Economics of Banking
Lecture 4

February 2023

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- Measuring risk
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- Coherent measures of risk (*not mandatory*)

Simple measures 1

Notional-amount approach:

Sum of the values of the individual assets

– possibly weighted by factor representing riskiness

Example: Risk-weighted assets in regulation according to Basel I – III

Simple measures 2

Factor sensitivity:

Change in portfolio value caused by change in risk factors

– or better:

Percentagewise change in portfolio value caused by change in risk factors

– that is, *elasticity* of the value wrt. the risk factor

Example: Duration (to be treated later today)

Risk measures based on loss distribution

How can a probability distribution be summarized in one or two numbers?

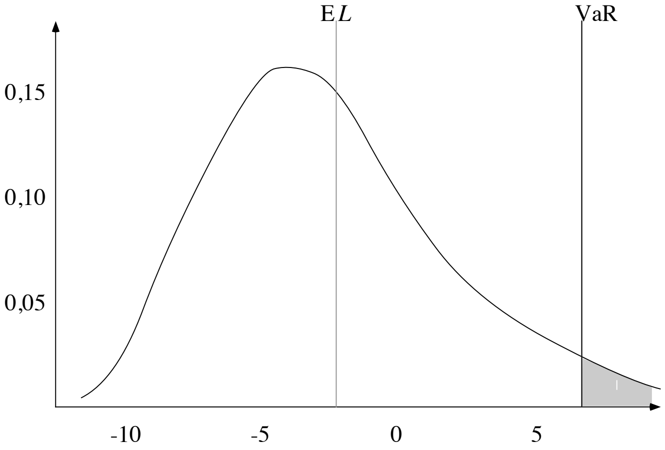
Maximal (except in very unlikely situations) loss can be measured by

Value at Risk:

$$\mathrm{VaR}_\alpha = \inf\{l \in R \mid F_L(l) \geq \alpha\}.$$

where F_L is the (cumulative) loss distribution

Value at Risk



Shortcomings of VaR

Losses above VaR occur with small probability, but how large are these losses?

An estimate of this can be obtained by **Expected Tail Loss**

$$\mathrm{ETL}_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_u f_L(u)\, du.$$

That is, ETL is the conditional mean of VaR for all probabilities $\geq \alpha$.

ETL works better than VaR and is now replacing VaR as popular risk measure.

(ETL is a *coherent* risk measure)

Scenario-based measures

Stress-testing:

Worst possible case for given risk factor changes $C = \{x_1, \ldots, x_n\}$:

Let $w = (w_1, \ldots, w_n)$ be weights, with $w_j \in [0, 1]$.

Risk of a portfolio is

$$\psi_{[C,w]} = \max \left\{ w_1 l_{[t]}(x_1), \ldots, w_n l_{[t]}(x_n) \right\} .$$

Many risk measures used in practice have this form.

Bonds and yields

We consider the assessment of risk on a portfolio of (default-free) *bonds*

A zero-coupon gives payoff 1 at the date T (the maturity).

At date $t < T$, the bond has a $p(t, T)$.

The *yield to maturity* $y(t, T)$ is

$$p(t, T) = e^{-(T-t)y(t,T)}$$

$y(t, T)$ is the interest rate so that $p(t, T)$ is present value at t of 1 paid at T. Then $p(T, T) = 1$.

The graph of the map $T \mapsto y(t, T)$ is the *yield curve* at time t.

Risk factors

Portfolio of d bonds with maturity T_i and prices $p(t, T_i)$,

with λ_i bonds of maturity T_i.

Take the yields $y(t, T)$ as risk factors.

Model of profits and losses is

$$V_t = \sum_{i=1}^{d} \lambda_i p(t, T_i) = \sum_{i=1}^{d} \lambda_i e^{-(T_i - t) y(t, T_i)},$$

Loss distribution

Loss L_{t+1} is

$$L_{t+1} = -\sum_{i=1}^{d} \lambda_i p(t, T_i) \left(y(t, T_i) - (T_i - t) x_{t+1,i} \right).$$

$x_{t+1,i} = y(t+1, T_i) - y(t, T_i)$ is the change in risk factor for type i.

Gap analysis

Notional measures: **Gap analysis**
Split assets and liabilities in

- fixed interest rate
- variable interest rate

and consider

- *fixed interest rate gap*
- *variable interest rate gap*

The variable interest can be subdivided: 1 month LIBOR, 3 months LIBOR etc.

Problems with gap analysis

- gap analysis neglects uncertainties in volume and maturity
- gaps give no information about assets and liabilities such as implicit options (in-balance) or guarantees (off-balance),
- gap measures tend to neglect the many different types of interest rate
- the gaps neglect the flows within the time limits set

Duration 1

Simple sensitivity measure: How does market value change with interest rates?
Market value at time 0 of a bond with maturity t_n is:

$$V = \sum_{t=0}^{t_n} Y_t (1+y)^{-t}$$

where Y_t is the payment at t. Differentiating wrt. y yields

$$\frac{\partial V}{\partial y} = -\sum_{t=0}^{t_n} t Y_t (1+y)^{-(t+1)}.$$

Duration 2

Define the (Macaulay)-*duration* D as the elasticity of V with respect to the payoff rate $1 + y$:

$$D = -\frac{\partial V}{\partial y}\frac{1+y}{V}.$$

Then

$$D = -\left[\sum_{t=0}^{t_n} tY_t(1+y)^{-(1+t)}\right]\frac{1+y}{V} = \frac{1}{V}\sum_{t=0}^{t_n} tY_t(1+y)^{-t} = \sum_{t=0}^{t_n} tw_t,$$

where

$$w_t = \frac{Y_t(1+y)^{-t}}{V}.$$

Duration matching 1

Asset and liability management (ALM) over T years:

(i) Assets A_j with maturity t_j and interest rate r_j, $j = 1, \dots, m$

(ii) Liabilities $L - k$, with maturity t_k and intest rate r_k, $k = 1, \dots, n$.

At any t_j (t_k), market interest rate is i_j (i_k). Define time units such that $T = 1$.

NPV of assets **at** $t = 1$ is:

$$V_A^1 = \sum_{j=1}^{m} A_j(1 + r_j)^{t_j}(1 + i_j)^{1-t_j}.$$

Duration matching 2

Assume that the interest rate structure has a **parallel lift** of size λ. Then

$$
\begin{aligned}
\frac{\partial V_A^1}{\partial \lambda} &= \sum_{j=1}^{m} A_j (1+r_j)^{t_j} (1-t_j)(1+i_j)^{-t_j} \\
&= \sum_{j=1}^{m} \frac{A_j (1+r_j)^{t_j}}{(1+i_j)^{t_j}} (1-t_j) \\
&= V_A (1 - D_A),
\end{aligned}
$$

with V_A the NPV of assets at $t = 0$ and D_A duration of assets.

Duration matching 3

Repeating the procedure for the liabilities, we get

$$\frac{\partial V_L^1}{\partial \lambda} = \sum_{k=1}^{n} L_k (1 + r_k)^{t_k} (1 - t_k)(1 + i_k)^{-t_k} = V_L(1 - D_L),$$

The portfolio is immune against shifts in the interest rate structure if

$$V_A(1 - D_A) = V_L(1 - D_L),$$

which is the principle of **duration matching**

Shortcomings

Duration matching can be used only for small changes in interest rates

If larger, use (Macaulay-)*convexity* defined as

$$K = \sum_{t=0}^{t_n} (t^2 + t) w_t,$$

so that

$$\frac{\partial^2 V}{\partial y^2} = \frac{VK}{(1+y)^2}.$$

Then

$$\triangle V = -\frac{VD}{1+y} \triangle y + \frac{1}{2} \frac{VK}{(1+y)^2} (\triangle y)^2.$$

General theory of risk measures

Given n possible future states of the world.

A *risk* is now a vector X with n components. $\mathcal{G}$ is the set of all risks.

A *measure of risk* ρ assigns to each $X \in \mathcal{G}$ a number $\rho(X)$.

An *acceptance set* $\mathcal{A}$ is a subset of $\mathcal{G}$ of "reasonable" risks.

Example: $\mathrm{VaR}_\alpha(X) = -\inf\{y \mid \mathrm{P}(\{\omega \mid X(\omega) \leq yr\}) > \alpha\}$ is a risk measure

$\mathcal{A}_\rho = \{X \mid \mathrm{VaR}_\alpha(X) \leq 0\}$ is an acceptance set

Conditions for acceptance sets

A1 $R^n_+ \subset \mathcal{A}$.

A2 $\mathcal{A} \cap \mathbb{R}^n_{--} = \emptyset$.

A3 $\mathcal{A}$ is convex.

A4 $\mathcal{A}$ is a cone.

r a given ("very safe") risk. Given an acceptance set $\mathcal{A}$, define risk measure

$$\rho_{\mathcal{A},r}(X) = \inf \{m \mid mr + X \in \mathcal{A}\}.$$

Properties of risk measures

- T For all X andα, $\rho(X+\alpha r)=\rho(X)-\alpha$.
- S For all X, Y, $\rho(X+Y) \leq \rho(X)+\rho(Y)$.
- P For all $\lambda \geq 0$ and X, $\rho(\lambda X)=\lambda\rho(X)$.
- M If $X \leq Y$, then $\rho(Y) \leq \rho(X)$.

A risk measure satisfying T,S,P and M is **coherent**

Characterization of coherent risk measures

Let $\mathcal{A}$ be an acceptance set satisfying A 1 – 4.

Then $\rho_{\mathcal{A},r}$ is a risk measure satisfying properties T, S, P and M.

Conversely,

if ρ satisfies T, S, P and M, then $\mathcal{A}_\rho = \{X \mid \rho(X) \leq 0\}$ satisfies A 1 – 4.