Lecture Note 11: Walras and General Equilibrium

Marie-Esprit Léon Walras (1834 – 1910) was educated at the Ecole des Mines, tried his hand as a novel author (Francis Saveur, 1858) without particular success, and from 1860 and onwards he became increasingly interested in economics. In 1870 he was appointed professor in Lausanne. His main work *Eléments d'économie politique pure* appeared 1874 – 77, followed by later works which however were less path-breaking than this work.

Walras' system of economics consists of three parts, namely

(1) pure economics, dealing with exchange and markets,

(2) applied economics, dealing with industrial production,

(3) social economics, which deals with questions of property.

Each of these parts are concerned with *social wealth,* defined as all things, materal or immaterial, which are scarce in the sense of being useful to us and only available in limited quantities. In the three subfields, one studies social wealth from of vue of value in exchange, in industry, and in property. While in the study of exchanges, the guiding principle is utility, this does not hold in part 3, where justice should be the fundamental guide in the distribution of social wealth. From a contemporary viewpoint, pure economics would be the positive study of the market mechanism, whereas applied economics is the normative theory of optimal allocation, and social economics the normative theory of income distribution.

According to Walras, pure economics leads to the principle of laissez-faire by which maximum utility is achieved through free competition. This principle must therefore also be applied to problems of applied economics, at least as long as no public interest is involved. Walras took care to emphasize that it would be an error to advocate a transfer of public services to private industry, and he insisted on nationalization of all landed property.

The theory of markets, or in Walras' terminology: pure economics, is what was elaborated in most details. Much of the theory of exchanges and of supply and demand could be taken over from predecessors, and Walras was well aware of the contributions by Cournot, but Walras contributed with the introduction of marginal utility, sharing the honour of its discovery with the other marginalists Jevons and Menger (although also here, earlier authors had these or similar concepts). But the really path-breaking contribution of Walras it the formulation of the problem of market equilibrium, described through a system of equations, the solution of which would give the equilibrium prices and quantities. The equations describe how demand and supply of commodities depend on market prices of *all* commodities. To this must be added equations taken from the equilibrium of production, giving demand and supply of factors of production. Walras considers also capital in connection with production, but here capital is social wealth which is not used up, that is land, human capital and capital goods proper. This contrasts with the way that the classical writers and Marx considered capital, as consisting of both constant and variable capital, and it points forward to the way in which contemporary economics treats capital. The final part of the system of equations comes from Walras' theory of money and was introduced only at a later stage of the research.

The existence of a general equilibrium. Setting up the system of equations which should be satisfied in an equilibrium immediately raises the question of whether there is a solution at all? A similar question arises when considering price formation by demand and supply in a single market, but here it can usually (even if not always) be answered in an easy way. This is not the case when there is a very large number of equations, at least as many as there are commodities. In a simple version, we have a system

$$\zeta_h(\mathbf{p}) = 0,\ h = 1, \dots, l, \tag{1}$$

where $\zeta_h(\mathbf{p})$ is the excess demand (supply minus demand) for commodity h, depending on the in the price vector $\mathbf{p} = (p_1, \dots, p_l)$, for $h = 1, \dots, l$. Since only relative prices matter for demand and supply, one may select one of the commodities, say number l, as *numeraire* (the terminology is taken from Walras), setting $p_l = 1$, so that there are $l - 1$ unknown commodity prices to be determined.

Walras used two different approaches to argue that the system will have a solution. The first one consisted in counting equations and unknowns in (1). At a first glance, this systems seems overdetermined since there are l equations but only $l - 1$ unknowns. However, using Walras' law

$$\sum_{h=1}^{l} p_h \zeta_h(\mathbf{p}) = 0$$

which was more or less known even before Walras (it follows from the fact that consumers use all their income for buying commodities), we have that if any $l - 1$ of the excess demands ζ_h are zero at some $\mathbf{p}$, then so is the last of them, meaning that $l - 1$ of the equations in (1) are enough.

Even so, matching the number of equations and unknown is not enough to secure that the system has a solution, but this was as far as one could go at the time of Walras. The necessary formal tools came around only in the beginning of the 20th century (Brouwer's fixed-point theorem) and they were used in the context of economics only with another delay of almost fifty years. An account of the long way from Walras' statement of the problem to its solution can be found in Arrow and Hahn (1971) or in Ingrao and Israel (1990). Many economists pointed out the need for a proof of existence, actually also a Danish economist (Frederik Zeuthen) was involved in the debate, noticing that one needs not only a solution of (1) but a *nonnegative* one. The first existence proofs were given by several authors more or less at the same time (Gale, Debreu, McKenzie, Nikaido) using basically the same approach. One of the stumbling blocks on the road was the fact that demand at a given price (that is, the best possible bundle for a consumer) need not be unique but may have many solutions, so that one has to work with demand not as a function (one price vector goes to one commodity bundle) but a correspondence (one price gives rise to may possible commodity bundles). Later approaches has made it possible to avoid the notions of demand (and utility) altogether.

The tâtonnement process can be considered as an economic argumentation for the existence of a solution – if excess demand for some commodity is positive, then we expect its price to rise, and if it is negative, the price would fall, thus getting closer to the equilibrium where excess demand is zero for all commodities. This can be formulated as a system of differential equations

$$\frac{\mathrm{d}p_h(t)}{\mathrm{d}t} = a_h \zeta_h(\mathbf{p}(t)), h = 1, \ldots, l, \tag{2}$$

defining a time path $\mathbf{p}(t)$ of the price vector starting at some arbitrary $\mathbf{p}(0)$. Intuitively, this path would take us to an equilibrium price. Unfortunately, it is not easy to show that this is actually the case. The first general result was obtained only much later (Allais, 1943) and under rather strong assumptions (gross substitution).

What is perhaps worse is that the tâtonnement process if taken literally presupposes that no trades are carried out before equilibrium is achieved. The traders may be considered as brought together in a market place, and there is an auctioneer (usually called a 'Walrasian' auctioneer) crying out prices on all the commodities, then collecting excess demands, correcting and crying a new price. Apart from the picturesque details, the tricky part is that all traders remains passive throughout the process. If instead they were allowed to carry out some trades, they would all have new demand and supply functions after these trades, and the process (2) could not be maintained. Instead, one would have some kind of *non-tatônnement process* (with some trade and partial updating of endowments) for approaching equilibrium prices. The research in non-tâtonnement processes has however not been promising and they have been largely abandoned after some initial activity in the 1960-70s.

It has been argued that Walras got his inspiration to the equilibrium concept from mechanics, whereas other parts of physics, notably thermodynamics, might have given rise to another way of formulating a general equilibrium (Smith and Foley, 2008).

Time in the Walrasian system. The general equilibrium model of Walras treats a static economy, and apart from the tâtonnement process, Walras did not study what happens over time. He does however treat capital and in this way he is led to the introduction of money which otherwise is totally absent.

Here is a simple version of Walras' model with capital goods, taken from Negishi. There are two goods, a consumption good and capital, which together with labour is used in production. We let x_1 and x_2 be output of the two goods, and define total income as

$$Y = w(a_1x_1 + a_2x_2) + q(b_1x_1 + b_2x_2),$$

where a_i and b_i are the input coefficients of labour and capital in production of the two goods, w and q are wages and the payment for hiring services of capital. We assume constant returns to scale, so that profits are 0, and therefore

$$p_1 = wa_1 + qb_1,$$
$$p_2 = wa_1 + qa_2,$$

where p_1 and p_2 are the prices of the two goods. In both markets, demand must equal supply,

$$D(p_1, p_2, w, q, Y) = x_1$$
$$H = x_2,$$

where $D(p_1, p_2, w, q, Y)$ is the demand for consumption goods, depending on all prices and income, and H is the demand for new capital goods (that is investment). Since also factor markets must balance, we have

$$a_1x_1 + a_2x_2 = L,$$
$$b_1x_1 + b_2x_2 = K,$$

where L and K are the given amounts of labour and capital.

Since there is no money in the model, we assume that there are capitalists owning capital and selling their services to firms. Capitalists save part of the revenue obtained when buying new machines, so that

$$p_2H = S(p_1, p_2, w, q, Y)$$

where $S(p_1, p_2, w, q, Y)$ is total savings in the economy.

The above equations, eight in total, describe what may be considered a *temporary equilibrium* in the economy considered (equilibrium in the current market, the future is described by the model). The unknown to be determined are Y, w, q, x_1, x_2, p_2, and H, since as always one commodity, say number 1, can be chosen as numeraire. But as usual, the eight equations are actually only seven, since Walras' law holds,

$$Y = p_1 D(p_1, p_2, w, q, Y) + S(p_1, p_2, w, q, Y)$$

for all values of p_1, p_2, w, q and Y. Thus the system is not overdetermined (even if this, as always, is no guarantee for existence of solutions).

The somewhat unrealistic assumption that saving takes place directly in the form of machines can be done away with: Walras introduced another commodity E which consists of a perpetual income of one unit of the numeraire good. The price of this good can be written as $\frac{1}{i}$ and defines the implicit interest rate of the perpetuity. This commodity is sold by firms wishing to buy new capital goods, and it is bought by capitalists wishing to save, so in this sense the commodity has some of the functions of money. Now savings depend on i rather than on q as above, so we have a new equilibrium condition

$$p_2 H = S(p_2, w, i, Y)$$

(remember that $p_1 = 1$ since good 1 is numeraire). Similarly, we have now

$$D(p_2, w, i, Y) = x_1$$

where the demand for consumptions good now depends on i instead of q. Finally, we must have that the rate of income derived from capital goods should be equal to the interest rate,

$$\frac{q}{p_2} - d = i,$$

where d is a given rate of depreciation of capital goods (buying a unit of capital and renting it out should give the same income as one would get from E). We have added an equation (the last one) and an unknown, keeping the parity.

We may consider this as a temporary equilibrium (concerned only with the present period) just as above, but it may be somewhat farfetched given that we have introduced a perpetuity and compare its payoff to that of capital. For this to make sense, we may think of the economy as stationary, everything repeats itself each period, but then we get yet another equation, namely

$$H = dK,$$

(investment only replaces downworn equipment, it doesn't increase the capital stock). Now the system is overdetermined, so for a solution to exist at all, we must have one more variable to be determined in equilibrium, and the obvious choice is K, the existing capital stock, which now must have a particular size in order for the economy to remain in the stationary state.

The two alternative approaches outlined here has later been developed into either a theory of temporary equilibria or a neoclassical growth theory. Both are based on the initial formulations given by Walras.

Vilfredo Pareto (1848 – 1923) followed Walras as professor in Lausanne and became himself the initiator of a specific Italian tradition in (mathematically oriented) economics. His main contributions to economics were the following two (closely interrelated) extensions of the Walrasian approach, inspired by Edgeworth:

(a) Use of indifference curves in the analysis of consumer behavior and demand, and a closer scrutiny of the concept of a utility function. More particularly, Pareto noticed that only "larger" or "smaller" utility levels mattered for the analysis, giving rise to the notion of *ordinal* utility functions, as distinct from *cardinal* utility functions, where the numerical value would have a meaning. Nowadays, one would characterize a utility function as ordinal if any increasing transformation of the utility function would be just as good, and cardinal if only increasing affine transformations (multiplication by a positive number and addition of a constant) would be just as good.

(b) The notion of Pareto optimality: Since the numerical values of the individual utilities have no meaning, it is similarly meaningless to maximize their sum, as was done in previous considerations of a social optimum. Realizing this, Pareto proposed the by now wellknown principle, according to which a social state is considered better than another only if everybody is better off (or slightly more general: if everybody is as well off and at least one is better off). Pareto optimality occurs when no improvement is possible according to this principle.

Pareto has also given name to a probability distribution, which he considered as describing data on income distribution. Nowadays, the Pareto distribution is mostly used in other contexts, such as financial economics and computer science. A popular approach to the Pareto distribution is the so-called 80:20 rule – eighty percent of the individuals have 20% of the income (and vice versa), and if the 20% richest are selected, then again 20% of these hold 80% of the income in this group. Its density on $[x_m, \infty)$ with $x_m > 0$ is

$$f(x) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}},$$

where $\alpha > 0$ is a parameter (around 1.15 if the 80:20 rule holds).

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