Lecture Note 8

Marx's Capital somewhat modernized

Since Marx' economics is so well adapted for a treatment à using linear models (of which we have seen several examples already), there has been many such contributions over the time, dating almost from the appearance of Marx's main work. We consider a simple model of this kind.

Assume that there are only two goods in the economy considered, namely machines (good number 1) and a wage good (number 2). Both goods are produced using machines and labour, with coefficients a_i and l_i, $i = 1, 2$, for machines and labour. The (labour) values of the two goods are then given as indirect and direct labour content,

$$v_i = a_i v_1 + l_i, i = 1, 2. \tag{1}$$

Now, one unit of labour can be (re-)produced using w units of the wage good, so that its value is wv_2. Inserting, we get that surplus value (left-hand side minus right-hand side) in the two sectors are

$$\begin{aligned} v_1 - a_1 v_1 - wv_2 l_1 &= l_1 - wv_2 l_1 = l_1(1 - wv_2) \\ v_2 - a_2 v_1 - wv_2 l_2 &= l_2 - wv_2 l_2 = l_2(1 - wv_2), \end{aligned} \tag{2}$$

from which it follows that the rates of exploitation (surplus value divided by value of direct labour) e_i in the two sectors satisfy

$$e_1 = \frac{l_1(1 - wv_2)}{wv_2 l_1} = \frac{1 - wv_2}{wv_2} = \frac{l_2(1 - wv_2)}{wv_2 l_2} = e_2,$$

so that $e_1 = e_2 = e$, the rate of exploitation is the same in the two sectors. We can then write the equations in (1) as

$$v_i = \underbrace{a_i v_1}_{\text{constant capital}} + \underbrace{wv_2 l_i}_{\text{variable capital}} + \underbrace{ewv_2 l_i,}_{\text{surplus value}} \quad i = 1, 2. \tag{3}$$

where the split into constant capital, variable capital and surplus value becomes apparent.

We collect the coefficients (how much do we need of good i to produce good j) in a matrix

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 \\ wl_1 & wl_2 \end{pmatrix}.$$

Using this notation, we can write the amount of the two goods we have to use as input if we want to produce the output vector $\mathbf{x} = (x_1, x_2)$ as

$$\mathbf{Ax},$$

and the net product (output minus input) in the economy is therefore

$$\mathbf{x} - \mathbf{Ax} = (\mathbf{I} - \mathbf{A})\mathbf{x}.$$

Technically, labour has disappeared here, due to the fixed relation between labour and the wage good, so we are left with a two-good closed production model.

A matrix $\mathbf{A}$ is said to be *productive* if there is some positive vector $\mathbf{y} \neq 0$ such that $\mathbf{x} - \mathbf{Ax} > 0$ (where $>$ is taken coordinatewise). It can be shown that if $\mathbf{A}$ is productive, then $(\mathbf{I} - \mathbf{A})$ is regular (has an inverse), and its inverse $(\mathbf{I} - \mathbf{A})^{-1}$ has only nonnegative elements. It follows that if $\mathbf{A}$ is productive and $\mathbf{c} = (c_1, c_2)$ is a vector with positive coordinates, then $\mathbf{y} = \mathbf{c}(\mathbf{I} - \mathbf{A})^{-1}$ is nonnegative in both coordinates and nonzero. This shows that if a matrix is productive, so that there is a positive vector $\mathbf{v}$ with $\mathbf{v} - \mathbf{vA} > 0$, meaning that surplus value is generated.

Profit and rent. In order to consider the redistribution of surplus products between industry and agriculture (and in this way to introduce a theory of rent), we extend the model so that there are now two goods produced using these goods and labour, and gross production (x_1, x_2) satisfies

$$\begin{aligned} x_1 &= a_{11}x_1 + a_{12}x_2 + y_1 \\ x_2 &= a_{21}x_1 + a_{22}x_2 + y_2, \end{aligned} \tag{4}$$

where a_{ij} as usual are the input requirements of good j in the production og good i, and y_i the net output, $i, j = 1, 2$. The values are given by

$$\begin{aligned} v_1 &= a_{11}v_1 + a_{21}v_2 + l_1 \\ v_2 &= a_{12}v_1 + a_{22}v_2 + l_2, \end{aligned} \tag{5}$$

where l_i is labour requirement per unit output. Total labour employment is then $L = l_1x_1 + l_2x_2$. Multiplying each equation in (4) with the relevant v_i and adding, we get that

$$\begin{aligned} v_1x_1 + v_2x_2 &= a_{11}v_1x_1 + a_{12}v_1x_2 + v_1y_1 + a_{21}v_2x_1 + a_{22}v_2x_2 + v_2y_2 \\ &= (a_{11}v_1 + a_{21}v_2)x_1 + (a_{21}v_1 + a_{22}v_2)x_2 + v_1y_1 + v_2y_2, \end{aligned}$$

so that

$$v_1 y_1 + v_2 y_2 = (v_1 - a_{11}v_1 - a_{21}v_2)x_1 + (v_2 - a_{21}v_1 - a_{22}v_2)x_2 = l_1 x_1 + l_2 x_2 = L,$$

giving that

$$L = v_1 y_1 + v_2 y_2.$$

We assume that wages take the form of a bundle (b_1, b_2) of the two goods, so that (subsistence or socially given) wages are $v = b_1 v_1 + b_2 v_2$, and the rate of exploitation is therefore

$$e = \frac{L - vL}{vL} = \frac{1 - v}{v} \tag{6}$$

(just as in the previously considered model). After transformation of values to production prices, we get that

$$\begin{aligned} p_1 &= (1 + r)(a_{11}p_1 + a_{21}p_2 + l_1 w) \\ p_2 &= (1 + r)(a_{12}p_1 + a_{22}p_2 + l_2 w), \end{aligned} \tag{7}$$

where $w_1 = b_1 p_1 + b_2 p_2$.

The production prices and the rate of profit as given in (7) comes from equalization of profits among different sectors, and here the distribution of the surplus among sectors matter, so that final demand (y_1, y_2) is not independent of the production prices and the demand of capitalists in different sectors. On the other hand, values are determined independent of final demand. This is one of the points for Marx in separating the determination of values (in Chapter 1) from the determination of prices of production (in Chapter 3). The distribution between wage earners and capitalists is independent of the distribution among capitalists (there is a dichotomy between exploitation of surplus and distribution of surplus).

Things become more complicated when we consider the distribution between capitalists and landowners, allowing (as Marx did) for decreasing quality of soil. Assuming that good 1 is an agricultural product and allowing for diminishing returns to scale, the system (4) is rewritten as

$$\begin{aligned} x_1 &= A_{11}(x_1)x_1 + a_{12}x_2 + y_1 \\ x_2 &= A_{21}(x_1)x_1 + a_{22}x_2 + y_2, \end{aligned} \tag{8}$$

where $A_{11}(x_1)$ and $A_{21}(x_1)$ is the *average* input requirement at the production x_1, assumed to be increasing in x_1. The values, now, are given by

$$\begin{aligned} v_1 &= a_{11}(x_1)v_1 + a_{21}(x_1)v_2 + l_1(x_1) \\ v_2 &= a_{12}v_1 + a_{22}v_2 + l_2, \end{aligned} \tag{9}$$

where $a_{11}(x_1)$ and $a_{21}(x_2)$ are the *marginal* input requirements and $l_1(x_1)$ is marginal use of direct labour at the output level x_1. This comes from the way in which values are defined, namely as direct and indirect inputs of labour when the latter is not assisted by the extra fertility of soil which is not at the margin. Total employment of labour is $L = L_1(x_1)x_1 + l_2x_2$, where $L_1(x_1)$ is *average* labour input in the production of good 1, and proceeding as above, we get from (8) and (9) that

$$v_1y_1 + v_2y_2 = L + (a_{11}(x_1) - A_{11}(x_1))v_1x_1 + (a_{21}(x_1) - A_{21}(x_1))v_2x_2 + (l_1(x_1) - L_1(x_1))x_1,$$

showing that value of net output is larger than the input of embodied labour, giving rise to what Marx called a "false social value" created by labour *assisted by land.*

Wages are defined in terms of values as before, so that exploitation should be defined as

$$e = \frac{v_1y_1 + v_2y_2 - vL}{vL}. \tag{10}$$

Here the dichotomy breaks down, since values cannot be determined independent of (x_1, x_2) which again depends on the demands of the capitalists (in agriculture and industry). In order to recover the rate of exploitation as defined in (6), some redistribution of surplus value to landowners must take place, and this transfer of surplus value is imagined to be accomplished by competition of capitalists. In this way, we arrive at a theory of differential rent, but it cannot be separated from the theory of labour values, they must be determined simultaneously.

The transformation problem. So far we have been concerned only with values, but now we turn to production prices p_1, p_2 of machines and the wage good, which should cover the cost of both constant and variable capital, allowing for a uniform rate of profit, so that

$$p_i = (1 + r)(a_ip_1 + wl_ip_2), i = 1, 2.$$

Even if this seems to be another approach than that used above with values, they are connected by what is known in the literature as the *Fundamental Marxian Theorem:* There is a price system $\mathbf{p} = (p_1, p_2)$ such that

$$p_i > a_ip_1 + wl_ip_2, i = 1, 2, \tag{11}$$

if and only if $e > 0$.

Indeed, from (11) we get that the transposed matrix of $\mathbf{A}$ is productive, so that also $\mathbf{A}$ is productive, and there is some $\mathbf{x}$ with $\mathbf{x} > \mathbf{A}\mathbf{x}$. Taking the sum of values in both sectors, we have that

$$v_1x_1 + v_2x_2 - (v_1(a_1x_1 + a_2x_2) + v_2(wl_1x_1 + wl_2x_2)) = e(wl_1x_1v_2 + wl_2x_2v_2) > 0,$$

where we have used (3). Conversely, if $e > 0$, we have from (3) that

$$v_i > a_i v_1 + w l_i v_2, i = 1, 2,$$

and we can define $p_i = k v_i$, $i = 1, 2$ for some $k > 0$.

According to Marx, the value of production measured at production prices should be equal to the sum as calculated using the labour values. This gives rise to problems since values are independent of actual production whereas profit rates and hence production prices are not. However, choosing the production in a particular and rather significant way will give the Marxian version: Production prices $\mathbf{p} = (p_1, p_2)$ should satisfy

$$\mathbf{p} = (1 + r)\mathbf{pA}, \tag{12}$$

and since $\mathbf{A}$ is productive, there is a production vector $\mathbf{y} > 0$ such that

$$\mathbf{y} = (1 + r)\mathbf{Ay}. \tag{13}$$

Indeed, rewriting (13) as $(\mathbf{I} - (1 + r)\mathbf{A})\,\mathbf{y} = 0$ we see that $\mathbf{y}$ should be an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $(1 + r)^{-1}$. A classical result about matrices with only nonnegative elements (the Perron-Frobenius theorem) states that $\mathbf{A}$ a maximal positive eigenvalue $\lambda(\mathbf{A})$. If $\mathbf{A}$ is productive, then $\lambda(\mathbf{A}) < 1$ and we get (13) with $1 + r = \lambda(\mathbf{A})^{-1}$.

Multiplying (13) by the value vector $\mathbf{v} = (v_1, v_2)$ (considered as a row) we obtain that

$$v_1 y_1 + v_2 y_2 = (1 + r)\left[v_1(a_1 y_1 + a_2 y_2) + v_2(w l_1 y_1 + w l_2 y_2)\right]],$$

and solving for r we get

$$\begin{aligned} r &= \frac{v_1 y_1 + v_2 y_2 - \left[v_1(a_1 y_1 + a_2 y_2) + v_2(w l_1 y_1 + w l_2 y_2)\right]}{v_1(a_1 y_1 + a_2 y_2) + v_2(w l_1 y_1 + w l_2 y_2)} \\ &= \frac{e(v_2 w l_1 y_1 + v_2 w l_2 y_2)}{v_1(a_1 y_1 + a_2 y_2) + v_2(w l_1 y_1 + w l_2 y_2)}, \end{aligned}$$

where we have used the expression in (3). If we introduce the notation $V = v_2 w l_1 y_1 + v_2 w l_2 y_2$ and $C = v_1(a_1 y_1 + a_2 y_2)$ for variable and constant capital at the production $\mathbf{y}$, then we get that

$$r = \frac{eV}{C + V}, \tag{14}$$

which is Marx' formular for the profit rate.

The debate about the transformation problem and its dynamical version. The transformation of values to prices as described by Marx using (14) was critized by L. von Bortkiewicz (1868 – 1931), who argued that it only transformed output values to output prices, keeping input values as they were, in our terminology that Marx used the formula

$$\mathbf{p} = (1 + r)\mathbf{v}\mathbf{A}.$$

However, Marx's text indicates that he was aware of this problem and explained by involving the time dimension of production. In this way, determination of production prices is seen as a process

$$\mathbf{p}_{t+1} = (1 + r)\mathbf{p}_t\mathbf{A},\ t = 0, 1, 2, \ldots.$$

It can be shown that the sequence of prices following the process converge to some $\mathbf{p}$ which then obviously must satisfy (12).

It remains to find out how we arrive at rate of profit which is the same in all sectors. Following Marx we consider the relation between money which is turned into capital and after production reestablished as (more) money. If money used in sector i is M_i, then

$$M_i = (p_1a_1 + p_2wl_i)c_i = q_ic_i,\ i = 1, 2,$$

where c_i is the number of units of capital, constant and variable, and q_i the value of one unit of capital. If current relative price $\frac{p_1}{p_2}$ is higher than what is given in (12), then the rate of profit is higher in sector 1 than in 2, and money capital moves from sector 2 to sector 1. We have that

$$m_i = \frac{\mathrm{d}M_i}{\mathrm{d}t} = c_i\frac{\mathrm{d}q_i}{\mathrm{d}t} \text{ and } \frac{\mathrm{d}q_i}{\mathrm{d}t} = a_i\frac{\mathrm{d}p_1}{\mathrm{d}t} + wl_i\frac{\mathrm{d}p_2}{\mathrm{d}t},\ i = 1, 2,$$

so that

$$\begin{pmatrix} m_1 & m_2 \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}p_1}{\mathrm{d}t} & \frac{\mathrm{d}p_2}{\mathrm{d}t} \end{pmatrix}\begin{pmatrix} c_1a_1 & c_2a_2 \\ c_1wl_1 & c_2wl_2 \end{pmatrix}$$

Solving for the price derivatives with Cramer's rule, we get that

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = \frac{m_1wl_2c_2 - m_2wl_1c_1}{c_1c_2|\mathbf{A}|}$$
$$\frac{\mathrm{d}p_2}{\mathrm{d}t} = \frac{m_2a_1c_1 - m_1a_2c_2}{c_1c_2|\mathbf{A}|}$$

where

$$|\mathbf{A}| = a_1wl_2 - a_2wl_1 = wl_1l_2\left(\frac{a_1}{wl_1} - \frac{a_2}{wl_2}\right)$$

is the determinant of $\mathbf{A}$. The fraction $\frac{a_i}{wl_i}$ is a measure of what Marx calls *the organic composition of capital*. If $\frac{a_1}{wl_1} < \frac{a_2}{wl_2}$, then $|\mathbf{A}| < 0$, and from $m_1 > 0$ and $m_2 < 0$ we get that $\dfrac{\mathrm{d}p_1}{\mathrm{d}t} < 0$, $\dfrac{\mathrm{d}p_2}{\mathrm{d}t} > 0$, and the price relation will move towards the equilibrium. If $\dfrac{a_1}{wl_1} > \dfrac{a_2}{wl_2}$, so that $|\mathbf{A}| > 0$, then the prices move away from equilibrium, and one has to involve also the changes in c_1 and c_2. It can be shown that also in this case the dynamical system will move towards the equilibrium.

The falling rate of profit. One of the most controversial statements in Capital is that about the long-run tendency of the rate of profit to fall. In the treatment of this topic in the literature, the (verbal) reasoning of Marx is usually formulated as follows: The rate of profit is defined as

$$r = \frac{S}{C+V} = \frac{\frac{S}{V}}{\frac{C}{V}+1},$$

where, as before, S, C and V are surplus value, constant and variable capital. If the organic composition of capital $\frac{C}{V}$ rises while the rate of surplus value $\dfrac{S}{V}$ remains constant, the rate of profit must fall. However, the rate of surplus value may not be constant, on the contrary one might expect it to rise as since the increase in the organic composition of capital should allow for more surplus to be extracted from each worker. An alternative approach would be to consider an upper bound for r,

$$r = \frac{S}{C+V} < \frac{V+S}{C+V} < \frac{V+S}{C},$$

and with $V+C$, living labour, declining compared to embodied labour, we eventually get a falling r, even if it may move upwards for some time.

Since we are dealing with a capitalist economy, changes in the organic composition of capital, or in the ratio of living to embodied labour, must occur only when it is advantageous for capitalists to adopt techniques which imply this. To consider this, we return to the simple model with two goods, a producer's good and a wage good. We let the production price of the wage good be 1 and choose units such that exactly one unit of the wage good reproduces labour, and we get the equation system

$$\begin{aligned} p &= (1+r)(a_{11}p + a_{12}) \\ 1 &= (1+r)(a_{21}p + a_{22}) \end{aligned} \tag{15}$$

where a_{12} and a_{22} are the input of the wage good needed to produce one unit of good 1 and good 2. We now introduce a new technique in the production of the wage good, changing a_{21} and a_{22} to some a'_{21}, a'_{22}. For the capitalists to adopt this technology, it must be the case that

$$a'_{21}p + a'_{22} < a_{21}p + a_{22}. \tag{16}$$

After the introduction of the new technique, we have a new system of production prices (again normalizing so that the price of the wage good is 1) and a new profit rate such that

$$\begin{aligned} p' &= (1+r')(a_{11}p' + a_{12}) \\ 1 &= (1+r')(a'_{21}p' + a'_{22}) \end{aligned} \tag{17}$$

Isolating a_{12} in (15) and (17), we get that

$$p\left(\frac{1}{1+r} - a_{11}\right) = p'\left(\frac{1}{1+r'} - a_{11}\right),$$

showing that p and r must move in the same direction. But we have also from (15) and (17) that

$$(1+r')(a'_{21}p' + a'_{22}) = (1+r)(a_{21}p + a_{22}) > (1+r)(a'_{21}p + a'_{22}),$$

where the inequality follows from (16), so that if $r' < r$, then we must have $p' > p$, contradicting that they move in the same direction, consequently it must be the case that $r' > r$, the profit rate increases.

This result, known in the literature as the Shibata-Okishio theorem, has been seen as a refutation of the Marxian theory of the falling profit rate (note that it holds also when $a'_{21} > a_{21}, a'_{22} < a_{22}$, so that the organic composition of capital increases). On the other hand, the simple model does not fully capture what Marx wrote, since it follows a tradition of formulating the theory in a static equilibrium framework. possibly more in the style of Walras than of Marx.

Marx on international trade. While labour is exploited by capital in each country, there may also be exploitation of capitalists in poorer countries by those of richer countries. The following is a simple Marxian extension of the Ricardian international trade model. There are two countries and two internationally traded goods, plus a third good which cannot be traded internationally. Labour is specific for each country and cannot be moved.

It so happens that country 1 specializes in the production of good 1 and country 2 in the production of the other one. Production uses only labour, with coefficient a_{ij} for the labour input in the production of good j in country i. The production prices must satisfy the following equations, where good 1 has been chosen as numeraire,

$$\begin{aligned} 1 &= (1+r)a_{11}(w+p) \\ p &= (1+s)a_{22}(v+p) \\ w &= (1+r)a_{13}(w+p) \\ v &= (1+s)a_{23}(v+p) \end{aligned} \tag{18}$$

Here p is the price of good 2, w and v the price of good 3 in country 1 and 2, and r (s) is the profit rate in country 1 (2). It is assumed that labour needs exactly 1 unit of good 2 and 1 unit of good 3 for reproduction, so that $w + p$ is the labour cost in country 1 and $v + p$ the cost in country 2.

With the given production pattern country 1 must have a comparative advantage in the production of good 1, so that $\frac{a_{11}}{a_{12}} < \frac{a_{21}}{a_{22}}$. We assume further that $a_{11} > a_{21}$ (so that country 2 is more productive even in the production of good 1), and it then follows that $a_{12} > a_{22}$ (something we shall need in a moment).

Since good 2 is internationally transferable, we let x_i be the quantity of good 2 needed (directly or indirectly) to reproduce 1 unit of labour. Then

$$x_i = 1 + y_i,$$

where y_i is the quantity of good 2 needed to reproduce one unit of good 3, and this quantity in its turn is found from

$$y_i = a_{i3}x_i,$$

since x_i reproduces one unit of labour and one unit of good 3 demands a_{i3} units of labour. Since good 2 can be traded, we must have $x_1 = x_2$.

Now we notice that p must satisfy $\frac{a_{22}}{a_{21}} \leq p \leq \frac{a_{12}}{a_{11}}$ since trade occurs at this price. We use the left of these inequalities, replacing a_{21} with the larger a_{11} to get that

$$\frac{a_{22}}{a_{11}} < p \quad \text{or} \quad pa_{11} > a_{22}.$$

We now divide the first equation in (18) by the third, and similarly the second by the fourth to obtain that

$$\frac{1}{w} = \frac{a_{11}}{a_{13}}, \quad \text{and} \quad \frac{p}{v} = \frac{a_{22}}{a_{23}}.$$

Dividing the second equation here by the first and using that $a_{13} = a_{23}$ we obtain that

$$pa_{11} = a_{22}\frac{v}{w}.$$

Now we use that $pa_{11} > a_{22}$, and we conclude that $\frac{v}{w} > 1$, so that $v > w$. The wage rate is higher in country 2 (the richer country) than in country 1, even if labour in both cases get the same subsistence bundle. Using finally that

$$\frac{v}{w} = \frac{1+s}{1+r}\,\frac{v+p}{w+p}$$

and noticing that $\dfrac{v+p}{w+p} < \dfrac{v}{w}$ (adding the same number in the numerator and denominator will decrease a fraction if it is > 1), we find that $s > r$: The profit rate is larger in the rich country. All taken together, we have a case where capitalists in one country can exploit capitalists in another country.

References:

von Bortkiewicz, L. (1907). Wertrechnung und Preisrechnung im Marxschen System, Archiv für Sozialwissenschaft und Sozialpolitik 25, 10–51.