Payments I

When several agents have to transfer payments to each other, some of the outwards flows may be offset by payments received from other agents, meaning that the total sum of money to be transferred may differ considerably from the net results of these payments. The way in which payments are carried out may also matter for the transmission of shocks in the system – if one agent fails to pay the liabilities, this may trigger failures elsewhere in the system, and liquidity troubles may look differently when dealing with gross or with net payments.

The following simple model of settling payment liabilities is due to Eisenberg and Moe (2001). Suppose that there are n agents with mutual liabilities L_{ij} (specifying what i should pay to j, $i, j = 1, \dots, n$ (where $L_{ii} = 0$ for all i). A payment vector is a vector $p = (p_1, \dots, p_n)$ with $p_i \geq 0$ for each i. We let

$$\overline{p}_i = \sum_{j=1}^{n} L_{ij}$$

be the total liability of agent i. Assuming that whatever payment is made, it will be distributed proportionally among all agents, we define the liability shares as

$$\Pi_{ij} = \begin{cases} \dfrac{L_{ij}}{\overline{p}_i} & \overline{p}_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

The agent i may have some cash flow e_i from outside the system, and the total case flow to i resulting from a payment vector p becomes

$$\sum_{j=1}^{n} p_j \Pi_{ji} + e_i.$$

Subtracting the payment p_i we get the net position of agent i given the payment vector p.

What happens to the liabilities once a payment p has been chosen? This of course depends on the character of the payment, and since its purpose is to eliminate liabilities, we should be more precise on this aspect: A *clearing payment* is a payment vector p^* which satisfies the two conditions of

(a) *limited liability:* $p_i^* \leq \sum_{j=1}^n p_j^* \Pi_{ji} + e_i$ (an agent cannot pay more than what is available for payments),

(b) *absolute parity:* Either $p_i^* = \overline{p}_i$ or $p_i^* = \sum_{i=1}^n p_j^* \Pi_{ji} + e_i$ (the liabilities are paid out fully if possible, otherwise the agent pays the cash flow).

Since the liabilities can have all possible magnitudes, a clearing payment does not necessarily turn all the liabilities into 0s, but it goes as far as possible – whenever the debt position is negative, all available sources are used to cancel them.

It is not obvious that clearing payments exist, so one needs to check this. Consider for this the mapping F from the set $P = \{p \mid 0 \leq p_i \leq \overline{p}_i, i = 1, \dots, n\}$ to itself defined by

$$F_i(p) = \min\left\{\sum_{j=1}^n p_j \Pi_{ji} + e_i, \overline{p}_i\right\}, i = 1, \dots, n,$$

giving the smallest of the two possibilities in (a) and (b). This mapping is continuous, and consequently it has a fixed p oint, that is a payment vector p^0 with $F(p^0) = p^0$ *. It is easy to see that p^0 satisfies (a) and (b) and thus is a clearing payment.

What may somewhat surprising is that the two conditions (a) and (b) for determination of a clearing payment does not give a unique solution, indeed there may be several clearing vectors. It can be shown, however, that any two clearing payments p^*, p^0 will satisfy the condition

$$\max\left\{0, \sum_{j=1}^n p_j^* \Pi_{ji} + e_i - \overline{p}_i\right\} = \max\left\{0, \sum_{j=1}^n p_j^0 \Pi_{ji} + e_i - \overline{p}_i\right\},$$

which may be interpreted as stating that the *value of equity* is the same for all clearing payments.

We shall return to this matter when considering systemic risk.

References

Eisenberg, L. and T.H.Moe (2001), Systemic risk in financial systems, Management Science 47, 236–249.

*For nerds only: We have here used Brouwer's fixed-point theorem, which here is shooting sparrows with cannons. Eisenberg and Moe use the weaker fixed-point theorem by Tarski.