Economics of Banking
Lecture 8

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Capital Asset Pricing Model

Market with n securities with random returns $\widetilde{r}_I$ with mean r_i,

covariance matrix $\Sigma = (\sigma_{ij})_{i=1}^{n}{}_{j=1}^{n}$.

Portfolio $x = (x_1, \ldots, x_n)$.

Investor with initial wealth W_0 gets

$$\widetilde{W} = \left(W_0 - \sum_{i=1}^{n} x_i\right) r + \sum_{i=1}^{n} x_i \widetilde{r}_i$$

(r is the risk-free rate of interest), mean and variance are

$$\mu(x) = \mathrm{E}\widetilde{W} = \left(W_0 - \sum_{i=1}^{n} x_i\right) r + \sum_{i=1}^{n} x_i r_i = W_0 r + \sum_{i=1}^{n} x_i (r_i - r)$$

$$\sigma^2(x) = \mathrm{Var}\,\widetilde{W} = x^t \Sigma x$$

CAPM2

Assume that U depends only on $U(\mu(x), \sigma^2(x))$. First order conditions are

$$U'_\mu \frac{\partial \mu(x)}{\partial x_i} + U'_{\sigma^2} \frac{\partial \sigma^2(x)}{\partial x_i} = U'_\mu(r_i - r) + 2U'_{\sigma^2} \sum_{j=1}^{n} \sigma_{ij} x_j = 0$$

for $i = 1, \ldots, n$, in matrix notation,

$$-\gamma \rho^t + \Sigma x = 0,$$

where

$$\gamma = -\frac{U'_\mu}{2U'_{\sigma^2}}$$

is MRS between mean and variance, $\rho = (r_1 - r, \ldots, r_n - r)$,.

Σ has an inverse Σ^{-1}, so we get

$$x^t = \gamma \Sigma^{-1} \rho.$$

CAPM3

Optimal portfolios of all investors are proportional

Therefore, they are all proportional to the *market portfolio*

Two-fund separation: Each investor holds a combination of

- money (risk-free asset)
- market portfolio

The proportion of the two depends on the attitude towards risk

Options

Options are rights to buy (or to sell) a security at a given price and date.

Example: European call option on asset R with maturity T, exercise price K.

The value of this option is

$$V^{BS}(s, R, r, \sigma, K, T) = R\Phi(d_1) - Ke^{-r(T-s)}\Phi(d_2).$$

(the Black-Scholes formula), where

The BS formula

$\Phi(\cdot)$ is standard normal distribution

r is the riskfree rate of interest

σ the volatility of R (follows a Geom. Brownian Motion),

and

$$d_1 = \frac{\ln\left(\frac{R}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-s)}{\sigma\sqrt{T-s}}, \; d_2 = d_1 + \sigma\sqrt{T-s}.$$

Risk assessment

For the risk assessment of an option, we need to **identify risk factors:**

- Value of security R
- discount rate r
- volatility σ

Changes in risk factors indicated by Δ

The Greeks

Then linearized loss is

$$L_{t+1}^{\Delta} = -(V_s^{BS}\Delta + V_R^{BS} R_t \Delta R_{t+1} + V_r^{BS}\Delta r + V_\sigma^{BS}\Delta\sigma).$$

Partial derivatives are "the Greeks":

- V_R^{BS} (*delta*) is price risk
- V_s^{BS} (*theta*) is time decay risk
- V_ρ^{BS} (*rho*) is discount rate risk
- V_σ^{BS} (*vega*) (!) is volatility risk.

Bank holding a portfolio

Bank specialized in holding portfolios. Initial capital K, portfolio x.

Assets are joint normally distributed with covariance matrix Σ.

In the next period, the capital has changed to

$$\widetilde{A} = K + \sum_{i=1}^{n} x_i \widetilde{r}_i.$$

For simplicity, riskfree rate of interest is 0.

Bank maximizes $U(\mu, \sigma^2)$, optimal portfolio is

$$x = \gamma \Sigma^{-1} \rho, \quad \text{with} \quad \gamma = -\frac{U'_\mu}{2U'_{\sigma^2}}.$$

Probability of default

Losses so big that A becomes negative happens with probability

$$P\{A < 0\} = P\left\{\frac{A - \mu(x)}{\sigma(x)} < -\frac{\mu(x)}{\sigma(x)}\right\} = \Phi\left(-\frac{\mu(x)}{\sigma(x)}\right)$$

Insert $\mu(x) = K + x^t\rho$ and $\sigma(x) = \sqrt{x^t\Sigma x}$:

$$P\{A < 0\} = \Phi\left(-\frac{K + x^t\rho}{\sqrt{x^t\Sigma x}}\right).$$

Define (weighted) **capital ratio** as

$$\kappa(x) = \frac{K}{\sum_{i=1}^n \alpha_i x_i},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a weight vector.

Relevance

Probability of ruin, $\mathrm{P}\{A<0\}$, is a decreasing function of κ:

Normalize the portfolio by $\hat{x}=\frac{1}{\sigma(x)}x$, then

$$\mathrm{P}\{A<0\}=\Phi\left(-\frac{K}{\sigma(x)}-\hat{x}^t\rho\right)=\Phi\left(-(\alpha^t\hat{x})\kappa(x)-\hat{x}^t\rho\right), \qquad (*)$$

where we have used that

$$\kappa(x)=\frac{K}{(\alpha^t\hat{x})\sigma(x)}.$$

Then $(*)$ is decreasing in κ.

Regulating by capital ratios

If the bank maximizes $U(\mu(x), \sigma^2(x))$ under the constraint

$$\kappa(x) \geq \bar{\kappa}, \text{ or equivalently:} \alpha^t x \bar{\kappa} \leq K,$$

then first order conditions are

$$U'_\mu \rho_i + 2U'_{\sigma^2} \sum_{j=1}^{n} \sigma_{ij} x_j - \lambda \alpha_i, \; i = 1, \ldots, n,$$

with solution

$$x^0 = \Sigma^{-1} \left(\gamma\rho + \hat{\lambda}\alpha \right), \quad \gamma = -\frac{U'_\mu}{2U'_{\sigma^2}}, \; \hat{\lambda} = \frac{\lambda}{2U'_{\sigma^2}}.$$

Two fund separation ceases to hold, market may not be efficient.

Next topic

Securitization and shadow banking

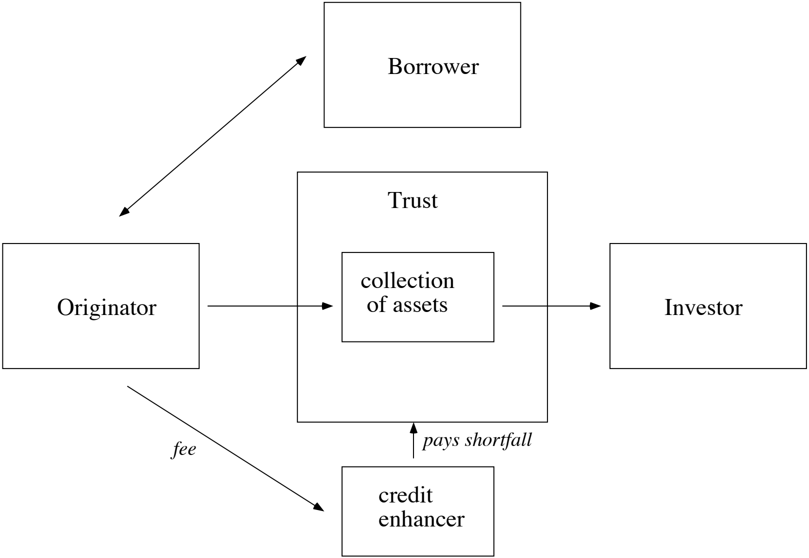
Types of securitization

Securitization (transformation of loans into securities) has many forms:

- Pass-through
- Asset-backed bonds
- Collateralized mortgage obligations (CMOs)

All versions of the same construction:

Pass-through



Simple model of securitization

Bank finances projects for one period

Initial capital W, can finance $kW(= 1)$ projects

Effort of the bank e_l or e_L, (= probability of getting high outcome y_H, otherwise y_L, with $y_H > 1 > y_L$,

effort cost $h(e_H) > h(e_L)$.

Effort is desirable for society:

$$e_H y_H + (1 - e_H) y_L - (1 + r) h(e_H) > e_L y_H + (1 - e_L) y_L - (1 + r) h(e_L). \quad (*)$$

Traditional banking or securitization?

Bank can fund the loans at interest rate r_D, expected profits as evaluated at $t = 0$ are

$$(1 + r)^{-1}[ey_H + (1 - e)y_L - (1 - W)(1 + r_D)] - h(e).$$

By $(*)$, bank chooses e_H.

Securitization: Sell loans in the market at price $p(e^*)$ such that

$$p(e^*) = (1 + r)^{-1}[e^* y_H + (1 - e^*)y_L]$$

In equilibrium, $e^* = e_L$

What is best?

From $r_D \leq r$ we get that

$$(1+r)^{-1}[e_H y_H + (1-e_H)y_L - (1+r_D)] - h(e_H) \\ \geq (1+r)^{-1}[e_H y_H + (1-e_H)y_L - (1+r)] - h(e_H),$$

and $(*)$ gives

$$e_H y_H + (1-e_H)y_L - (1+r) - (1+r)h(e_H) \\ > e_L y_H + (1-e_L)y_L - (1+r) - (1+r)h(e_L),$$

so that

$$(1+r)^{-1}[e_H y_H + (1-e_H)y_L - (1+r_D)] - h(e_H) > p(e_L) - 1 - h(e_L),$$

traditional banking is best, securitization used only due to capital constraint.

Risks from securitization

Economy over three periods $t = 0, 1, 2$.

Investment projects at $t = 0$ or at $t = 1$: 1 invested yields a payoff Y at $t = 2$.

Projects are financed by loans, and the bank demands a fee f when initiating the investment.

Bank has equity E_0 at $t = 0$ (in cash),

Loans may be kept by the bank or they may be sold as securities which pay exactly 1 at $t = 2$.

Bank must keep a fraction $d < 1$ of securities as collateral for loans.

$P_0 = 1$ and P_1 is the security price at $t = 0$ and $t = 1$. **Assume** $P_1 < 1$ (prices known at $t = 1$).

Role of capital ratio

Banks are subject to capital regulation:

Equity must constitute a fraction h of total assets, or equivalently, of total liabilities, equity plus loans,

$$\frac{E_t}{E_t + L_t} \geq h, \;\; t = 0, 1,$$

The bank can use funding at $t = 0$ up to E_0/h, initiating $N_0 = E_0/dh$ projects.

Bank has a security portfolio $N_0 d = \frac{E_0}{h}$ at $t = 0$.

This are the assets of the bank, and it has borrowed $L_0 = E_0(1 - h)/h$.

What happens at $t = 1$

Equity is $E_1 = E_0 P_1$ (profits were paid out at $t = 0$ as dividends).

Capital constraint violated, bank must sell assets S at price P_1 and pay back loans.

Assets after the sale must be $(N_0 d - S)P_1$, E_1 is this minus loans $L_0 - P_1 S$.

Capital ratio is $\dfrac{(N_0 d - S)P_1 - (L_0 - P_1 S)}{(N_0 d - S)P_1}$, insert $L_0 = E_0(1-h)/h$, $N_0 d = E_0/h$, get

$$S = \frac{E_0}{h} \cdot \frac{1 - P_1}{P_1} \cdot \frac{1-h}{h}$$

Bank has a loss $S(1 - P_1)$ in this period.

Was it profitable after all?

Three possibilities:

(1) Securitization: One unit of equity yields:

$$\frac{f}{dh} - \left[\frac{1}{h} \cdot \frac{1-P_1}{P_1} \cdot \frac{1-h}{h}\right](1-P_1)$$

Alternatively, the bank may stay liquid until $t = 1$ and then do either:

(2) Traditional banking: Projects funded by loans, profit $\frac{1}{h}f$,

(3) Take unsecured loans, buy securities at $t = 1$, profit is $\frac{1}{h}\frac{1-P_1}{P_1}$.

Depending on parameter values, (a) may well be the best option.