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Simple measures 1

**Notional-amount approach:**

Sum of the values of the individual assets
– possibly weighted by factor representing riskiness

Example: Risk-weighted assets in regulation according to Basel I – III

Simple measures 2

**Factor sensitivity:**

Change in portfolio value caused by change in risk factors
– or better:
Percentagewise change in portfolio value caused by change in risk factors
– that is, *elasticity* of the value wrt. the risk factor

Example: Duration (to be treated later today)
Risk measures based on loss distribution

How can a probability distribution be summarized in one or two numbers?

Maximal (except in very unlikely situations) loss can be measured by

**Value at Risk:**

\[
\text{VaR}_\alpha = \inf \{ l \in R \mid F_L(l) \geq \alpha \},
\]

where \( F_L \) is the (cumulative) loss distribution.
Risk measures

Shortcomings of VaR

Losses above VaR occur with small probability, but how large are these losses?

An estimate of this can be obtained by **Expected Tail Loss**

$$\text{ETL}_\alpha = \frac{1}{1 - \alpha} \int_0^1 \text{VaR}_u f_L(u) \, du.$$ 

That is, ETL is the conditional mean of VaR for all probabilities $\geq \alpha$.

ETL works better than VaR and is now replacing VaR as popular risk measure.

(ETL is a **coherent** risk measure)

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Risk measures

Scenario-based measures

**Stress-testing:**

Worst possible case for given risk factor changes $C = \{x_1, \ldots, x_n\}$:

Let $w = (w_1, \ldots, w_n)$ be weights, with $w_j \in [0, 1]$.

Risk of a portfolio is

$$\psi_{[C,w]} = \max \left\{ w_1 l_t(x_1), \ldots, w_n l_t(x_n) \right\}.$$

Many risk measures used in practice have this form.
We consider the assessment of risk on a portfolio of (default-free) bonds A zero-coupon gives payoff 1 at the date $T$ (the maturity).

At date $t < T$, the bond has a $p(t, T)$.

The *yield to maturity* $y(t, T)$ is

$$p(t, T) = e^{-(T-t)y(t, T)}$$

$y(t, T)$ is the interest rate so that $p(t, T)$ is present value at $t$ of 1 paid at $T$. Then $p(T, T) = 1$.

The graph of the map $T \mapsto y(t, T)$ is the *yield curve* at time $t$.

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**Risk factors**

Portfolio of $d$ bonds with maturity $T_i$ and prices $p(t, T_i)$, with $\lambda_i$ bonds of maturity $T_i$.

Take the yields $y(t, T)$ as risk factors.

Model of profits and losses is

$$V_t = \sum_{i=1}^{d} \lambda_i p(t, T_i) = \sum_{i=1}^{d} \lambda_i e^{-(T_i-t)y(t, T_i)},$$
Loss distribution

Loss $L_{t+1}$ is

$$L_{t+1} = -\sum_{i=1}^{d} \lambda_i p(t, T_i) (y(t, T_i) - (T_i - t)x_{t+1,i}).$$

$x_{t+1,i} = y(t + 1, T_i) - y(t, T_i)$ is the change in risk factor for type $i$.

Gap analysis

Notional measures: **Gap analysis**
Split assets and liabilities in
- fixed interest rate
- variable interest rate

and consider
- **fixed interest rate gap**
- **variable interest rate gap**

The variable interest can be subdivided: 1 month LIBOR, 3 months LIBOR etc.
Problems with gap analysis

- gap analysis neglects uncertainties in volume and maturity
- gaps give no information about assets and liabilities such as implicit options (in-balance) or guarantees (off-balance),
- gap measures tend to neglect the many different types of interest rate
- the gaps neglect the flows within the time limits set

**Duration 1**

Simple sensitivity measure: How does market value change with interest rates?
Market value at time 0 of a bond with maturity $t_n$ is:

$$V = \sum_{t=0}^{t_n} Y_t(1 + y)^{-t}$$

where $Y_t$ is the payment at $t$. Differentiating wrt. $y$ yields

$$\frac{\partial V}{\partial y} = -\sum_{t=0}^{t_n} tY_t(1 + y)^{-t+1}.$$
Define the (Macaulay)\textit{-duration} $D$ as the elasticity of $V$ with respect to the payoff rate $1 + y$:

$$D = - \frac{\partial V}{\partial y} \frac{1 + y}{V}.$$ 

Then

$$D = - \sum_{t=0}^{t_n} t Y_t (1 + y)^{-(1+t)} \frac{1 + y}{V} = \frac{1}{V} \sum_{t=0}^{t_n} t Y_t (1 + y)^{-t} = \sum_{t=0}^{t_n} t w_t,$$

where

$$w_t = \frac{Y_t (1 + y)^{-t}}{V}.$$ 

### Duration matching 1

Asset and liability management (ALM) over $T$ years:

(i) Assets $A_j$ with maturity $t_j$ and interest rate $r_j$, $j = 1, \ldots, m$

(ii) Liabilities $L_k$, with maturity $t_k$ and interest rate $r_k$, $k = 1, \ldots, n$.

At any $t_j$ ($t_k$), market interest rate is $i_j$ ($i_k$). Define time units such that $T = 1$.

NPV of assets at $t = 1$ is:

$$V_A^1 = \sum_{j=1}^{m} A_j (1 + r_j)^{\gamma} (1 + i_j)^{1-t_j}.$$
Assume that the interest rate structure has a **parallel lift** of size $\lambda$. Then

$$\frac{\partial V^1_A}{\partial \lambda} = \sum_{j=1}^{m} A_j(1 + r_j)^{t_j}(1 - t_j)(1 + i_j)^{-t_j}$$

$$= \sum_{j=1}^{m} \frac{A_j(1 + r_j)^{t_j}}{(1 + i_j)^{t_j}}(1 - t_j)$$

$$= V_A(1 - D_A),$$

with $V_A$ the NPV of assets at $t = 0$ and $D_A$ duration of assets.

Repeating the procedure for the liabilities, we get

$$\frac{\partial V^1_L}{\partial \lambda} = \sum_{k=1}^{n} L_k(1 + r_k)^{t_k}(1 - t_k)(1 + i_k)^{-t_k} = V_L(1 - D_L),$$

The portfolio is immune against shifts in the interest rate structure if

$$V_A(1 - D_A) = V_L(1 - D_L),$$

which is the principle of **duration matching**.
Shortcomings

Duration matching can be used only for small changes in interest rates.

If larger, use (Macaulay-)convexity defined as

\[ K = \sum_{t=1}^{T} (t^2 + t)w_t, \]

so that

\[ \frac{\partial^2 V}{\partial y^2} = \frac{VK}{(1+y)^2}. \]

Then

\[ \Delta V = -\frac{VD}{1+y} \Delta y + \frac{1}{2} \frac{VK}{(1+y)^2} (\Delta y)^2. \]
Conditions for acceptance sets

A1 \( \mathbb{R}_+^n \subset \mathcal{A} \).

A2 \( \mathcal{A} \cap \mathbb{R}_-^n = \emptyset \).

A3 \( \mathcal{A} \) is convex.

A4 \( \mathcal{A} \) is a cone.

Let \( r \) be a given ("very safe") risk. Given an acceptance set \( \mathcal{A} \), define risk measure

\[
\rho_{\mathcal{A},r}(X) = \inf \{ m \mid mr + X \in \mathcal{A} \}.
\]

Properties of risk measures

T For all \( X \) and \( \alpha \), \( \rho(X + \alpha r) = \rho(X) - \alpha \).

S For all \( X, Y \), \( \rho(X + Y) \leq \rho(X) + \rho(Y) \).

P For all \( \lambda \geq 0 \) and \( X \), \( \rho(\lambda X) = \lambda \rho(X) \).

M If \( X \leq Y \), then \( \rho(Y) \leq \rho(X) \).

A risk measure satisfying T, S, P and M is coherent.
Let $\mathcal{A}$ be an acceptance set satisfying $A 1 – 4$.

Then $\rho_{\mathcal{A}, r}$ is a risk measure satisfying properties T, S, P and M.

Conversely,

if $\rho$ satisfies T, S, P and M, then $\mathcal{A}_\rho = \{X \mid \rho(X) \leq 0\}$ satisfies $A 1 – 4$. 