Payments II – Application to Systemic Risk

1. Introduction

Following Glasserman and Young (2015), we may use the formalism of Eisenberg and Moe (2001) to consider the impact of a random shock in the financial system. As before, we have the liabilities L_{ij}, $i, j = 1, \dots, n$ to the other agents, to which we add possible outside liabilities b_i, giving us a vector $\overline{p} = (\overline{p}_1, \dots, \overline{p}_n)$ of total liabilities of the agents,

$$\overline{p}_i = \sum_{j=1}^{n} L_{ij} + b_i$$

for each i. Furthermore, we have a vector $e = (e_1, \dots, e_n)$ of outside payments or *assets* of the agents. The *net worth* of agent i is

$$w_i = e_i + \sum_{j=1}^{n} L_{ji} - \overline{p}_i. \tag{1}$$

As before, we introduce the relative liabilities $\Pi_{ij} = L_{ij}/\overline{p}_i$ (if $\overline{p}_i \neq 0$ and 0 otherwise), notice that the row sums may be ≤ 1 due to possible liabilities outside the financial system.

A random shock in the system is a vector $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$ of random variables $\tilde{x}_i$ with values in the interval $[0, e_i]$, $i = 1, \dots, n$, interpreted as a sudden reduction in the outside assets of the agents. If the shock to agent i is sufficiently large, the value of the assets may become smaller than the liabilities, and i defaults, so that the net worths of all the other agents will be affected. Their liabilities must consequently be reduced as well, since they cannot be satisfied.

Given a realization $x = (x_1 \dots, x_n)$ of the random shock, a *clearing vector* $p(x)$ is a solution to the equations system

$$p_i(x) = \min\left\{\overline{p}_i, \sum_{j=1}^{n} p_j(x)\Pi_{ji} + e_i - x_i\right\}, i = 1, \dots, n. \tag{2}$$

It is seen that the clearing vector represents a way of writing down the liabilities of the agents in a way which is consistent with relative liabilities. It can be shown that under weak assumptions on the system, the clearing vector is determined uniquely.

2. Contagion

The consequences of a shock may be assessed once we know the distribution of $\tilde{x}$. For this, we let

$$\beta_i = \frac{\overline{p}_i - b_i}{\overline{p}_i} \tag{3}$$

be the share of internal liabilities out of the total liabilities of agent i, representing the degree to which the financial problems of agent i effects the other agents. It can be considered as measuring the *financial connectivity* of agent i. We assume that $w_i > 0$ (otherwise i would already be insolvent) and that $w_i \leq c_i$ (since otherwise i could never default). We let

$$\lambda_i = \frac{e_i}{w_i}$$

be agent i's leverage of outside assets (different from total leverage which includes all assets. We then have the following result:

Proposition 1. *Suppose that only agent i is subject to a random shock, so that $\tilde{x}_j = 0$ for $j \neq i$, and that no agent is in default before the shock. If D is a set of agents with $i \notin D$, then the probability that a shock causes all agents in D to default is bounded by*

$$P\left\{\tilde{x}_i \geq w_i + \frac{1}{\beta_i}\sum_{j\in D} w_j\right\},$$

and contagion is impossible if

$$\sum_{j\in D} w_j > w_i\beta_i(\lambda_i - 1).$$

Proof: Let $D(x)$ be the set of agents defaulting from a realization x_i of $\tilde{x}_i$, so that $D \subset D(x)$. Clearly we may restrict attention to cases where $i \in D(x)$, which therefore is nonempty.

Define the shortfall at j as $s_j = \overline{p}_j - p_j(x)$. We have that for each h that

$$\begin{aligned}\sum_{j=1}^n s_j\Pi_{jh} = \sum_{j=1}^n \overline{p}_j\Pi_{jh} - \sum_{j=1}^n p_j(x)\Pi_{jh} = \sum_{j=1}^n L_{jh} - [p_h(x) + e_h + w_h] \\ = \left[w_h + \overline{p}_h - e_h\right] - [p_h(x) + e_h + w_h] = s_h + w_h - x_h,\end{aligned} \tag{4}$$

where we have used first (2) and then (1). In matrix form, this can be written as

$$x_{D(x)} - w_{D(x)} = s_{D(x)} - s_{D(x)}A_{D(x)} = s_{D(x)}\left(I_{D(x)} - \Pi_{D(x)}\right), \tag{5}$$

where all vectors and matrices are restricted to the set $D(xx)$, and where $I_{D(x)}$ is the corresponding unit matrix. Since the sum of elements in the jth row of $\left(I_{D(x)} - \Pi_{D(x)}\right)$ is at least $1-\beta_j$, we have that the sum of the elements of $s_{D(x)}\left(I_{D(x)} - \Pi_{D(x)}\right)$ is at least $\sum_{j\in D(x)} s_j(1-\beta_j)$ which trivially is $\geq s_i(1-\beta_i)$ since $i \in D(x)$.

Now (5) gives us that $s_i \geq x_i - w_i$, which is positive since i defaults, so that

$$\sum_{j\in D(x)} (x_j - w_j) = x_i - w_i - \sum_{j\in D(x)\setminus\{i\}} w_j \geq s_i(1-\beta_i) \geq (x_i - w_i)(1-\beta_i),$$

so that

$$\beta_i(x_i - w_i) \geq \sum_{j\in D(x)\setminus\{i\}} w_j \geq \sum_{j\in D} w_j. \tag{6}$$

The first statement of the proposition follows directly from (6). Rewriting (6) using that $x_i \leq e_i$, we get

$$\sum_{j\in D} w_j \leq \beta_i(e_i - w_i) = \beta_i w_i(1-\lambda_i),$$

which gives us the second part of the proposition. □

It should be noticed that the bounds were obtained without any specification of the distribution of the shock. It can be seen that the important parameters are the connectivity β_i and the degree of leverage of outside assets λ_i. Glaserman and Young estimate these parameters based on stress tests conducted by the European Banking Authority: For the 50 largest banks, average λ_i is 24.9, average λ_i is 15%, and the average value of $\beta_i(1-\lambda_i)$ is 3.2. It follows that a shock in any of these banks cannot result in a group of banks with a net worth greater than 3.2 times that of the defaulting bank.

3. Size of losses

So far the shock has been restricted to a single agent in the system, but shocks may occur simultaneously for many or possibly all agents, so we should consider also the effects of several simultaneous defaults. For this, we need to measure the systemic impact of a shock, and here one might use e.g. number of defaults, loss of bank capital, loss for the external sector etc. We shall use the total loss in value summed over all agents as well as the external sector, so that the loss at the shock x is

$$L(x) = \sum_{i=1}^{n} x_i + S(x), \tag{7}$$

where $S(x) = \sum_{i=1}^{n}(\overline{p}_i - p_i(x))$ is the indirect loss caused by the reduction in the liabilities to the external sector. A reasonable measure of the loss is obtained by averaging over all possible realizations,

$$L = \int \left(\textstyle\sum_{i=1}^{n} x_i + S(x)\right) \mathrm{d}\, F(x),$$

where F is the probability distribution function of the random shock.

The loss as given in (7) can be expressed in another way using the notion of shortfall $s_i = \overline{p}_i - p_i(x)$ already introduced. Let $D = D(x)$ be the default set at the shock x. Using subscript D to restrict vectors and matrices to the coordinates in D, we have from the previous section that

$$s_D = s_D \Pi_D - (w_D - x_D)$$

or

$$s_D(I_D - \Pi_D) = x_D - w_D$$

All elements of the matrix Π_D are nonnegative, the diagonal elements are 0, and the row sums are ≤ 1. Under the weak additional assumption that it is possible to connect any i and j from D with a chain corresponding to positive elements of Π_D (a property also known as irreducibility of Π_D), we have that $I_D - \Pi_D$ is invertible, and

$$(I_D - \Pi_D)^{-1} = I_D + \Pi_D + \Pi_D^2 + \cdots$$

Let $u(x)$ as the vector such that $u_i(x)$ is the sum of the ith row of $(I_D - \Pi_D)^{-1}$ if $i \in D$ and $u_i(x) = 0$ if $i \notin D$. Then (7) can be rewritten as

$$L(x) = \sum_{i=1}^{n} \min\{x_i, w_i\} + \sum_{i=1}^{n} (x_i - w_i) u_i(x).$$

Here the first term represents the direct losses to net worth to the agents, and the second term gives the missing payments caused by other agents' defaults. The coefficient $u_i(x)$ can be seen as a coefficient of amplification of the losses to agent i to losses to the financial system as such, so that it measures the systemic impact of agent i at the loss x.

To find a lower bound for $u_i(x)$, one may use the notion of cohesiveness: A set D

of agents is α-cohesive if

$$\sum_{j \in D} \Pi_{ij} \geq \alpha$$

for each $i \in D$, that is if every agent in D has at least α of its obligations within D. The *cohesiveness* of D, denoted α_D, is then the maximal α for which D is α-cohesive. We then have that

$$u_i(x) \geq \frac{1}{1 - \alpha_D}, \text{ all } x,$$

showing that the more cohesive the network, the greater is the amplification of losses.

An upper bound can be obtained using the connectivity coefficients β_i from (3): If $\beta_D = \max_{i \in D} \beta_i$, then

$$u_i(x) \leq \frac{1}{1 - \beta_D}, \text{ all } x$$

provided that $\beta_D < 1$.

4. References

Glasserman, P. and H.P.Young (2015), How likely is contagion in financial networks? Journal of Banking & Finance 50, 383 – 399.