Lecture Note 17:
A brief introduction to Sraffa's"Production of Commodities..."

The small (around 100 pages) book which appeared in 1960 but based on work done in the 1920s has been described both as a revolution in economic theory and as useless and outdated. Here is a brief overview of what happens in the first part (roughly the first 40 pages) of the book.

From the preface it can be seen that the book is seen by the author as an argument against *marginalism* and as such a rehabilitation of the classical authors and their approach to economics. Sraffa emphasizes that no assumption of constant returns to scale is used, since the analysis is restricted to a fixed case instance of production described by input and output. Indeed, any such assumption would imply that the analysis takes *changes* of input and output into consideration, so that some sort of marginalism enters through the backdoor. As we shall see, this viewpoint may be sustained but at a certain cost, since change in the formal specification of inout and output do occur and actually plays a crucial role.

We consider a productive system where k commodities are produced separately but using some, possibly all, of the commodities as input. The analysis proceeds in several steps:

1st step: Here we consider an abstract production system which functions under pure subsistence, so that the output of commodities is equal to the inputs. Following Sraffa, we give an example of a production in such a system (with three commodities):

	wheat	iron	pigs	output
wheat	280	12	18	450
iron	90	6	12	21
pigs	120	3	30	60
total use	450	21	60	

Since production is fully used up as inputs, there is no surplus. We are interested in prices which reflect this no-surplus situation, so that value of output equals value of inputs. Denoting the sectors as 1,2,3, input of sector j into sector i as z_{ij} and output in sector i as y_i, we get the three linear equations

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\begin{aligned}
p_1 z_{11} + p_2 z_{12} + p_3 z_{13} &= p_1 y_1 \\
p_1 z_{21} + p_2 z_{22} + p_3 z_{23} &= p_2 y_2 \\
p_1 z_{31} + p_2 z_{32} + p_3 z_{33} &= p_3 y_3
\end{aligned}
$$

with the three unknowns p_1, p_2, p_3. Since multiplying all prices with a constant gives a new solution, we can solve only for relative prices, which is as it should be since one of the equations can be derived from the two others due to equality of output and total use of input for each commodity. What matters for Sraffa is that relative prices are *uniquely* determined once we know the production in this subsistence economy.

2nd step: We now move to the case where there is a surplus, so that for some or all commodities, more is produced than what is used up. Here is an example of this situation, again taken from Sraffa (for the moment, neglect the column labeled "labor"):

	iron	coal	wheat	labor	output
iron	90	120	60	$\frac{3}{16}$	180
coal	50	125	150	$\frac{5}{16}$	450
wheat	40	40	200	$\frac{8}{16}$	480
total use	180	285	410	1	

The surplus or net product, 165 units of coal and 70 units of wheat, can be used by by the individuals of the economy. To find out how it should be distributed, one needs prices, giving rise to an income (the GDP of this economy) to be distributed among income receivers. Sraffa assumes that there are two types of income receivers, capital and labor, and that capital gets its income according to a *profit rate r*, the same for all sectors (here Sraffa argues that otherwise capital would move from one sector into another, an argumentation that presupposes a theory about the way in which production is organized), whereas labor is paid by a wage according to the share of labor present in each sector (as indicated in the example). The equations for determining prices, profit and wage rates are then

$$\begin{aligned} (p_1 z_{11} + \ldots + p_k z_{1k})(1 + r) + L_1 w &= p_1 y_1 \\ \vdots \qquad &= \quad \vdots \\ (p_1 z_{k1} + \ldots + p_k z_{kk})(1 + r) + L_k w &= p_k y_k \end{aligned}$$

We now have $k + 2$ variables, or, if we consider only relative prices, $k + 1$ variables. But there are only k equations (even though in this case they are independent), so we need to specify one variable in order to solve for the remaining variables, for example we can specify r and then solve for the prices and wages (measured relative to one of them).

Once we have prices, we can determine GDP (value of net product of commodities at this prices), and then we can determine the income distribution between capital and labor. But in order to determine prices, we have already chosen either profit or wage rate. And if we chose it in another way, we would have had different prices, different GDP and different income distribution.

Now we can see Sraffa's problem and where it enters the discussion for and against neoclassical theory of distribution. In order for payment according to marginal productivities of capital or of labor to make sense, capital and labor should be measured in a meaningful way, which is already a problem, noticed by many other authors. But Sraffa points out that there are problems even with measuring the income which is to be distributed – this cannot be done meaningfully without knowledge of the income distribution!

3rd step: The solution which Sraffa proposes, is to reintroduce *values* which largely disappeared with the marginalists, into the discussion. If there is a measure of value which is independent of the distribution of the surplus, then one may discuss the distribution of this surplus, measured in value terms, in a meaningful way. Taking one of the commodities or labor as the standard by which to measure values will not serve, exactly by the arguments above. So Sraffa sets out to find another standard of value.

To do so, we return (with Sraffa) to the previous example, where we multiply all entries in the second row by $\frac{3}{5}$ and all entries in the third row by $\frac{3}{4}$. This gives us the new table as below:

	iron	coal	wheat	labor	output
iron	90	120	60	$\frac{3}{16}$	180
coal	30	75	90	$\frac{3}{16}$	270
wheat	30	30	150	$\frac{6}{16}$	360
total use	150	225	300		

An immediate interpretation of what has happened would be that we reduced the production in sectors 2 and 3 while retaining that of sector 1. But this interpretation presupposes constant return to scale, which Sraffa would rather avoid, so we should perhaps think of it as emerging when we produce as before but throw away some output and input. Be this as it may, the new system looks almost as the one we had in Step 1, only now output is not equal to input, rather output is $\frac{6}{5}$ times input in each sector. This means that just as in Step 1, we can determine relative prices uniquely, since one of the equations can be found from the remaining ones due to this new constraint. In other words, we have way of finding price which is independent of assumptions about distribution of the surplus.

As a by-product, we have that the combination of iron, coal and wheat in the proportion 150:225:300 plays a particular role, and we may consider this combination of commodities as a new one, which we call the *standard commodity*. The rate by which output of the standard commodity increases compared to input is called the standard ratio and denoted by R (so that here we have $R = \frac{1}{5}$).

4th step: We found the revised production by a particular manipulation, but we have of course to check whether (a) this can always be done, and (b) there is only one way of doing it. For this, we reconsider what was actually done: Each line in the table was multiplied by some q_i so that the resulting inputs of each commodity j, that is $q_j z_{ij}$, multiplied by $1 + R$, sums to the final (revised) production of j, giving a system

$$\begin{aligned}(q_1 z_{11} + \ldots + q_k z_{k1})(1+R) &= q_1 y_1 \\ \vdots \qquad\qquad &= \ \vdots \\ (q_1 z_{1k} + \ldots + q_k z_{kk})(1+R) &= q_k y_k\end{aligned}$$

and checking whether there are such multipliers $q_1, \ldots, q_k$ and such a rate R amounts to finding a solution to this system. In matrix form, the system can be written as

$$\begin{pmatrix} q_1 & \cdots & q_k \end{pmatrix} \mathbf{Z}\,(1+R) = \begin{pmatrix} q_1 y_1 & \cdots & q_k y_k \end{pmatrix}$$

where $\mathbf{Z}$ is the matrix with (i, j)-element z_{ij}. This doesn't look too promising, but if we divide each of the equations above by y_i (corresponding to measuring input use relative to output, the same format as the input coefficients that we used when discussing Smith and Marx), giving a matrix $\widehat{\mathbf{Z}}$ with elements $\hat{z}_{ij} = \frac{z_{ij}}{y_j}$, we get rid of the coefficients on the righthand side and can rewrite the system as

$$\begin{pmatrix} q_1 & \cdots & q_k \end{pmatrix} \widehat{\mathbf{Z}} = \frac{1}{1+R} \begin{pmatrix} q_1 & \cdots & q_k \end{pmatrix}$$

Now we can recognize the problem as that of finding an *eigenvector* of the matrix $\widehat{\mathbf{Z}}$ (or, to be correct, to the transposed matrix $\widehat{\mathbf{Z}}'$), and now we may invoke the Perron-Frobenius theorem, saying that if all elements of $\widehat{\mathbf{Z}}$ are nonnegative (as they indeed are), then there is a unique greatest real eigenvalue with associated nonnegative eigenvector $q_(q_1, \ldots, q_k)$. We may even (under additional assumptions which are satisfied in our case) assume that the eigenvalue is nonzero, so that its inverse gives us the quantity $1 + R$. Although the Perron-Frobenius theorem was established around 1910, Sraffa was probably not aware of it, so he gives what essentially amounts to an alternative proof of the result.

In the example above, we already found the eigenvector to be $(1, \frac{3}{5}, \frac{3}{4})$, the eigenvalue is $\frac{5}{6}$, and we get the standard commodity as $(q_1 y_1, \ldots, q_k y_k)$, in this case $(180, 270, 360)$.

Concluding comments: So far we have found that to every configuration of inputs and outputs there is a particular standard commodity which may be used as a unit for defining values of commodities, and a standard rate R which gives the value of the surplus when revised so as to produce the standard commodity using the standard commodity as input. Now one can determine an associated income distribution as follows: Normalize the standard commodity so that input is 1. Then assign a share w to labor, so that Rw is the surplus going to labor. The remaining 1-w goes to capital and yields, so that capital gets

$$r = R(1 - w)$$

which is the rate of profit since value of input is 1.

This of course pertains only to the revised configuration and doesn't tell us how much goes to labor and how much to capital in the actual configuration. Here Sraffa proposes to use that w is price in terms of standard commodity of buying one unit of labor. In the actual configuration with given prices (which are no longer uniquely determined), we can then find the value of the standard commodity and then the value of labor, now measured in actual prices. In this sense it can be argued that we have achieved a way of deciding upon distributional issues (namely, in value terms) without being disturbed by the non-uniqueness of prices sustaining the given configuration.

Sraffa does not follow up on this issue, the rest of the book is concerned with extension of the treatment to situations of dated inputs and of joint production, as well as the case where more than one technique is available in each sector, so that switching techniques becomes relevant. The way in which the value is used remains rather obscure, which may be one of the reasons why Sraffa's work was not followed up by further contributions and new insights, but remained in the field of verbal discussions among theoreticians.

Game Theory arose out of discussions between mathematicians (among which **John von Neumann** (1903 – 1957) and **Emile Borel** (1871 – 1956)) in the late 1920s, and the real breakthrough (without which there would probably not have been an independent discipline of game theory) was the idea of von Neumann to introduce *mixed strategies* (determining the choice using a probability distribution). A cornerstone in early game theory is the book *Game Theory and Economic Behavior* by von Neumann and **Oscar Morgenstern** (1902 – 1977) from 1944, but when it appeared, there was already a well-established body of knowledge.

In the first decades, zero-sum games constituted the main field of interest, only in the late 50s non-zero sum games came to the forefront. The prisoners' dilemma game is a standard showpiece of game theory but less important in itself than as a stepping stone for further developments (e.g. of repeated games). It should be mentioned also that the 1944 book by von Neumann and Morgenstern used more than half of its pages on *cooperative games.*

The early history of game theory is described in Dimand and Dimand (1996).

References:

Dimand, M.A. and R.W.Dimand (1996), A history of game theory, Volume 1, Routledge, London and New York.

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