Solutions to Exercises in

Game Theory
Chapter 5

1. [The problem is unfortunately very badly formulated: What is proposed to be shown is obviously false. What should be shown is that the *sum* of marginal rates of substitution which equals not k but $1/k$.]

If the allocation is Pareto efficient, then it must maximize a weighted sum of individual utilities subject to the constraints given by the endowment and the production of public goods,

$$\max \sum_{i=1}^{n} \lambda_i u_i(x_i, y)$$
$$\sum_{i=1}^{n} x_i + \frac{1}{k} y = \sum_{i=1}^{n} \omega_i$$

with first order conditions

$$\lambda_i u'_{i1}(x_i, y) + \mu = 0,$$
$$\sum_{i=1}^{n} \lambda_i u'_{i2} + \mu \frac{1}{k} = 0$$

for $i = 1, \ldots, m$, from which we obtain that

$$\sum_{i=1}^{m} \frac{u'_{12}}{u'_{i1}} = \frac{1}{k}. \tag{1}$$

Given the concavity of the utility functions, an allocation satisfying (1) is Pareto efficient.

In a Nash equilibrium, each individual has chosen b_i so as to maximize utility given the messages of the other individuals, so that

$$\frac{\partial u_i}{\partial b_i} = -u'_{i1} \frac{\partial t_i}{\partial b_i} + k u'_{i2} = 0$$

or

$$\frac{u'_{i2}}{u'_{i1}} = \frac{1}{k} \frac{\partial t_i}{\partial b_i} = \frac{1}{k} \left[\frac{1}{n} + \frac{n-1}{n} 2(b_i - \mu_i) \right] = \frac{1}{k}.$$

Since (1) is satisfied, the allocation is Pareto efficient.

2. The situation is one where it matters whether the players choose simultaneously or whether there is a first and second mover. The interesting result obtains when the son must move first. In this case, the mother, choosing last, must select t so as to maximize $U(y_1(e)-t)+\alpha V(y_2(e)+t)$ for given e, with first order condition

$$-U'y'(y_1(e)-t)+\alpha V'(y_2(e)-t)=0. \tag{2}$$

Using implicit function theorem, one obtains

$$\frac{\mathrm{d}t}{\mathrm{d}e}=\frac{U''y_1'-\alpha V''y_2'}{U''+\alpha V''},$$

so that the first order condition for the son's maximum becomes

$$V'y_2'+V'\frac{\mathrm{d}t}{\mathrm{d}e}=V'y_2'+V'\frac{U''y_1'-\alpha V''y_2'}{U''+\alpha V''}=0,$$

and dividing out V' (assumed $\neq 0$) one gets

$$y_2'+\frac{U''y_1'-\alpha V''y_2'}{U''+\alpha V''}=0. \tag{3}$$

If $\alpha=1$ and it is assumed that $U''=V''$ and not zero, this reduces further to

$$\left(1-\frac{\alpha}{2}\right)y_2'+\frac{1}{2}U'=0$$

which in the case of $\alpha=1$ becomes $y_2'+y_1'=0$, so that the sum of incomes is maximized.

It is seen that an increase in α results in a smaller numerical value of the (negative) second term on the left-hand side of (3), and assuming y_2' increasing, this means that e must decrease to restore equality. Taken together with negativity of $\frac{\mathrm{d}t}{\mathrm{d}e}$, one gets that t is a increasing function of α.

For small enough values of α, optimization by the mother results in negative t if $y_1(e)$ is sufficiently small compared to $y_1(e)$ for all e.

3. Since the game is symmetric, we look for symmetric equilibria. It is obvious that the probability of choosing research outlays r greater than W must be 0. If firm 1 chooses research outlays r in the interval $[0,W]$, the expected net income is

$$W\cdot\mathrm{Prob}\{r_2\le r,\dots,r_n\le r\}-r=WF(r)^{n-1}-r.$$

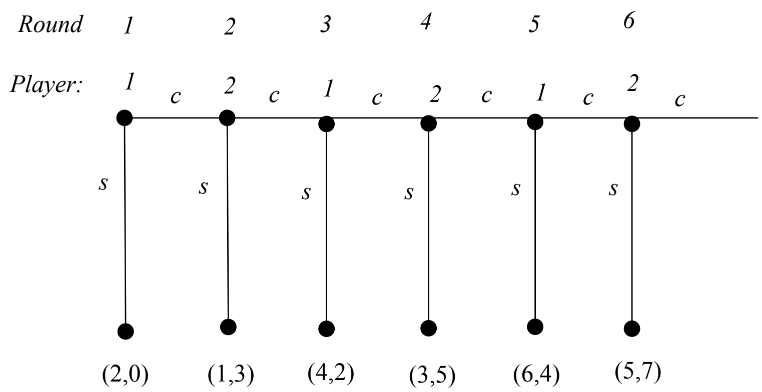
Assuming that the support of F is all of $[0,W]$, we can solve for $F(r)$ to get

$$F(r)=\left(\frac{r}{W}\right)^{\frac{1}{n-1}}$$

for $r\in[0,W]$ and $F(r)=1$ for $r\ge W$.

Strictly speaking, Theorem 1 does not apply, since the sets of pure strategies is not compact. However, as we see, the strategy sets may be truncated in an obvious way so as to make the theorem apply.

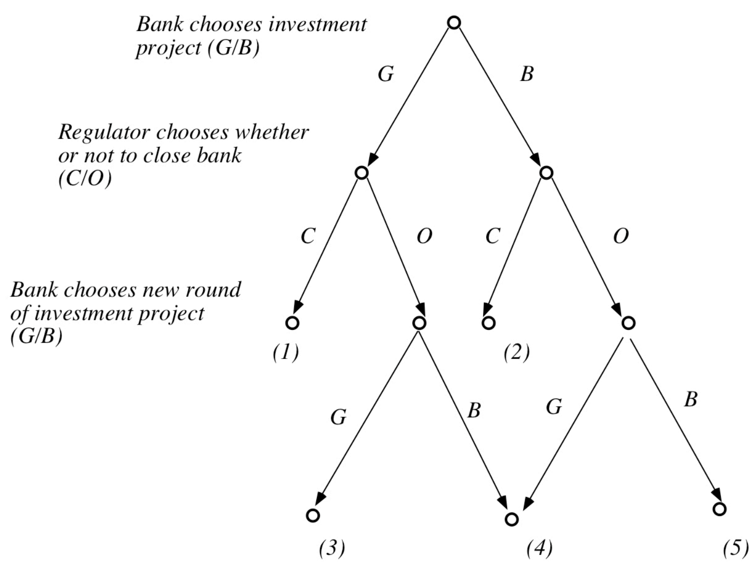
4. The extensive form is sketched in the figure below, where only the first 6 of 100 rounds are indicated.



A strategy of one of the players prescribes the choice in any of the 50 rounds where the player must choose (even if the game has actually stopped long before). Any strategy where the first player stops at round 1 and the second player stops at round 2 (or stops with a large enough probability to prevent player 1 from choosing c) is a Nash equilibrium. Most of these strategies will prescribe choices that the player would not want to make if the game actually reached this round (which it does not since it stopped in the initial round).

There is only one subgame perfect equilibrium, namely the one where each player chooses s at any possible round.

5. The game tree in the figure below has cost of regulator and gain of bank as payoffs (so that regulator minimizes payoff). If e.g. the bank chooses S (safe) and the regulator C (close down, not a very likely choice, but in principle a possibility), then the play results in the terminal node numbered (1) with a gain of r to the bank and a cost of C to the regulator. If the bank chose B and the regulator C (node (2)), then the payoff to the bank would be $p\rho$ and the cost to the regulator $C + (1 - p)$ (with probability $1 - p$, the investment fails and the regulator will have to compensate the depositors to the amount 1). Similarly for the remaining nodes.



To find the equilibria, consider first the case where the bank prefers one risky and one safe investment to two risky investments, which occurs when $p(\rho - r) > 2p^2\rho$ (the outcome in (4) is better than that in (5)) or $p < \frac{\rho+r}{2\rho}$. The regulator knows that if the bank has chosen B, then the bank will choose G next time if it is not closed down, and a comparison of the cost at nodes (2) and (4) shows that the regulator keeps the bank open. If the bank initially chooses S, then again keeping open is better than closing down.

Suppose now that two risky investments is better for the bank than one safe and one risky. If the first choice was B, then the regulator will be better off by closing down the bank if $C + (1 - p) < C(1 - p^2) + 2(1 - p)^2 + 2p(1 - p)(1 - \rho)$ or, equivalently, if $C < \frac{(1-p)(1-2p\rho)}{p^2}$. Otherwise, the regulator will keep the bank open after an initial choice of B.

It remains to consider the other case, where two risky investments are preferred to one risky and one safe. In this case, we must expect the bank to choose B if it gets to the second round, no matter what was chosen before. If the first choice was B, then the regulator will be better off by closing the bank if

$$C + (1 - p) < C(1 - p^2) + 2(1 - p)^2 + 2p(1 - p)(1 - \rho)$$

or, equivalently, if

$$C < \frac{(1 - p)(1 - 2p\rho)}{p^2}. \tag{4}$$

Otherwise, the regulator will keep the bank open after an initial choice of B. If (4)2) is satisfied, then the bank may G initially and then B. After an initial choice of G, the regulator will leave the bank open, even taking into consideration that B will be chosen next, unless $C < C(1 - p) + (1 - p)(1 - r)$, which can also be stated as

$$C < \frac{1 - p}{p}(1 - r).$$