Solutions to Exercises in

Game Theory
Chapter 6

1. [Typo: The formula for the optimal bid should be

$$\beta(x) = \frac{1}{G(x)} \int_0^x yg(y)\,\mathrm{d}y$$

with and y inside the integral and x as upper bound for the integral.] To show that β is a symmetric equilibrium strategy, we let the bidder act as if the value was z (rather than x). The expected payoff is then

$$G(z)(x - \beta(z)$$

(the probability of $\beta(z)$ being largest or (assuming monotonicity of β) of z being largest value, multiplied with the gain if winning), and this can be rewritten as

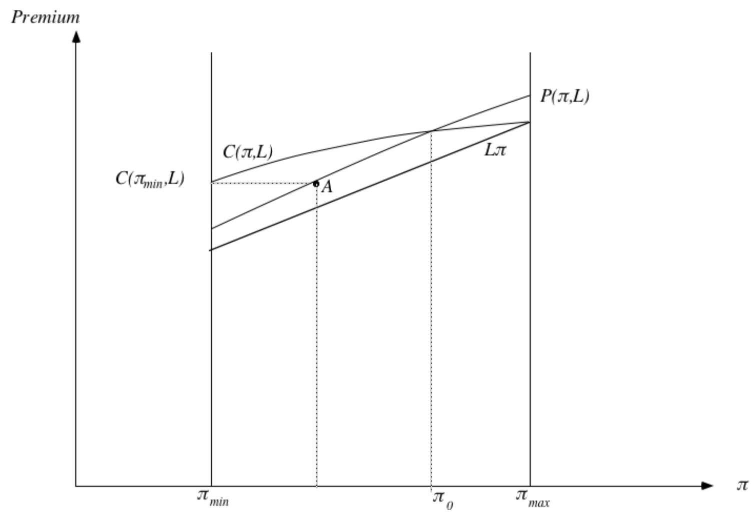
$$\begin{aligned} G(z)(x - \beta(z)) &= G(z)x - \int_0^z yg(y)\,\mathrm{d}y \\ &= G(z)x - G(z)z + \int_0^z G(y))\,\mathrm{d}y = G(z)(x - z) + \int_0^z G(y))\,\mathrm{d}y, \end{aligned}$$

where we have used integration by parts. Subtracting this from the expected gain using x, which is $\int_0^x yg(y)\,\mathrm{d}y$, we get the difference

$$G(z)(z - x) - \int_x^z G(y)\,\mathrm{d}y,$$

which is easily seen to be ≥ 0 for all values of z.

2. The figure below contains three curves, namely for each p (in the text p) (1) the average loss of an individual with risk p, (2) the average loss $C(p, L)$ of an individual with risk $\geq p$, and (3) the willingness to pay $P(p, L)$ of an individual with risk p.



The individuals for which $C(p;L) > P(p,L)$ will not buy insurance since they consider it as too expensive.

3. In the first step of this procedure (where we determine $t(a)$ for each e), we solve the problem of maximizing

$$\sum_{h=1}^{r} p_h(e)(y_h - t_h)$$

over all $(t_1, \dots, t_r)$ such that

$$\sum_{h=1}^{r} p_h(e)(v(t_h) - w(e)) = 0,$$
$$\sum_{h=1}^{r} p_h(e)(v_h(t_h) - w(e)) \geq \sum_{h=1}^{r} p_h(e')(v(t_h) - w(e')),\ \text{all } e'.$$

where the first constraint a participation condition (the 0 on the right-hand side represents the expected value of alternative engagement) and the second is the incentive compatibility constraint (the agent must be induced to deliver the effort e)..

Let $t^* = (t_1^*, \dots, t_r^*)$ be a solution, and let $K(e)$ be set of e' such that the last condition is fulfilled with equality. Restricting to the corresponding equations gives a maxinization problem with first order conditions

$$p_h(e) + \lambda v'(t_h^*) + \sum_{e' \in K(e)} \mu(e') v'(t_h^*)(p_h(e) - p_h(e')) = 0,$$

or

$$\frac{1}{v'(r_h^*)} = -\lambda - \sum_{e' \in K(e)} \mu(e') \frac{p_h(e) - p_h(e')}{p_h(e)},$$

for $h = 1, \dots, r$, where λ and $\mu(e')$, $e' \in K(a)$, are Lagrangian multipliers.

4. In a symmetric Bayesian Nash equilibrium, the bid function $b(x)$ (depending on the signal x received by the individual), assumed to be monotone, is the same for both, and the payoff is

$$\pi(b,x) = \int_0^{\beta^{-1}(b)} (v(x,y) - \beta(y))g(y|x)\,\mathrm{d}y$$

where $g(y|x)$ is the conditional density of the signal received by the other individual, given that x is the highest signal – the first individual wins if her bid is the highest and pays only the bid of the other individual. Writing the integral as

$$\int_0^{\beta^{-1}(b)} (v(x,y) - v(y,y))g(y|x)\,\mathrm{d}y$$

(using the both have the same bidding function), we use that $v(x,y)$ is $>$ or $<$ than $v(y,y)$ depending on whether $x > y$ or $x < y$. To maximize π, we need to keep all the positive and discard the negative contributions to the integral, and this is obtained by choosing $\beta^{-1}(b) = x$ or $b = \beta(x)$.

Suppose that

$$v(x,y) = \frac{1}{3}x + \frac{2}{3}y$$

(valuation depends with weight 1/3 on own signal and with weight 2/3 on that of the other bidder). Then the valuation of bidder 1 is greater than that of bidder 2 if and only if $y > x$. In this case bidder 2 has the highest bid and wins the auction, but bidder 2 values it higher, meaning that the second-price auction is not efficient.

Let $x > y$ be arbitrary, and consider the path from (y,x) to (x,y) given by

$$\gamma(t) = (1-t)(y,x) + t(x,y),\ t \in [0,1]$$

By the mean value theorem, there is some t^0 such that

$$\frac{\mathrm{d}v(\gamma(t))}{\mathrm{d}t}(t^0) = v(x,y) - v(y,x)$$

The left-hand side can be written as

$$v_1'(\gamma(t^0))(x-y) + v_2'(\gamma(t^0))(y-x) = (v_1' - v_2')(x-y),$$

and by single-crossing and our assumption $x > y$, we get that $v(x,y) > v(y,x)$, so that highest valuation also has highest bid.

5. If the buyer must propose a price initially, then utility is $\tau_B q - p$ if $p \geq \tau S\, q$ and 0 otherwise, so that trade will occur whenever $\tau_B \geq \tau_S$, and the proposed price is $\tau_S \frac{Q}{2}$.

When the seller proposes a price initially, the buyer will accept when expected value of q exceeds the proposed price, and since the quality must be in the interval $\left[0, \frac{p}{\tau_S}\right]$ with mean $\frac{p}{2\tau_S}$ and the buyer will accept when

$$p \leq \tau_B \frac{p}{2\tau_S}$$

or $\tau_B \geq 2\tau_S$. In this case, the seller will obtain maximal payoff if the price is set as $\tau_S Q$.