Opgave IV.1

a) + b)

$\bar{x} = 1092$ $s_x = 2476$ $N = 4440$ $n = 223$ $\bar{X} = 400$

$$\hat{R} = \frac{\bar{y}}{\bar{x}} = \frac{y}{x} \text{ salg af varen pr. arealenhed .}$$

$$\text{var}[\bar{y}_R] \cong \frac{N-n}{Nn}\tau_z^2 \qquad \hat{\text{var}}[\bar{y}_R] = \frac{N-n}{Nn}\hat{\tau}_z^2$$

$$\bar{y}_R = \frac{\bar{y}}{\bar{x}} \cdot \bar{X} = \hat{R} \cdot \bar{X}$$

$$\hat{\text{var}}[\hat{R}] = \frac{1}{\bar{x}^2}\frac{N-n}{Nn}\hat{\tau}_z^2 \text{ , } \hat{\tau}_z^2 = s_y^2 + \hat{R}^2 s_x^2 - 2\hat{R}\hat{\rho}_{xy}s_x s_y$$

Pålægschokolade:

$\bar{y} = 29118$ $s_y = 66344$ $\rho_{xy} = 0.8828$

$$\hat{R} = \frac{\bar{y}}{\bar{x}} = \frac{29118}{1092} = 26.66$$

$$\hat{\text{var}}[\hat{R}] = \frac{1}{\bar{X}^2} \cdot \frac{N-n}{Nn}\hat{\tau}_z^2$$

$$= \frac{1}{400^2} \cdot \frac{4440-223}{4440 \cdot 223} \cdot (66344^2 + 26.66^2 \cdot 2476^2 - 2 \cdot 26.66 \cdot 0.883 \cdot s_x \cdot s_y)$$

$$= 5.23^2 \Rightarrow \sqrt{\hat{\text{var}}(\hat{R})} = 5.23$$

$$\bar{y}_R = 29118 \cdot \frac{400}{1092} = 10666$$

$$\hat{var}[\bar{y}_R] = \frac{N - n}{Nn}\hat{\tau}_z^2$$

$$= \frac{4440 - 223}{4440 \cdot 223} \cdot (66344^2 + 26.66^2 \cdot 2476^2 - 2 \cdot 26.66 \cdot 0.8828 \cdot 2476 \cdot 66344)$$

$$= 2091.2^2 \Rightarrow \sqrt{var(\bar{y}_R)} = 2091.2$$

Chips:

$\bar{y} = 136$ $s_y = 337$ $\rho_{xy} = 0.9244$

$$\hat{R} = \frac{136}{1092} = 0.125$$

$$\hat{var}[\bar{y}_R] = \frac{4440 - 223}{4440 \cdot 223} \cdot \tau_z^2 = 8.4^2 \Rightarrow \sqrt{var(\bar{y}_R)} = 8.4$$

$$\bar{y}_R = 136 \cdot \frac{400}{1092} = 49.8$$

$$\hat{var}[\hat{R}] = \frac{1}{400^2} \cdot \frac{4440 - 223}{4440 \cdot 223} \cdot (337^2 + 0.125^2 \cdot 2476^2 - 2 \cdot 0.125 \cdot 0.9244 \cdot 2.496 \cdot 337)$$

$$= 0.021^2 \Rightarrow \sqrt{\hat{var}(\hat{R})} = 0.021$$

Shampoo:

$\bar{y} = 161$ $s_y = 396$ $\rho_{xy} = 0.8582$

$$\hat{R} = \frac{161}{1092} = 0.147$$

$$\text{ŧ}_z^2 = 396^2 + 0.147^2 \cdot 2476^2 - 2 \cdot 0.147 \cdot 0.8582 \cdot 396 \cdot 2476 = 41955$$

$$\hat{\text{var}}[\hat{R}] = \frac{1}{400^2} \cdot \frac{4440 - 223}{4440 \cdot 223} \cdot 41955 = 0.033^2 \Rightarrow \sqrt{\hat{\text{var}}(\hat{R})} = 0.033$$

$$\bar{y}_R = 400 \cdot 161 \cdot \frac{1}{1092} = 59$$

$$\hat{\text{var}}[\bar{y}_R] = \frac{4440 - 223}{4440 \cdot 223} \cdot 41955 = 13.4^2 \Rightarrow \sqrt{\text{var}(\bar{y}_R)} = 13.4$$

Instant kaffe:

$\bar{y} = 2581$ $s_y = 5282$ $\rho_{xy} = 0.8240$

$$\hat{R} = \frac{2581}{1092} = 2.4$$

$$\hat{\text{var}}[\hat{R}] = \frac{1}{400^2} \cdot \frac{4440 - 223}{4440 \cdot 223} \cdot 13215190 = 0.59^2 \Rightarrow \sqrt{\hat{\text{var}}(\hat{R})} = 0.59$$

$$\text{ŧ}_z^2 = 5282^2 + 2.4^2 \cdot 2476^2 - 2 \cdot 2.4 \cdot 0.8240 \cdot 2476 \cdot 5282 = 13215190$$

$$\bar{y}_R = 2581 \cdot \frac{400}{1092} = 945.4$$

$$\hat{var}[\bar{y}_R] = \frac{4440 - 223}{4440 \cdot 223} \cdot 13215190 = 237.2^2 \Rightarrow \sqrt{var(\bar{y}_R)} = 237.2$$

c)

$$\bar{y}_{REG} = \bar{y} + \beta_1(\bar{X} - \bar{x}) \text{ , } \hat{\beta}_1 = \hat{\rho}_{xy}\frac{s_y}{s_x}$$

$$\hat{var}(\bar{y}_{REG}) = \frac{N - n}{Nn}\left(s_y^2 + \hat{\beta}_1^2 s_x^2 - 2\hat{\beta}_1\hat{\rho}_{xy}s_x s_y\right)$$

OBS:

Formlen

$$var(\bar{y}_{REG}) = s_y^2(1 - \hat{\rho}_{xy}^2)$$

giver samme resultat, pånær små afrundingsfejl i begge regneudtryk, og den er langt lettere at indsætte i. I besvarelsen er dog brugt den mere besværlige formel. I praksis skal den "lettere" formel selvfølgelig anvendes.

Pålægschokolade:

$$\beta_1 = 0.8828 \cdot 66344/2476 = 23.65$$

$$\bar{y}_{REG} = 29118 \cdot 23.65(400 - 1092) = 12749$$

$$\hat{var}(\bar{y}_{REG}) = \frac{4440 - 223}{4440 \cdot 223} \cdot (66344^2 \cdot 2476^2 - 2 \cdot 23.65 \cdot 0.8828 \cdot 2476 \cdot 66344)$$

$$= 2033.9^2 \Rightarrow \sqrt{\hat{var}(\bar{y}_{REG})} = 2033.9$$

Chips:

$$\beta_1 = 0.9244 \cdot 337/2476 = 0.126$$

$$\bar{y}_{REG} = 136 + 0.126(400 - 1092) = 48.9$$

$$\hat{var}(\bar{y}_{REG}) = \frac{4440 - 223}{4440 \cdot 223} \cdot (337^2 + 0.126^2 \cdot 2476^2 - 2 \cdot 0.126 \cdot 0.9244 \cdot 337 \cdot 2476)$$

$$= 8.4^2 \Rightarrow \sqrt{\hat{var}(\bar{y}_{REG})} = 8.4$$

Shampoo:

$$\beta_1 = 0.8582 \cdot 396/2476 = 0.137$$

$$\bar{y}_{REG} = 161 + 0.137(400 - 1092) = 66$$

$$\hat{var}(\bar{y}_{REG}) = \frac{4440 - 223}{4440 \cdot 223} \cdot (396^2 + 0.137^2 \cdot 2476^2 - 2 \cdot 0.8582 \cdot 0.137 \cdot 396 \cdot 2476)$$

$$= 13.3^2 \Rightarrow \sqrt{\hat{var}(\bar{y}_{REG})} = 13.3$$

Instant kaffe:

$$\beta_1 = 0.8240 \cdot 5282/2476 = 1.758$$

$$\bar{y}_{REG} = 2581 + 1.758(400 - 1092) = 1364.6$$

$$\hat{var}(\bar{y}_{REG}) = \frac{4440 - 223}{4440 \cdot 223} \cdot (5282^2 + 1.758^2 \cdot 2476^2 - 2 \cdot 0.8240 \cdot 1.758 \cdot 5282 \cdot 2476)$$

$$= 173.8^2 \Rightarrow \sqrt{\hat{var}(\bar{y}_{REG})} = 195.3$$

Opgave IV.2

$$\hat{R} - R = \frac{\bar{y}}{\bar{x}} - R = \frac{1}{\bar{x}}(\bar{y} - R\bar{x}) = \frac{1}{\bar{x}} \cdot \bar{z} \simeq \frac{1}{\bar{X}} \cdot \bar{z} \; ;$$

$$\text{alm. approksimation: } \frac{1}{\bar{x}} \simeq \frac{1}{\bar{X}}$$

$$\text{var}(\hat{R}) = \text{var}(\hat{R} - R) \simeq \text{var}\left(\frac{1}{\bar{X}} \cdot \bar{z} \right) = \frac{1}{\bar{X}^2} \cdot \frac{N - n}{Nn} \cdot \tau_z^2$$

Taylorapproksimation:

$$\frac{1}{\bar{x}} = \frac{1}{\bar{X}} - \frac{(\bar{x} - \bar{X})}{\bar{X}^2} + \text{flere led} \simeq \frac{1}{\bar{X}} - \frac{(\bar{x} - \bar{X})}{\bar{X}^2}$$

$$\hat{R} - R = \frac{\bar{y}}{\bar{x}} - R = \frac{1}{\bar{x}}(\bar{y} - R\bar{x}) = \frac{1}{\bar{x}} \cdot \bar{z} \simeq \left(\frac{1}{\bar{X}} - \frac{\bar{x} - \bar{X}}{\bar{X}^2} \right) \cdot \bar{z}$$

$$\text{var}(\hat{R}) = \text{var}(\hat{R} - R) \simeq \text{var}\left\{ \left(\frac{1}{\bar{X}} - \frac{\bar{x} - \bar{X}}{\bar{X}^2} \right) \bar{z} \right\}$$

I dette udtryk opfattes faktoren med x-erne som en konstant, der kvadreret sættes uden for variansdannelsen

$$\text{var}(\hat{R}) \approx \left(\frac{1}{\bar{X}} - \frac{\bar{x} - \bar{X}}{\bar{X}^2} \right)^2 \text{var}(\bar{z}) = \left(\frac{1}{\bar{X}^2} + \frac{(\bar{x} - \bar{X})^2}{\bar{X}^4} - \frac{2}{\bar{X}^3}(\bar{x} - \bar{X}) \right) \text{var}(\bar{z})$$

Nu erstattes faktoren med x-erne med dens forventede værdi

$$\begin{aligned}
&\operatorname{var}(\hat{R}) = \\
&= E\left[\left(\frac{1}{\bar{X}^2} + \frac{(\bar{x} - \bar{X})^2}{\bar{X}^4} - \frac{2}{\bar{X}^3}(\bar{x} - \bar{X})\right)\operatorname{var}(\bar{z})\right] \\
&= \operatorname{var}(\bar{z}) \cdot \left(\frac{1}{\bar{X}^2} + \frac{1}{\bar{X}^4}E[(\bar{x} - \bar{X})^2] - \frac{2}{\bar{X}^3}(E[\bar{x}] - \bar{X})\right) \\
&= \operatorname{var}(\bar{z}) \cdot \left(\frac{1}{\bar{X}^2} + \frac{1}{\bar{X}^4}\operatorname{var}(\bar{x})\right) \\
&= \frac{N - n}{Nn}\tau_z^2\left(\frac{1}{\bar{X}^2} + \frac{1}{\bar{X}^4}\frac{N - n}{Nn}S_x^2\right)
\end{aligned}$$

dvs.

$$\operatorname{var}(\hat{R}) \simeq \left(\frac{1}{\bar{X}^2} + \frac{1}{\bar{X}^4}\frac{N - n}{Nn}s_x^2\right)\frac{N - n}{Nn}\tau_z^2$$

Pålægschokolade:

$$\begin{aligned}
&\operatorname{var}(\hat{R}) \\
&= \left[\frac{1}{400^2} + \frac{1}{400^4} \cdot \frac{4440 - 223}{4440 \cdot 223} \cdot 2476^2\right]\frac{4440 - 223}{4440 \cdot 223} \\
&\cdot (66344^2 + 26.66^2 \cdot 2476^2 - 2 \cdot 26.66 \cdot 0.8828 \cdot 2476 \cdot 66344) \\
&= \left[\frac{1}{370.9^2}\right]\frac{4440 - 223}{4440 \cdot 223} \cdot 1026637665 = 5.64^2 \Rightarrow \sqrt{\hat{\operatorname{var}}(\hat{R})} = 5.64
\end{aligned}$$

Dette kan sammenlignes med standardafvigelsen 5.23 ved den grove approksimation i opgave IV.1.

Chips:

$$\begin{aligned}
\hat{\operatorname{var}}(\hat{R}) &= \left[\frac{1}{370.9^2}\right]\frac{4440 - 223}{4440 \cdot 223} \\
&\cdot 337^2 + 0.125^2 \cdot 2476^2 - 2 \cdot 0.125\ 0.9244 \cdot 2476 \cdot 337) \\
&= 0.023^2 \Rightarrow \sqrt{\hat{\operatorname{var}}(\hat{R})} = 0.023
\end{aligned}$$

Dette kan sammenlignes med standardafvigelsen 0.021 ved den grove approksimation i opgave IV.1.

Shampoo:

$$\hat{\text{var}}(\hat{R}) = \left[\frac{1}{370.9^2}\right]\frac{4440 - 223}{4440 \cdot 223} \cdot 41955 = 0.036^2 \Rightarrow \sqrt{\hat{\text{var}}(\hat{R})} = 0.036$$

Dette kan sammenlignes med standardafvigelsen 0.033 ved den grove approksimation i opgave IV.1.

Instant kaffe:

$$\hat{\text{var}}(\hat{R}) = \left[\frac{1}{370.9^2}\right]\frac{4440 - 223}{4440 \cdot 223} \cdot 13215190 = 0.640^2 \Rightarrow \sqrt{\hat{\text{var}}(\hat{R})} = 0.640$$

Dette kan sammenlignes med standardafvigelsen 0.59 ved den grove approksimation i opgave IV.1.

Opgave IV.3

$$\kappa = 2477 \qquad n = 4556 \qquad y = \sum_{k=1}^{\kappa} Y_{k.} = 330410 \qquad K = 2300000 \qquad N = 4300000$$

$\bar{y}$: gennemsnitlige vinudgifter pr. kvartal pr. dansker over 14 år.

$$\hat{y}_{kl-1} = \frac{1}{N} \cdot \frac{K}{\kappa} \cdot \sum_{k=1}^{\kappa} Y_{k.} \qquad \hat{y}_{kl-R} = \frac{\sum_{k=1}^{\kappa} Y_{k.}}{\sum_{k=1}^{\kappa} N_k} = \frac{y}{n}$$

$$\text{var}(\bar{y}_{kl-R}) = \frac{K(K - \kappa)}{N^2\kappa} \cdot \tau_z^2 \,, \quad \tau_z^2 = \frac{1}{K - 1}\sum_{k=1}^{K} (Y_{k.} - N_k\bar{Y})$$

$$\text{var}(\bar{y}_{kl-1}) = \frac{K(K - \kappa)}{N^2\kappa} \cdot \frac{1}{K - 1}\sum_{k=1}^{K} (Y_{k.} - Y/K)^2$$

a)

$$\hat{\tau}_z^2 = \frac{1}{\kappa - 1}\sum_{k=1}^{\kappa}(Y_{k.} - N_k \cdot \bar{y}_{kl-R})^2 = 2.42 \cdot 10^4$$

$$\frac{1}{K - 1}\sum_{k=1}^{K}(Y_{k.} - Y_{kl-1}/K)^2 = 2.39 \cdot 10^4$$

$$\hat{var}(\bar{y}_{kl-R}) = \frac{2300000(2300000 - 2477)}{4300000^2 \cdot 2477} \cdot 2.42 \cdot 10^4 = 1.671^2$$

$$\bar{y}_{kl-R} = \frac{330410}{4556} = 72.52$$

$$\bar{y}_{kl-1} = \frac{1}{4300000} \cdot \frac{2300000}{2477} \cdot 330410 = 71.35$$

$$\hat{var}(\bar{y}_{kl-1}) = \frac{2300000(2300000 - 2477)}{4300000^2 \cdot 2477} \cdot 2.39 \cdot 10^4 = 1.661^2$$

b)

Udtagelse på klyngeniveau; interessevariabel y_k: vinudgifter pr. kvartal pr. husstand; hjælpevariabel x: antal personer over 14 år pr. husstand.

$$\hat{\rho}_{xy} = 0.06 \qquad s_x = 0.77 \qquad s_y = 310.93 \qquad \bar{X} = \frac{4300000}{230000} = 1.87$$

$$\hat{\beta}_1 = \hat{\rho}_{xy} \cdot \frac{s_y}{s_x} = 0.06 \cdot \frac{310.93}{0.77} = 24.23 \qquad \bar{x} = \frac{4556}{2477} = 1.84$$

$$\bar{y}_{REG} = \bar{y}_{SI} + \hat{\beta}_1(\bar{X} - \bar{x}) = \frac{230410}{2477} + 24.23(1.87 - 1.84) = 133.39 + 0.75 = 134.1$$

$$\hat{var}(\bar{y}_{REG}) = \frac{K - \kappa}{K\kappa} s_y^2 (1 - \hat{\rho}_{xy}^2) = \frac{2300000 - 2477}{2300000 \cdot 2477} \cdot 310.93(1 - 0.06^2) = 38.85$$

(eksklusiv usikkerhed ved bestemmelsen af β_1).

Opgave IV.4

$\kappa = 4736$ n = 9419 $y = Y_{1.} + ... + Y_{\kappa.} = 1080520000$ K = 2300000 N = 4300000

a)

$$R = \frac{Y}{X}: \text{ gennemsnitlig indkomst pr. person over 14 år .}$$

y_i: indkomst pr. husstand

x_i: personer over 14 år pr. husstand.

$$\hat{R} = \frac{\bar{y}}{\bar{x}} = \frac{y}{x} = \frac{1080520000}{9419} = 114717.1$$

Klyngeudvælgelse:

$Y_{k.}$: indkomst pr. husstand

N_k: personer over 14 år pr. husstand.

$$\bar{y}_{kl-R} = \frac{\sum_{k=1}^{\kappa} Y_{k.}}{\sum_{k=1}^{\kappa} N_k} = \frac{1080520000}{9419} = 114717.1$$

$$\bar{y}_{kl-1} = \frac{1}{N}\frac{K}{\kappa}\sum_{k=1}^{\kappa} Y_{k.} = \frac{2300000}{4300000} \cdot \frac{1}{4736} \cdot 1080520000 = 122033.9 \ .$$

Estimaterne i de to første tilfælde er ens, fordi de to estimatorer er identiske. At sidstnævnte adskiller sig fra de to øvrige kan skyldes den definitoriske usikkerhed på størrelsen $\frac{K}{N}$, d.v.s. K: antal husstande i populationen. K burde måske kun omfatte "pæne" husstande, der er villige til at oplyse om dette og hint til Gallup. Hvis K justeres ned, falder værdien af estimatet.

b)

$$\frac{1}{\kappa - 1}\sum_{k=1}^{\kappa} (Y_{k.} - \hat{Y}_{kl-1}/K)^2 = 196.44^2$$

$$\left.\begin{aligned} \hat{var}(\hat{R}) &= \frac{1}{\bar{X}^2}\frac{N-n}{Nn}\hat{t}_z^2 \\ \bar{X} = \frac{N}{K}\ , N = K\ , n &= \kappa \end{aligned}\right\} \quad \hat{var}(\hat{R}) = var(\bar{y}_{kl-R}) = \frac{K^2}{N^2}\frac{K-\kappa}{K\kappa}\hat{t}_z^2$$

$$= \frac{2300000^2}{4300000^2} \cdot \frac{2300000 - 4736}{2300000 \cdot 4736} \cdot 182.11^2 = 1.414^2 \ (1000kr.)^2$$

$$\hat{t}_z^2 = \frac{1}{\kappa - 1}\sum_{k=1}^{\kappa} (Y_{k.} - \bar{y}_{kl-R}N_k)^2 = 182.11^2 \qquad \left(= \frac{1}{n-1}\sum_{i=1}^{n} (y_i - \hat{R}x_i)^2 \right)$$

$$var(\bar{y}_{kl-1}) = \frac{2300000^2}{4300000^2}\ \frac{2300000-4736}{2300000\times 4736}\ 196.44^2 = 1.525^2$$

c)

To - trins klyngeudvælgelse

n_k = 1

$var(\bar{y}_{kl-1}) = 1.5257^2 + 2300000/4300000^2 \times 8126.7 = 10.17^2$

$var(\bar{y}_{kl-R}) = 1.414^2 + 2300000/4300000^2 \times 8126.7 = 10.15^2$

n_k = Gallup

$var(\bar{y}_{kl-1}) = 1.5257^2 + 2300000/4300000^2 \times 6795.0 = 9.32^2$

$var(\bar{y}_{kl-R}) = 1.414^2 + 2300000/4300000^2 \times 6795.0 = 9.30^2$