot x_1',\dots,p\cdot x_n')$ and $(p\cdot(x_1+x_1'),\dots,p\cdot(x_n+x_n'))$ satisfy the additivity condition. It follows now that it must be equal to the Shapley value.

Consider the economy $\mathcal{E}$ with two commodities and two consumers, where $u_1 = u_2 = u$ is given by $u(x_1,x_2) = x_1x_2$ for $x = (x_1,x_2)\imath\mathbb{R}^2_+$ and where $\omega_1 = (3,1)$, $\omega_2 = (1,3)$, and let the price be $p = \left(\frac{1}{2},\frac{1}{2}\right)$. For the allocation $x = (\omega_1,\omega_2)$ we have that $M(u_i,p,x_i) = \left(\sqrt{3},\ \sqrt{3}\right)$, and that $v(p,\{i\}) = \sqrt{3}$ for $i = 1,2$, whereas $v(p,S) = 2p\cdot(2,2) = 4$. Since $\phi_1(v(p,\cdot))+\phi_1(v(p,\cdot)) = 4$, we cannot have that $M(u_i,p,x_i) = \phi_i(v(\cdot))$ for $i = 1,2$.

4. The quantity $h(w,S)$ is well-defined by out assumption that $V(S) = K_S - \mathbb{R}^S_+$ for some compact set $K_S\subset\mathbb{R}^S$. Define a cooperative TU game v_w by $v_w(S) = h(w,S)w_i$ for $S\subseteq N$.Then $e(w,S) = e(S,h(w,N))\sum_{i\in S} w_i$, where $e(S,\cdot)$ is the excess of the TU game v_w as defined in Chapter 13 (p.231). The construction corresponds to restricting the cooperative game to deal only with imputations on the ray defined by w.

The nucleolus with respect to w is then defined by assigning to each player the nucleolus of v_w multiplied by w. Since v_w has a nonempty set of imputations, the nucleolus of v_w is nonempty and singlevalued, and so is the nucleolus of (N,V) (with respect to w).

If the nucleolus w.r.t. w is not in the core, then there must be a coalition S such that $h(w;N)w_S$ belongs to the interior of $V(S)$, a contradiction, so that the nucleolus must belong to the core.