Time-Series Analysis of the Solow Growth Model*

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Abstract This research propose a rigorous method to evaluate and test Solow (1956)'s growth model on the country level. I present a generalised version of the model, allowing for time-varying savings and population growth rates, and analyse its statistical properties under the assumption that the growth rates are subject to random shocks that are exogenous to the model. I argue that this structural time-series approach circumvents many of the problems associated with cross-section estimation of the Solow growth model identified by e.g. Durlauf and Quah (1999) and Durlauf et al. (2005). This procedure yields goodness-of-fit measures and p-values indicating the validity of the Solow model for each country. Through automatic model selection proceduces, the model will be applied to data from more than 150 countries available in the Penn World Tables (Heston et al., 2009). The obtained measures will then be regressed on various country characteristics and data quality measures, to provide a general picture of the circumstances under which the assumptions of the Solow model are met (if at all). In the present draft of the paper, I apply the model to each of the G7 countries and Denmark, Norway and Sweden. Overall, the Solow model describes the data from these 10 countries reasonably well.

Keywords Solow model, Time-series analysis, VAR, Cointegration

JEL Classification Codes C32, C51, O10, O40

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1 Introduction

How well does the Solow growth model (Solow, 1956) fit time series data? Surprisingly little has been said about this question. The vast majority of empirical investigations of the Solow model is based on cross-country regressions, most notably the literature following Mankiw et al. (1992). To evaluate the Solow model using cross-country data is associated with a host of problems related to missing variable bias, endogeneity and and strong assumptions about independence between the level of technology, the savings rate and the population growth rate (for a great overview, see Durlauf and Quah (1999) and Durlauf et al. (2005)). Furthermore, these cross-country studies often impose the strong assumption that the economies are in the steady state predicted by the Solow model to obtain a linear model that can be estimated. One problem with this assumption is that the regression equation is almost identical to an equation that can be derived from an accounting identity using only an assumption of constant factor shares and a constant capital-output ratio, whereby the Solow regression equation becomes tautological (see Felipe and McCombie (2005)). As researchers became aware of the problems associated with testing and estimating the Solow model using cross-country data, growth regressions in the style of Mankiw et al. (1992) has come out of fashion, although there are recent examples such as Brock and Taylor (2010).

To avoid the problem with unobserved heterogeneity, Islam (1995) succesfully apply a panel-data approach. Unfortunately this approach does not take cointegration between the variables into account. This problem can in turn be solved by performing time-series analysis as I will show in this article.

I aim to answer the question in the first paragraph by developing a general time-series framework based on the Solow model, and then applying it to yearly time-series data for the seven G7 countries and Denmark, Norway and Sweden. This framework allows for unit roots in the population growth rate and the savigns rate. The goal is first and foremost to test if the restrictions implied by the Solow model can be said to hold. In other words, the goal is to test if the Solow model delivers a reasonable description of the data. If this turns out to the be the case, a second goal is then to obtain the estimates of the model parameters and compare them to *a priori* assumptions.

To my knowledge, only one other article investigates the Solow growth model using time-series data, namely Kalaitzidakis and Korniotis (2000). Unfortunately, there are some problems with the authors' analysis. First of all, they interpret the steady state relationship between output per capita, the savings rate and the population growth rate as a cointegrating relation, but this ignores the transitional dynamics inherent in the Solow model. Second of all, they forget to include a time trend term in the equation, probably because it is also (rightfully) dropped in the cross-section analysis by Mankiw et al. (1992). The framework developed in the present article takes the transitional dynamics

of the Solow model into account, although an approximation around the steady state is necessary to linearise the model.

In the empirical part of the article, I use data from the Penn World Table (Heston et al., 2009). This database contains yearly observations of the relevant variables from 1950 to 2007 for most modern economies, thus delivering a reasonable number of observations on a long enough time period to study economic growth.

Unit root tests of the savings rate and the population growth rate often reveal these variables to approximately follow unit root processes, see for example Jones (1995) and Kalaitzidakis and Korniotis (2000). In this article, I extend the Solow model to allow for the savings rate and the population growth rate to be arbitrarily close to unit root processes. I formulate a VAR model in the growth rate of per capita output, the savings rate and the population growth rate based upon the theoretical model. This kind of structural CVAR model is explained in Møller (2008). I extend the model, adding greater flexibility and allow for more than one lag in the VAR. I then derive the conditions that must be satisfied for there to be cointegration between the variables. This provides a general framework for testing and estimating the Solow model using time series data.

In the next section, I present the generalised version of the Solow model in continuous time. I show the model has a stable non-trivial steady state when the differential equations describing the evolutions of the savings rate and the population growth rate are both stable. I then eliminate the unobservable variables from the model to form a model in the growth rate of per capita output, the savings rate and the growth rate of population.

2 The theory model

The Solow model describes a closed economy without government spending in the long run. The textbook version of the model, as found in e.g. Acemoglu (2009), assumes the savings rate and population growth rate to be constant. I formulate a Solow model in continuous time with time-varying savings rate, $s(t)$, and population growth rate, $n(t)$. The rates will be governed by autonomous differential equations that are continuous time analogues of autoregressive processes.

Total output, $Y(t)$ is given by the Cobb-Douglas production function

$$Y(t) = K(t)^{\lambda}(A(t)L(t))^{\mu}, \quad \lambda, \mu \in (0,1),\ Y(t), K(t), L(t) > 0, \tag{1}$$

where $K(t)$ is aggregate capital input, $L(t)$ is the total labour input and $A(t)$ is the labour-augmenting technological progress. The stock of technology grows with the constant rate g, i.e.

$$\dot{A}(t) = gA(t). \tag{2}$$

The fundamental law of motion describes the evolution of capital in the equilibrium by

$$\dot{K}(t) = s(t)Y(t) - \delta K(t), \quad \delta \in (0,1), \tag{3}$$

where $s(t)$ is the fraction of output that goes to savings and thus investment, since the economy is closed. The parameter δ is the depreciation rate of capital.

The total labour input grows with the rate $n(t)$

$$\dot{L}(t) = n(t)L(t). \tag{4}$$

In the long run, the growth rate of total labour input equals the growth rate of the population. I now turn to specify the evolution of the savings rate and the population growth rate.

The savings rate is given by the first-order linear ordinary differential equation

$$\dot{s}(t) = \bar{s} - (1-\rho_s)s(t), \qquad \bar{s} > 0,\ \rho_s \in (0,1). \tag{5}$$

This equation says that the savings rate will converge to its long-run equilibrium value. To see this more formally, let the symbol f^* denote the non-trivial steady state value of a function of time, $f(t)$ in case it is unique and note that $\dot{s} = 0$ implies $s^* = \bar{s}/(1-\rho_s)$. For a given differentiable function, $f(t)$, let $g_f(t) \equiv \partial f(t)/\partial t$ denote the growth rate of $f(t)$. The growth rate of $s(t)$ is given by

$$g_s(t) = \bar{s}/s - (1-\rho_s) = -(1-\rho_s)(1 - s^*/s(t)).$$

Whenever the savings rate is below (above) the steady state value, the savings rate will increase (decrease). The convergence rate will decrease over time, as $s(t)$ gets close to its steady state, i.e. as $s^*/s(t)$ gets close to 1. Independently of this, a value of ρ_s closer to 1, means a lower convergence rate. Two different economies can therefore have the same steady state savings rate but different convergence rates. Imagine two different economies, economy A and economy B, with the same steady state savings rate $s^*_A = s^*_B$. If economy A has a more rigid savings system, the savings rate in economy B will converge with a slower rate towards the steady state than in economy B, such that $\rho_{s,A} > \rho_{s,B}$. Since the steady state savings rates are equal in the two economies, the constant term must be different, i.e. $\bar{s}_A < \bar{s}_B$. Thus, an economy with a very slow convergence rate and a steady state savings rate will tend to have a very low value of $\bar{s}$. If an economy has a very slow convergence rate, no matter how far $s(t)$ is from its steady state, i.e. if ρ_s is close to 1, any shock to the savings rate will take a long time to be corrected, whereby the savings rate will appear non-stationary. Note that when $\rho_s = 1$ either $s^* = \bar{s} = 0$ or there is no

steady state. When $\bar{s} \neq 0$ and $\rho_s = 1$, the differential equation is unstable, and since the savings rate is bounded, this case can be ruled out.

The population growth rate is modelled analogously by

$$\dot{n}(t) = \bar{n} - (1 - \rho_n)n(t), \qquad \bar{n} > 0,\ \rho_n \in (0, 1). \tag{6}$$

The model consists of equation (1)–(6). I will now show that it has a unique, local, asymptotically stable, non-trivial steady state. This is not only a desirable property in a theoretical sense, but also allow for a point to approximate an equation around later. Combining equation (1) and (3) yields

$$g_K(t) = z(t) - \delta \tag{7}$$

where $z(t) \equiv s(t)Y(t)/K(t) = s(t)K(t)^{\lambda-1}(A(t)L(t))^{\mu}$. Define a balanced growth path by $g_K(t) = g_K^*$ where g_K^* is a constant. In the balanced growth path equation (7) implies $g_K^* = z(t) - \delta \Rightarrow g_z(t) = 0$. The growth rate of $z(t)$, $g_z(t)$ is given by

$$g_z(z) = g_s(t) - (1 - \lambda)g_K(t) + \mu(g + n(t)).$$

Inserting equation (7) and rearranging gives

$$\dot{z}(t) = (g_s(t) - (1 - \lambda)(z(t) - \delta) + \mu(g + n(t)))z(t) \tag{8}$$

A steady state of the system is given by $(\dot{z}(t), \dot{s}(t), \dot{n}(t)) = (0, 0, 0)$. Equation (5) implies $s^* = \bar{s}/(1 - \rho_s) > 0$ and equation (6) implies $n^* = \bar{n}/(1 - \rho_n) > 0$. According to equation (8) there are two steady state values of $z(t)$, namely the trivial $z(t) = 0$ and the non-trivial $z(t) = z^* = \mu/(1 - \lambda)(g + n^*) + \delta$ (using $g_s^* = 0$). The non-trivial steady state is thus given by $(z(t), s(t), n(t)) = (z^*, s^*, n^*)$.

The three differential equations (8), (5) and (6) characterize the dynamical system. The Jacobian, J, is given by

$$J = \begin{pmatrix} -(1-\lambda)\delta - \mu(g + \frac{\bar{n}}{1-\rho_n}) & 0 & \mu\delta + \frac{\mu^2}{1-\lambda}(g + \frac{\bar{n}}{1-\rho_n}) \\ 0 & -(1 - \rho_s) & 0 \\ 0 & 0 & -(1 - \rho_n) \end{pmatrix},$$

where $(\partial f/\partial g)^*$ denotes the partial derivative of f with respect to g evaluated at the steady state. The eigenvalues of J are $-(1 - \lambda)\delta - \mu(g + \bar{n}/(1 - \rho_n))$, $-(1 - \rho_s)$ and $-(1 - \rho_n)$ and since they are all negative, the steady state is asymptotically stable.

3 The related VAR and CVAR models

It is convenient to exclude the variables $A(t)$, $K(t)$ and $L(t)$ from the system, since $A(t)$ is unobservable and estimates of $K(t)$ are rare. The goal is therefore now to obtain a linear discrete time version of the model in the three variables g_Y, s_t, n_t and then to formulate it as a VAR model. I first derive an equation relating g_Y to s_t, n_t and the parameters. This equation describes the transitional dynamics of the economy. To obtain a linear equation, I approximate it around the steady state. I then approximate the continuous functions with their discrete time counterparts.

3.1 A linear discrete time version of the theory model

Note first that equation (1), (3), (4) and (2) implies

$$g_Y(t) = \lambda g_K(t) + \mu(g + n(t)). \tag{9}$$

For this equation to be linear in $s(t)$, I will now derive a linear approximation of $g_K(t)$ around the steady state.

Since $z(t) = s(t)Y(t)/K(t)$ and in the steady state $z(t) = z^*$ and $s(t) = s^*$, the term $Y(t)/K(t)$ must be constant in the steady state. Let us denote the steady state value of this term by $(Y/K)^*$. We can then write

$$(Y/K)^* = z^*/s^* = ((\mu/(1-\lambda))(g + n^*) + \delta)/s^* \tag{10}$$

Note that when there are constant returns to scale (CRS), $\lambda + \mu = 1$, we get $(Y/K)^* = (g+n^*+\delta)/s^*$ completely analogous to the textbook Solow model found in e.g. Acemoglu (2009), the only difference being the terms s^* and n^* that now reflects the steady state values of the differential equations in (5) and (6). Using this we can approximate $z(t)$ around the steady state allowing $s(t)$ to vary by writing $z(t) = s(t)(Y/K)^*$. In effect, using equation (10), an approximation of equation (7) is given by

$$g_K(t) = s(t)((\mu/(1-\lambda))(g + n^*) + \delta)/s^* - \delta. \tag{11}$$

Inserting this equation into equation (9) yields

$$g_Y(t) = \theta s(t) - \lambda\delta + \mu(g + n(t)), \tag{12}$$

where $\theta \equiv \lambda((\mu/(1-\lambda))(g + n^*) + \delta)/s^*$. The data on output in the Penn World Tables are measured in per capita terms. To account for this note that $g_Y - n(t)$ is the growth

rate of per capita output. Equation (12) can then be written as

$$g_{Y/L}(t) = \theta s(t) - (1-\mu)n(t) - \lambda\delta + \mu g. \tag{13}$$

Equation (5), (6) and (13) constitute a linear model in the three observable variables. The discrete time stochastic model (or the "empirical model") is then given by

$$g_{Y/L,t} = \theta s_t - (1-\mu)n_t - \lambda\delta + \mu g + \varepsilon_g, \quad \varepsilon_g \sim i.i.N(0, \sigma_g^2) \tag{14}$$

$$s_t = \bar{s} + \rho_s s_{t-1} + \varepsilon_{s,t}, \quad \varepsilon_s \sim i.i.N(0, \sigma_g^2) \tag{15}$$

$$n_t = \bar{n} + \rho_n n_{t-1} + \varepsilon_{n,t}, \quad \varepsilon_n \sim i.i.N(0, \sigma_g^2). \tag{16}$$

3.2 The VAR model

The VAR model with k lags (denoted VAR(k)) can be written on the so-called Error-Correction-Form as

$$\Delta x_t = \Pi x_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \Phi D_t + \varepsilon_t, \tag{17}$$

where x_t is a vector of variables, D_t is a vector of deterministic components, the ε_t are independent and normally distributed shocks with zero mean and identical covariance matrices. The characteristic polynomial related to the VAR is

$$C(q) = (1-q)I - \Pi q - \sum_{i=1}^{k-1} \Gamma_i (1-q), \tag{18}$$

and it is a general result that when one or more of the roots of the characteristic polynomial are 1, i.e. there are unit roots, the system is non-stationary.

The empirical model consists of equation (14), (15) and (16). It can be written as a VAR(1) on the Error-Correction-Form, equation (17), with $x_t = (g_{Y/L,t}, s_t, n_t)'$,

$$\Pi = \begin{pmatrix} -1 & \theta\rho_s & -(1-\mu)\rho_n \\ 0 & -(1-\rho_s) & 0 \\ 0 & 0 & -(1-\rho_n) \end{pmatrix}, \Phi = \begin{pmatrix} \theta\bar{s} - (1-\mu)\bar{n} + \mu g - \lambda\delta \\ \bar{s} \\ \bar{n} \end{pmatrix},$$

the scalar $D_t = 1$ and empty Γ_i matrices for $i = 1, \ldots, k-1$. When all three variables are stationary, the Solow model should therefore be estimated using the VAR model, equation (17), with restrictions on the Π and Φ matrices corresponding to the expressions above. There are eight coordinates with combinations of the ten parameters, $\lambda, \mu, \rho_s, \rho_n, \bar{n}, n^*, \bar{s}, s^*, g, \delta$ (remember the definition of θ above). In accordance with Mankiw et al. (1992) and Kalaitzidakis and Korniotis (2000), I will assume $g + \delta$ to be

known, in which case the remaining parameters can be identified. It is possible to impose a restriction like $\lambda + \mu = 1$ on the system, even though it corresponds to a non-linear combination of coordinates of Π, by using the results from Boswijk and Doornik (2004), which are implemented in PcGive.

In the next section, I show that when the savings rate and/or the population growth rate are non-stationary, they will cointegrate with the growth rate of per capita output. I show how these cointegrating relations look, and what restrictions should be imposed on the system.

3.3 The CVAR model

The matrix Π has reduced rank when the savings rate or the population growth rate is integrated. When either variable is integrated, they turn out to cointegrate with the growth rate of per capita output.

A matrix has reduced rank when its determinant is zero. The determinant of Π is $-(1-\rho_s)(1-\rho_n)$. The characteristic polynomial, equation (18), is

$$C(q) = -(1-q\rho_s)(1-q\rho_n)$$

with the roots $q = \rho_s^{-1}$ and ρ_s^{-1}. The rank of Π is r when there are $p-r$ unit roots, where p is the dimension of x. Thus $\rho_s \neq 1 \wedge \rho_n \neq 1$ if and only if Π has full rank. In other words, when s_t and n_t are both I(0), all variables in the model are I(0), since also g_{yt} will be I(0). In that case, we have three stable cointegrating relations in the system, namely the trivial case where all of the variables cointegrate with themselves, and we can estimate the model using the OLS estimator. Furthermore, it appears from the roots of the characteristic polynomial that $(\rho_s \neq 1, \rho_n \neq 1) \Leftrightarrow r = 3$, $(\rho_s = 1, \rho_n = 0) \vee (\rho_s = 0, \rho_n = 1) \Leftrightarrow r = 2$ and $(\rho_s = 1, \rho_n = 1) \Leftrightarrow r = 1$. The rank of Π is at least 1. When some of the variables cointegrates, $r < p$, it is possible to decompose Π into two $p \times r$ matrices of rank r, α and β, such that $\alpha\beta' = \Pi$. The β matrix contains the cointegrating relations and the α matrix the loadings. This decomposition is unique up to a chosen normalisation, i.e. for any matrix of full rank, Q, we can renormalize the decomposition into $\tilde{\alpha} \equiv \alpha Q$ and $\tilde{\beta} \equiv Q^{-1}\beta'$ such that $\alpha\beta' = \alpha Q Q^{-1}\beta' = \tilde{\alpha}\tilde{\beta}'$. One way to decompose Π into α and β is is to let α be a basis for the column space of Π and let $\beta = (\alpha(\alpha'\alpha)^{-1})'\Pi$. A convenient renormalization matrix, Q, is given by $Q = (R[1, r; p+1, p+r])^{-1}$ where R is the reduced row echelon form of $(\beta' \mid I_r)$ and $X[m_1, m_2; n_1, n_2]$ denotes the submatrix of a matrix X consisting of rows m_1 to m_2 and columns n_1 to n_2 of X.

Furthermore, when Π has reduced rank, we can decompose the deterministic term, Φ into a part related to the cointegrating relations, η, i.e. a part in the span of α, and a residual part, γ, i.e. a part not in the span of α, such that $\Phi = \alpha\eta + \gamma$. Let the $m \times (m-n)$ matrix of full column rank $A_\perp$ denote the orthogonal complement of an $m \times n$ matrix A

of full column rank, i.e. a matrix satisfying $A'_\perp A = 0$. For a given matrix A, $A_\perp$ is not unique: it is possible to write $A_\perp = \Lambda V$ where Λ is a particular orthogonal complement of A and V is an arbitrary, square, non-singular matrix. A simple choice of $A_\perp$ is the null space of A'. Using the identity $I = \alpha(\beta'\alpha)^{-1}\beta' + \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp$, it is possible to write $\eta = \alpha(\beta'\alpha)^{-1}\beta'\Phi$ and $\gamma = \beta_\perp(\alpha'_\perp\beta_\perp)^{-1}\alpha'_\perp\Phi$.

Using these procedures, the α, β, η and γ matrices can be found in each of the three cases of cointegration. In the first case, the savings rate is integrated, $s_t \sim I(1)$, and

$$\alpha = \begin{pmatrix} -1 & -(1-\mu)\rho_n \\ 0 & 0 \\ 0 & -(1-\rho_n) \end{pmatrix}, \beta' = \begin{pmatrix} 1 & -\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\eta = \begin{pmatrix} \bar{n}(1-\mu)(1-\rho_n)^{-1} + \lambda\delta - \mu g) \\ -(1-\rho_n)^{-1}n \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} \theta\bar{s} \\ \bar{s} \\ 0 \end{pmatrix}.$$

It appears from the β' matrix that this case yields two cointegrating relations: the growth rate of per capita output and the savings rate is cointegrated and positively related, $g_{Y/L} - \theta s_t \sim I(0)$, and the population growth rate is cointegrated with itself, $n_t \sim I(0)$.

In the second case the population growth rate is integrated, $n_t \sim I(1)$ and

$$\alpha = \begin{pmatrix} -1 & \theta\rho_s \\ 0 & -(1-\rho_s) \\ 0 & 0 \end{pmatrix}, \beta' = \begin{pmatrix} 1 & 0 & 1-\mu \\ 0 & 1 & 0 \end{pmatrix},$$

$$\eta = \begin{pmatrix} -\bar{s}\theta(1-\rho_s)^{-1} + \lambda\delta - \mu g) \\ -(1-\bar{\rho}_s)^{-1}s \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} -(1-\mu)\bar{n} \\ 0 \\ \bar{n} \end{pmatrix}.$$

What can be seen here is that there is again two cointegrating relations: one between $g_{Y/L}$ and n_t in which they are negatively related, $g_{Y/L} + (1-\mu)n_t \sim I(0)$, and one in which n_t cointegrates with itself, $n_t \sim I(0)$.

Finally, in the third case the savings rate and the population growth rate are both integrated, $s_t, n_t \sim I(1)$, and

$$\alpha = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \beta' = \begin{pmatrix} 1 & -\theta & 1-\mu \end{pmatrix}, \eta = \lambda\delta - \mu g \text{ and } \gamma = \begin{pmatrix} \theta s - (1-\mu)n \\ s \\ n \end{pmatrix}.$$

In this case there is only one cointegrating relation, namely $g_{Y/L} - \theta s_t + (1-\mu)n_t \sim I(0)$.

In all three cointegration cases γ is not empty, meaning that the deterministic term is not restricted to the cointegrating relations (i.e. the span of α). In the empirical part

of the article, I have chosen to model the deterministic term simply as an unrestricted constant.

In the next section I extend the empirical model by allowing for more than one lag in the VAR and by allowing the growth rate of per capita output to depend on its own lagged values.

4 The generalised VAR and CVAR models

The full empirical model extends the above in two ways. First of all, I allow for more than one lag in the VAR(k), i.e. $k \geq 1$. Second of all, I allow for lagged growth rates in per capita output to be a determinant of the present growth rate (this does not change the conclusions derived above about stability and does not enter into the expression for θ). The reason for doing this is to capture the possible effect of business cycles on the growth rate of output. The results in this section are analogous to those from the previous section, but they are more general. The full empirical model is given by the following equations

$$\tilde{g}_{Y/L,t} = \theta s_t - (1-\mu)n_t + \mu g - \lambda\delta + \sum_{i=1}^{k} \rho_{g,i}\tilde{g}_{Y/L,t-i} + \varepsilon_g,\ \varepsilon_g \sim i.i.N(0,\sigma_g^2),$$

$$s_t = \bar{s} + \sum_{i=1}^{k} \rho_{s,i} s_{t-i} + \varepsilon_s,\ \varepsilon_g \sim i.i.N(0,\sigma_s^2),$$

$$n_t = \bar{n} + \sum_{i=1}^{k} \rho_{n,i} n_{t-i} + \varepsilon_s,\ \varepsilon_g \sim i.i.N(0,\sigma_s^2).$$

The model implies the following matrices of the Error-Correction-Model formulation of the VAR, equation (17),

$$\Pi = \begin{pmatrix} -(1-\bar{\rho}_g) & \theta\check{\rho}_s & -(1-\mu)\check{\rho}_n \\ 0 & -(1-\check{\rho}_s) & 0 \\ 0 & 0 & -(1-\check{\rho}_n) \end{pmatrix},$$

where $\check{\rho}_g \equiv \sum_{i=1}^{k} \rho_{g,i}$, $\check{\rho}_s \equiv \sum_{i=1}^{k} \rho_{s,i}$ and $\check{\rho}_n \equiv \sum_{i=1}^{k} \rho_{n,i}$,

$$\Gamma_i = \begin{pmatrix} -\sum_{j=i+1}^{k} \rho_{g,j} & -\theta\sum_{j=i+1}^{k} \rho_{s,j} & -(1-\mu)\sum_{j=i+1}^{k} \rho_{n,j} \\ 0 & -(1-\sum_{j=i+1}^{k} \rho_{s,j}) & 0 \\ 0 & 0 & -(1-\sum_{j=i+1}^{k} \rho_{n,j}) \end{pmatrix}$$

for $i = 1, \ldots, k$ and the deterministic component

$$\Phi = \begin{pmatrix} \theta\bar{s} - (1-\mu)\bar{n} + \mu g - \lambda\delta \\ \bar{s} \\ \bar{n} \end{pmatrix},$$

with $D_t = 1$. The determinant of Π is $-(1-\check{\rho}_g)(1-\check{\rho}_s)(1-\check{\rho}_n)$ and the characteristic polynomial, equation (18), is

$$C(q) = \left(\sum_{i=1}^{k}((i-1)q - (i-2))q\rho_{g,i} - 1\right) \left(\sum_{i=1}^{k}((i-1)q - (i-2))q\rho_{s,i} - 1\right) \left(\sum_{i=1}^{k}((i-1)q - (i-2))q\rho_{n,i} - 1\right),$$

which can be seen by inspection to have unit roots exactly for $\bar{\rho}_g = 1$, $\bar{\rho}_s = 1$ and $\bar{\rho}_n = 1$, analogous to the simpler model. Assuming $\bar{\rho}_g \in (0,1)$, not at odds with the data, we get again three cases of cointegration: $(\bar{\rho}_s \neq 1, \bar{\rho}_n \neq 1) \Leftrightarrow r = 3$, $(\bar{\rho}_s = 1, \bar{\rho}_n \neq 1) \vee (\bar{\rho}_s \neq 1, \bar{\rho}_n = 1) \Leftrightarrow r = 2$ and $(\bar{\rho}_s = 1, \bar{\rho}_n = 1) \Leftrightarrow r = 1$.

When $s_t, n_t \sim I(0)$, i.e. $\check{\rho}_s, \check{\rho}_n \in (0,1)$ the rank of Π is $r = 3$ and the model should be estimated using the restricted Π and Γ_i matrices above (the Γ_i matrices are common to all the four cases).

When Π has reduced rank, $r < p$, using the procedures described in section 3.3 yields the following matrices of the CVAR. When $s_t \sim I(1)$ and $n_t \sim I(0)$ the rank is $r = 2$ and the matrices are

$$\alpha = \begin{pmatrix} -(1-\check{\rho}_g) & -(1-\mu)\check{\rho}_n \\ 0 & 0 \\ 0 & -(1-\check{\rho}_n) \end{pmatrix}, \beta' = \begin{pmatrix} 1 & -\frac{\theta}{1-\check{\rho}_g} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\eta = \begin{pmatrix} \frac{\bar{n}(1-\mu)}{(1-\rho_g)(1-\rho_n)} + \frac{\bar{s}\theta\rho_g}{(1-\rho_g)^2} + \frac{\lambda\delta - \mu g}{1-\rho_g} \\ -\frac{\bar{n}}{1-\check{\rho}_n} \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} \frac{\theta\bar{s}}{1-\check{\rho}_g} \\ \bar{s} \\ 0 \end{pmatrix}.$$

Compared with the simpler model, the term $1 - \rho_g$ now appears in the cointegrating relation as well as in the first coordinate of the α matrix. The fact that $g_{Y/L}$ is now allowed to depend on its own lagged values means that the cointegrating relation between $g_{Y/L}$ and s_t needs more of a counter-reaction in the savings rate to keep the relation stationary. Note that the summed coefficients of the lagged variables, $\check{\rho}_s$ and $\check{\rho}_n$ enter in the same way as their simpler counterparts ρ_s and ρ_n. The same is true for $\check{\rho}_g$ although it can not be seen here, since I did not include the lagged value of $g_{Y/L}$ in the simple model. The η matrix looks the same as before, except now $\check{\rho}_g$ enter and there is an extra term

related to $\bar{s}$. When $s_t \sim I(0)$ and $n_t \sim I(1)$ the rank is $r = 2$ and the matrices are

$$\alpha = \begin{pmatrix} -(1-\check{\rho}_g) & \theta\check{\rho}_s \\ 0 & -(1-\check{\rho}_s) \\ 0 & 0 \end{pmatrix}, \beta' = \begin{pmatrix} 1 & 0 & \frac{1-\mu}{1-\check{\rho}_g} \\ 0 & 1 & 0 \end{pmatrix},$$

$$\eta = \begin{pmatrix} -\frac{\bar{n}(1-\mu)\rho_g}{(1-\rho_g)^2} - \frac{\theta\bar{s}}{(1-\rho_g)(1-\rho_s)} + \frac{\lambda\delta-\mu g}{1-\rho_g} \\ -\frac{\bar{s}}{1-\check{\rho}_s} \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} -\frac{1-\mu}{1-\check{\rho}_g} \\ 0 \\ \bar{n} \end{pmatrix}.$$

Again the matrices correspond to those from the simpler model, and the same general comments can be said as above. Finally when $s_t \sim I(1)$ and $n_t \sim I(1)$ the rank is $r = 1$ and the matrices becomes

$$\alpha = \begin{pmatrix} -(1-\check{\rho}_g) \\ 0 \\ 0 \end{pmatrix}, \beta' = \begin{pmatrix} 1 & -\frac{\theta}{1-\check{\rho}_g} & \frac{1-\mu}{1-\check{\rho}_g} \end{pmatrix},$$

$$\eta = \begin{pmatrix} \bar{n}(1-\mu)\check{\rho}_g - \theta\bar{s}\check{\rho}_g + \frac{\lambda\delta-\mu g}{1-\check{\rho}_g} \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} \frac{\lambda s-(1-\mu)n}{1-\check{\rho}_g} \\ \bar{s} \\ \bar{n} \end{pmatrix}.$$

I have now developed a complete statistical framework to estimate and test the Solow model using time series data. To sum up, the framework covers four different cases:

Case 1 Stationarity. Corresponds to a restricted VAR (see p. 10).

Case 2 $s_t \sim I(1)$. Corresponds to a restricted CVAR (see p. 11).

Case 3 $n_t \sim I(1)$. Corresponds to a restricted CVAR (see p. 11).

Case 3 $s_t, n_t \sim I(1)$. Corresponds to a restricted CVAR (see p. 12).

5 Data

I use data from the Penn World Tables Version 6.3 (Heston et al., 2009). For a description of the data and methods used to generate this dataset, see Summers and Heston (1991). The data set includes time series data on the three variables for a wide range of countries for the years 1951 to 2007 (1971–2007 for Germany). As a first step of applying the model, I have restricted the analysis to include the seven G7 countries: Canada, France, Germany, Italy, Japan, the United Kingdom and the United States of America and furthermore Denmark, Norway and Sweden. The inclusion of the G7 countries allow me to compare

the results with those of Kalaitzidakis and Korniotis (2000), although they used the first public version of the dataset covering the shorter period 1950–1988.

For the growth rate of per capita output, I use the variable *grgdpch* since the Penn World Table authors recommends it. The savings rate is the investment share of real GDP, the variable *ki* divided by 100. The population growth rate is calculated on the basis of the population variable *POP*.

6 Estimation

6.1 Applying the model

I will now explain all the steps in estimating the model using the data for USA. In accordance with the methodological guidelines in Juselius (2006), I first fit a general unrestricted VAR model in order to determine the number of cointegrating relations, i.e. the rank of Π. Using CATS in RATS version 2 (Dennis et al., 2005), I perform the automated lag length determination procedure, with a maximum of five lags, indicating a lag length of $k = 4$ or $k = 2$. For each extra lag included in the model, nine more variables are introduced (if I could restrict the Γ matrices, this would instead be five). Therefore, a lag length of four corresponds to 18 more variables in the model, compared to a lag length of two. Since this is a lot of variables compared to the number of observations, I choose a lag length of two. I then perform a residual analysis, testing for normality and autocorrelation (see Dennis (2006) page 176–177 for details). There are no problems with neither autocorrelation or non-normality. Had there been problems, I would look for large residuals defined by having a t value larger than three, and deal with them by including a break or some kind of dummy. I then consider the recursively calculated fluctuation test of the eigenvalues to check for volatility in the model. This test reveals no problems with non-constancy of the eigenvalues.

Having now a well-specified unrestricted VAR(2) model, I perform the rank test. This test is a series of trace tests of which the first tests if all roots are unit roots, i.e. $r = 0$. If this hypothesis is not rejected, the rank test indicates a rank of $r = 0$. If it is rejected, one proceeds to test if all roots, but one, are unit roots, i.e. $r = 1$, and so on. Using the data for USA, the Bartlett-corrected p-value for the hypothesis of $r = 0$ is 0%, for $r = 1$ it is 57% and for $r = 2$ it is 65% (see Table 1). This points towards a rank of $r = 1$. I then calculate the recursive trace test with the standard values in CATS. The plots indicates two cointegrating relations, i.e. $r = 2$. Thus the correct rank of Π could be either one or two. Although the estimates are only relevant under the correct choice of rank, I estimate the model in all four cases.

Before turning to estimate the model under all the four cases derived in section 4, consider the following. When the model is estimated under full rank, it returns estimates

for $\check{\rho}_s$ and $\check{\rho}_n$ as well as $\bar{s}$ and $\bar{n}$. Therefore, it is possible to find s^* and n^*. Using these and assuming the size of g and δ, it is possible to find λ from the estimates of $\Pi_{1,2}$, $\Pi_{2,2}$, $\Pi_{1,3}$ and $\Pi_{3,3}$. If $s_t \sim I(1)$ then $\rho_s = 1$ and the steady state value s^* is undefined. Therefore, when the assumption $s_t \sim I(1)$ is imposed, it is not directly possible to recover λ from θ, because if s^* really does not exist, the expression for θ on page 6 is no longer valid. Keep in mind, that imposing $s_t \sim I(1)$ is simply a way of finding out how to impose restrictions on the matrices under reduced rank. It is true, that if s_t really do follow a random walk, it has no steady state. However, the interpretation here is that s_t is so persistent, that assuming $s_t \sim I(1)$ is more appropriate, statistically, than assuming $s_t \sim I(0)$. Therefore, when $s_t \sim I(1)$, I use the estimate of s^* calculated under the assumption of full rank. The same considerations applies to n_t and n^*.

I use PcGive version 13.1 to estimate the model under the four different cases with the following procedure. Starting with Case 1, I estimate the VAR model imposing $\Pi_{2,1} = \Pi_{2,3} = \Pi_{3,1} = \Pi_{3,2} = 0$. It is not, to my knowledge, possible to restrict the Γ_i matrices at the same time as the Π matrix, at least not under the more complex restrictions that will be imposed later, so these are the only restrictions put on the system. The regression yields the following estimated Π and Φ matrices

$$\Pi = \begin{pmatrix} \mathbf{-0.785} & \mathbf{-0.319} & \mathbf{-1.942} \\ 0 & -0.074 & 0 \\ 0 & 0 & \mathbf{-0.072} \end{pmatrix}, \Phi = \begin{pmatrix} 0.110 \\ 0.018 \\ 0.001 \end{pmatrix},$$

where coordinates with t-values larger than 2 are marked in bold typeface (t-values for Φ are not available). The restrictions are accepted with a p-value of 22%. To derive an implied estimate of λ one needs to assume something about g and δ. In accordance with Mankiw et al. (1992), I assume $g + \delta = 0.05$. The estimates imply $\check{\rho}_g = 0.21$, $\check{\rho}_s = 0.93$, $\check{\rho}_n = 0.93$, $s^* = 0.24$, $n^* = 0.01$, $\mu = -1.09$ and $\lambda = 0.73$. The estimate of $\mu = -1.09$ is contrary to the *a priori* assumption of $\mu \in (0, 1)$. It should be noted though, that for all the estimates that are *derived* from the coordinates of Π and Φ, and do not appear directly, it is unknown whether they are significant or not. For example, it is known that the estimate of $-(1 - \check{\rho}_g)$ is significant, but it is not known if the estimate of $\check{\rho}_g$ is significant. It is possible to find the t-values of all the derived estimates, but since the calculations will be rather involved in some cases, I have not done this. Thus, the estimate of $\mu = -1.09$ might as well be insignificant, in which case it would not count as heavily against the Solow model. In any case, the rank tests above indicated a reduced rank, and thus the estimate of the full rank model are irrelevant.

The model predicts a long-run savings rate of 24% and a long-run population growth rate of 1%, which is quite reasonable. The average savings rate is 23% and the average

Table 1: Rank tests

Rank	CAN	DEN	FRA	GER	ITA	JAP	NOR	SWE	UK	USA
Trace test p-value (percent)										
$r = 2$	58	41	36	31	84	61	56	23	44	65
$r = 1$	2	57	0	29	29	55	11	20	41	57
$r = 0$	0	1	0	19	10	0	3	5	15	0
Recursive trace test indication										
	-	1/2/3	2	2/3	2	1	2	-	3	2

population growth rate is 1%, which indicates that the rates are not very far from their steady state values.

Most importantly, the estimates indicates that both s_t and n_t are very persistent since both $\check{\rho}_s$ and $\check{\rho}_n$ are close to one. This makes sense given the indication of a reduced rank.

Continuing to Case 2, I estimate the model under the assumption of $s_t \sim I(1)$, i.e. $\check{\rho}_s = 1$. I impose a rank of 2 on $\Pi = \alpha\beta'$ and impose the restrictions $\alpha_{2,1} = \alpha_{2,2} = \alpha_{3,1} = \beta'_{1,3} = \beta_{2,1} = \beta_{2,2} = 0$ and $\beta_{1,1} = \beta_{2,3} = 1$. The regression gives the following estimates

$$\alpha = \begin{pmatrix} \mathbf{-0.785} & \mathbf{-1.942} \\ 0 & 0 \\ 0 & -0.072 \end{pmatrix}, \beta' = \begin{pmatrix} 1 & \mathbf{0.238} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \Phi = \begin{pmatrix} 0.081 \\ 0.001 \\ 0.001 \end{pmatrix},$$

implying $\check{\rho}_g = 0.21$, $\check{\rho}_n = 0.93$, $n^* = 0.01$, $\mu = -1.09$ and $\lambda = 0.64$, where I have assumed $s^* = 0.24$ which was the estimate of s^* from before. The restrictions are accepted with a p-value of 16%.

Estimating Case 3, $n_t \sim I(1)$, i.e. $\check{\rho}_n = 1$, I impose a rank of $r = 2$ and $\alpha_{2,1} = \alpha_{3,1} = \alpha_{3,2} = \beta_{1,2} = \beta_{2,1} = \beta_{2,3} = 0$ and $\beta_{1,1} = \beta_{2,2} = 1$. The estimates are

$$\alpha = \begin{pmatrix} \mathbf{-0.785} & \mathbf{0.320} \\ 0 & -0.075 \\ 0 & 0 \end{pmatrix}, \beta' = \begin{pmatrix} 1 & 0 & \mathbf{2.582} \\ 0 & 1 & 0 \end{pmatrix}, \Phi = \begin{pmatrix} 0.112 \\ 0.018 \\ 0.000 \end{pmatrix},$$

implying $\check{\rho}_g = 0.21$, $\check{\rho}_s = 0.93$, $s^* = 0.24$, $\mu = -1.03$ and $\lambda = 0.74$, where I have assumed $n^* = 0.01$. The restrictions are accepted with a p-value of 6%.

Finally I estimate the model in Case 3, $s_t, n_t \sim I(1)$, i.e. $\check{\rho}_s = \check{\rho}_n = 1$. I impose a rank of $r = 1$, $\alpha_2 = \alpha_3 = 0$ and $\beta_1 = 1$. The estimates are

$$\alpha = \begin{pmatrix} \mathbf{-0.785} \\ 0 \\ 0 \end{pmatrix}, \beta' = \begin{pmatrix} 1 & \mathbf{0.238} & \mathbf{2.582} \end{pmatrix}, \Phi = \begin{pmatrix} 0.082 \\ 0.001 \\ 0.000 \end{pmatrix},$$

Table 2: Estimates without the CRS restriction, $\mu + \lambda \gtreqless 1$

	$I(1)$	p	$\check{\rho}_g$	$\check{\rho}_s$	$\check{\rho}_n$	s^*	n^*	μ	λ
Canada	s_t	2%	0.28		0.88		0.01	-0.03	0.99
Denmark		46%	0.25	0.94	0.89	0.32	0.00	1.87	3.03
France	s_t	6%	0.23		0.73		0.01	3.12	0.23
Germany	s_t, n_t	21%	0.22					0.03	0.99
Italy	s_t, n_t	20%	0.51					3.14	0.32
Japan	n_t	9%	0.28	0.91		0.37		2.73	3.01
Norway	s_t	54%	0.26		0.93		0.00	0.75	-0.89
Sweden	s_t, n_t	2%	0.32					-0.12	1.13
UK	s_t	11%	0.32		0.87		0.00	0.78	-4.04
USA	s_t	16%	0.21		0.93		0.01	-1.09	0.64

implying $\check{\rho}_g = 0.21$, $\mu = -1.03$ and $\lambda = 0.73$, where I have assumed $s^* = 0.24$ and $n^* = 0.01$. The restrictions are accepted with a p-value of 7%.

All in all the restrictions implied by the Solow model are accepted in all four cases, and the estimates are very robust towards the choice of rank for the USA.

Most commonly, the aggregate production function is assumed to have CRS. In this case, it means that $\lambda + \mu = 1$. I also estimate the model imposing this extra restriction. Besides the restrictions on the α and β matrices described above, I impose the following restrictions. In case 1, I impose $\Pi_{1,2} = -\Pi_{1,3}(0.05 + n^*)/(s^*(\Pi_{3,3} + 1))(\Pi_{2,2} + 1)$, in case 2 I impose $\beta_{1,2} = \alpha_{1,2}(0.05 + n^*)/(s^*(\alpha_{3,2} + 1)\alpha_{1,1})$, in case 3 I impose $\alpha_{1,2} = \beta_{1,3}\alpha_{1,1}(\alpha_{2,2} + 1)(0.05 + n^*)/s^*$ and in case 4 I impose $\beta_2 = -\beta_3(0.05 + n^*)/s^*$. I use the values of s^* and n^* implied by the estimates from case 1 without the assumption of CRS. Using these extra restrictions I estimate the model again for each of the four cases.

In case 1 the restrictions are not accepted with a p-value of 1%. The implied estimates are $\check{\rho}_g = 0.27$, $\check{\rho}_s = 1.01$, $\check{\rho}_n = 0.92$, $\mu = 0.94$ and $\lambda = 0.06$. In case 2 the restrictions are not accepted with a p-value of 1%. The implied estimates are $\check{\rho}_g = 0.27$, $\check{\rho}_n = 0.92$, $\mu = 0.98$ and $\lambda = 0.02$. In case 3 the restrictions are not accepted with a p-value of 0%. The implied estimates are $\check{\rho}_g = 0.26$, $\check{\rho}_s = 1.01$, $\mu = 0.90$ and $\lambda = 0.10$. Finally, in case 4 the restrictions are also not accepted with a p-value of 0%. The implied estimates are $\check{\rho}_g = 3.84$, $\mu = 0.96$ and $\lambda = 0.04$. The fact that the relative size of λ and μ now changes and that p-values is low, is a sign that the model has a poor fit with the data. It can be concluded, that the problem is related to the size of μ which does not conform to the *a priori* assumption. Note also the unreasonably high estimate of $\check{\rho}_g$ in case 4, indicating that this parameter had to be "adjusted" a lot for the system to conform to CRS.

All the estimates for the ten countries can be found in Table 4 in the appendix.

As was explained earlier, the rank tests indicated a rank of one or two. Since the results are nicer given a rank of two, and since the p-value is highest when s_t is the integrated variable, I regard the result from case 2 as the estimates most likely to be correct. The

Table 3: Estimates with the CRS restriction, $\mu + \lambda = 1$

Country	$I(1)$	p	$\check{\rho}_g$	$\check{\rho}_s$	$\check{\rho}_n$	μ	λ
Canada	s_t	0%	0.33		0.87	1.17	-0.17
Denmark		60%	0.25	0.95	0.89	1.94	-0.94
France	s_t	0%	0.52		0.71	0.46	0.54
Germany	s_t, n_t	32%	2.29			-0.22	1.21
Italy	s_t, n_t	2%	1.65			0.58	0.43
Japan	n_t	15%	0.29	0.91		2.63	-1.63
Norway	s_t	68%	0.26		0.93	0.70	0.26
Sweden	s_t, n_t	3%	2.20			0.59	0.41
UK	s_t	18%	0.32		0.86	0.85	0.15
USA	s_t	1%	0.27		0.92	0.98	0.02

estimates from case 2 are therefore chosen to be presented in Table 2 and 3, representing the no-CRS and CRS case, respectively.

6.2 Results

Performing the procedure explained in the previous subsection for each of the ten countries, I end up with the results in Table 2 and 3. Both the no-CRS and the CRS restrictions are generally accepted, and the estimates generally have the correct sign. Three out of ten estimates of μ and one out of ten estimates of λ are negative without the CRS restriction and vice versa with the CRS restriction. The average estimate of λ and μ without the CRS restriction is 0.54 and 1.12, respectively.

However, the estimates of λ and μ are often outside the *a priori* assumption of $\lambda, \mu \in (0, 1)$. This contrasts with Kalaitzidakis and Korniotis (2000) who get only estimates satisfying this assumption (given an assumption of CRS).

7 Conclusion

I generalised the Solow model by allowing the savings rate and the population growth rate to vary over time and showed that the model is stable. I then derived three equations in the growth rate of per capita output, the savings rate and the population growth rate. These three equations where then formulated as a VAR model on Error-Correction-Form. I showed under which conditions the system of variables cointegrates, and derived the implied restrictions on the CVAR model under each condition. I then estimated the model using data for the seven G7 countries and Denmark, Norway and Sweden. The restrictions on the VAR and the cointegrating relations in the CVAR implied by the Solow model were generally statistically accepted, although the estimates of the parameters of the Cobb-Douglas production function where not generally between zero and one, and

did not sum to one. Overall, the Solow model does seem to describe the data reasonably well.

Future research could focus on obtaining t-values for the implied estimates, estimate the model for all the countries in the Penn World Table, perform robustness checks with regards to the assumption of $g + \delta = 0.05$, calculate the trace correlations (which can be loosely considered analogous to the R^2 from cross-section regressions) and estimate the model using the growth rate of the labour force instead of the growth rate of population. The model fit measures can be regressed on various country charactaristics such as the level of GDP per capita, population size, openness, continent, indstrialization rate, fertility rate, etc. as well as data quality and abundance.

A Complete estimation results

Table 4: Complete estimation results

	Without CRS restriction, $\mu+\lambda \gtreqless 1$								With CRS restriction, $\mu+\lambda=1$					
$I(1)$	p	$\check{\rho}_g$	$\check{\rho}_s$	$\check{\rho}_n$	s^*	n^*	μ	λ	p	$\check{\rho}_g$	$\check{\rho}_s$	$\check{\rho}_n$	μ	λ
							Canada							
	2%	0.28	1.00	0.88	0.28	0.01	-0.03	0.99	0%	0.31	1.19	0.87	0.78	0.22
s_t	2%	0.28		0.88		0.01	-0.03	0.99	0%	0.33		0.87	1.17	-0.17
n_t	0%	0.28	0.94		0.27		-0.23	0.96	0%	1.00	1.07		0.60	0.40
s_t, n_t	91%	0.28					-0.23	0.97	0%	0.12			1.01	-0.01
							Denmark							
	46%	0.25	0.94	0.89	0.32	0.00	1.87	3.03	60%	0.25	0.95	0.89	1.94	-0.94
s_t	55%	0.25		0.89		0.00	1.86	968.24	66%	0.27		0.89	1.57	-0.57
n_t	9%	0.25	0.93		0.30		2.28	5.09	14%	0.26	0.92		2.19	-1.19
s_t, n_t	40%	0.25					2.28	7.55	43%	2.10			1.66	-0.66
							France							
	1%	0.23	0.91	0.73	0.28	0.01	3.12	0.00	0%	0.43	0.86	0.73	1.32	-0.32
s_t	6%	0.23		0.73		0.01	3.12	0.23	0%	0.52		0.71	0.46	0.54
n_t	0%	0.23	0.89		0.28		3.23	-0.11	0%	1.00	0.92		1.34	-0.34
s_t, n_t	0%	0.23					3.23	0.39	0%	-1.04			0.65	0.35
							Germany							
	11%	0.22	0.85	0.73	0.26	0.00	-1.09	-0.15	4%	0.43	0.84	0.77	1.50	-0.50
s_t	20%	0.22		0.75		0.00	-0.97	2.04	28%	0.17		0.77	-0.64	1.63
n_t	1%	0.22	0.85		0.25		0.03	0.90	3%	0.18	0.87		0.13	0.86
s_t, n_t	21%	0.22					0.03	0.99	32%	2.29			-0.22	1.21
							Italy							
	2%	0.51	0.87	0.96	0.32	0.00	3.28	-2.43	2%	0.61	0.81	0.96	2.34	-1.34
s_t	0%	0.51		0.97		0.00	3.27	-0.13	0%	0.78		0.97	0.63	0.37
n_t	1%	0.51	0.87		0.32		3.14	-2.59	1%	0.61	0.82		2.25	-1.26
s_t, n_t	20%	0.51					3.14	0.32	2%	1.65			0.58	0.43
							Japan							
	12%	0.28	0.92	0.97	0.37	0.00	2.63	2.93	20%	0.29	0.91	0.97	2.58	-1.58
s_t	1%	0.28		0.97		0.00	2.63	-2.29	0%	0.39		0.96	1.50	-0.50
n_t	9%	0.28	0.91		0.37		2.73	3.01	15%	0.29	0.91		2.63	-1.63
s_t, n_t	5%	0.28					2.73	-7.78	2%	-0.31			1.50	-0.50
							Norway							
	29%	0.26	0.92	0.92	0.33	0.00	0.71	24.90	41%	0.26	0.92	0.92	0.95	0.05
s_t	54%	0.26		0.93		0.00	0.75	-0.89	68%	0.26		0.93	0.70	0.26
n_t	15%	0.26	0.93		0.33		1.21	-1.33	23%	0.26	0.93		0.92	0.08
s_t, n_t	29%	0.26					1.21	0.03	45%	1.85			0.72	0.28
							Sweden							
	0%	0.32	0.88	0.83	0.25	0.00	-0.40	0.73	0%	0.34	0.93	0.84	1.03	-0.03
s_t	2%	0.32		0.84		0.00	-0.39	-0.01	2%	0.33		0.84	0.66	0.34
n_t	0%	0.32	0.90		0.25		-0.11	0.89	0%	0.31	0.94		0.89	0.11
s_t, n_t	2%	0.32					-0.12	1.13	3%	2.20			0.59	0.41
							United Kingdom							
	2%	0.32	0.93	0.87	0.24	0.00	0.78	2.02	3%	0.31	0.94	0.87	1.19	-0.19
s_t	11%	0.32		0.87		0.00	0.78	-4.04	18%	0.32		0.86	0.85	0.15
n_t	4%	0.32	0.92		0.22		0.76	1.82	0%	1.00	0.93		1.46	-0.46
s_t, n_t	18%	0.32					0.76	-0.75	33%	1.87			0.84	0.16
							United States of America							
	22%	0.21	0.93	0.93	0.24	0.01	-1.09	0.73	1%	0.27	1.01	0.92	0.94	0.06
s_t	16%	0.21		0.93		0.01	-1.09	0.64	1%	0.27		0.92	0.98	0.02
n_t	6%	0.21	0.93		0.24		-1.03	0.74	0%	0.26	1.01		0.90	0.10
s_t, n_t	7%	0.21					-1.03	0.73	0%	3.84			0.96	0.04

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