Re-exam MakØk 2

~~11/10-10~~ 3/2-2012

~~E.2) = C.2 - 2011~~

1)

a) (*) is essentially just national product accounting:

$$\dot{K} = Y - cL - \delta K$$

(**) is the K-R rule expressed in terms of $\tilde{c} \equiv c/T$ and using that in equilibrium, $r_t = f'(\tilde{k}) - \delta$.

(***) is the TVC expressed in terms of $\tilde{a}_t \equiv a_t / T_t \equiv \frac{A_t}{T_t L_t} = \frac{K_t}{T_t L_t} \equiv \tilde{k}_t$.

δ = cap. depreс. rate

g = rate of technical progress.

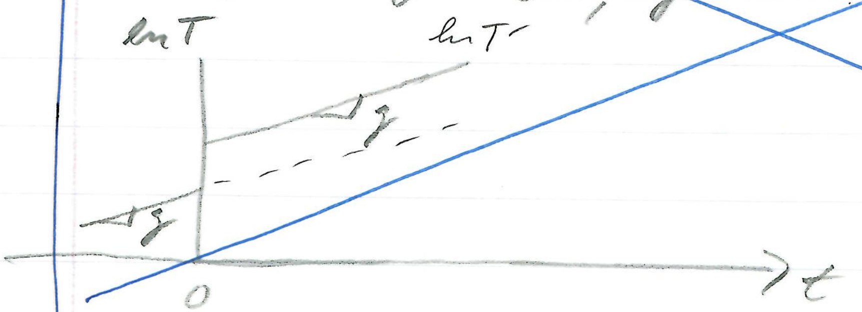
n = population growth rate

θ = elast. of marg. utility of cons. (desire for cons. smoothing)

ρ = pure rate of time preference (rate of impatience)

b) see p. 338. Yes, divergent paths don't satisfy TVC.

c) $T_0 \to T_0' > T_0$, g unchanged



b) At time t the firm solves

$$\max_{K,L} \pi = F(K,TL) - (r+\delta)K - wL$$

FOC's:

$$\frac{\partial \pi}{\partial K} = F_K(K,TL) - (r+\delta) = 0 \qquad (1)$$

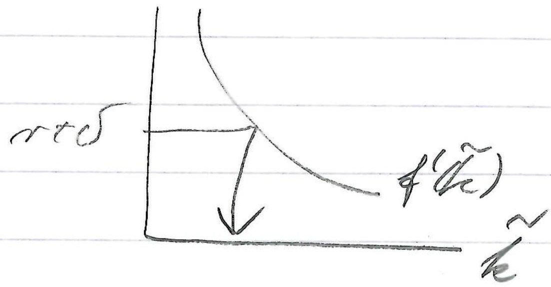
$$\frac{\partial \pi}{\partial L} = F_L(K,TL)T - w = 0 \qquad (2)$$

The desired capital-labor ratio, $k \equiv K/L$, is found the following way:

$$Y = F(K,TL) = TL\,F(\tilde{k},1) \equiv TL\,f(\tilde{k}),$$

where $f' = F_K > 0, \overbrace{\quad}^{f''<0}$. So (1) $\Rightarrow$

$$f'(\tilde{k}) = r+\delta$$



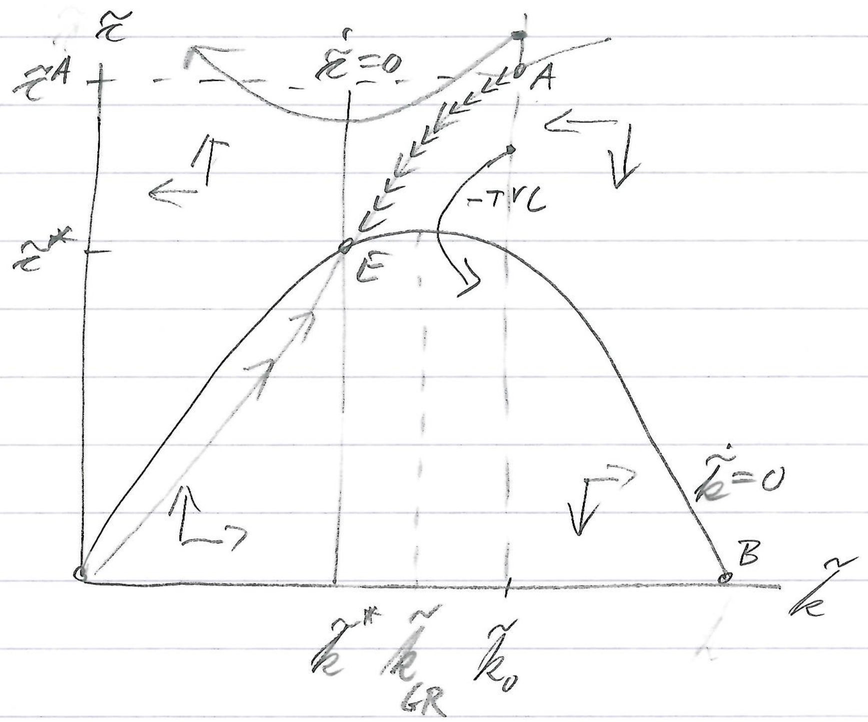
$$k = \tilde{k} \cdot T$$

c) $\dot{\tilde{k}} \gtreqless 0$ for $\tilde{c} \lesseqgtr f(\tilde{k}) - (\delta+g+n)\tilde{k}$

$\dot{\tilde{c}} = 0$ for $\tilde{c} = 0$ or $f'(\tilde{k}) - \delta = \rho + \theta g$

i.e., $\tilde{k} = \tilde{k}^*$ where $f'(\tilde{k}^*) = \delta + \rho + \theta g$

$\dot{\tilde{c}} \gtrless 0$ for $\tilde{c} > 0$ and $\tilde{k} \lessgtr \tilde{k}^*$



We know that $f(0) \geq 0$. The phase diagram, as drawn here, represents the case $f(0) = 0$.

At time $t=0$ the economy, $(\tilde{k}, \tilde{c})$, must be at some point at the line $\tilde{k} = \tilde{k}_0$ since $\tilde{k}_0$ is pre-determined.

The steady state, E, is seen to be a saddle point. At time $t=0$ general equilibrium with perfect foresight requires that the economy is at the point A (the crossing point of the line $\tilde{k} = \tilde{k}_0$ and the saddle path), i.e., $\tilde{c}_0 = \tilde{c}^A$ is required.

If $\tilde{c}_0 < \tilde{c}^A$, the generated evolution of $(\tilde{k}, \tilde{c})$ will be diverging and not satisfy the TVC of the households.

If $\tilde{c}_0 > \tilde{c}^A$, — " —

...... not satisfy the NPG of the households.

Hence, the divergent paths can be ruled out as equilibrium paths with perfect foresight.

We are left with the convergent path AE along the saddle path as the solution. Indeed, this path satisfies the TVC, (3), since it gives $\tilde{k}_t \to \tilde{k}^*$ for $t \to \infty$ so that

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s) - \delta - g - n) ds}$$

$$= \tilde{k}^* e^{-[f'(\tilde{k}^*) - \delta - g - n]t} = 0$$

in view of $f'(\tilde{k}^*) - \delta = \rho + \theta g > g + n$, an inequality following from the condition

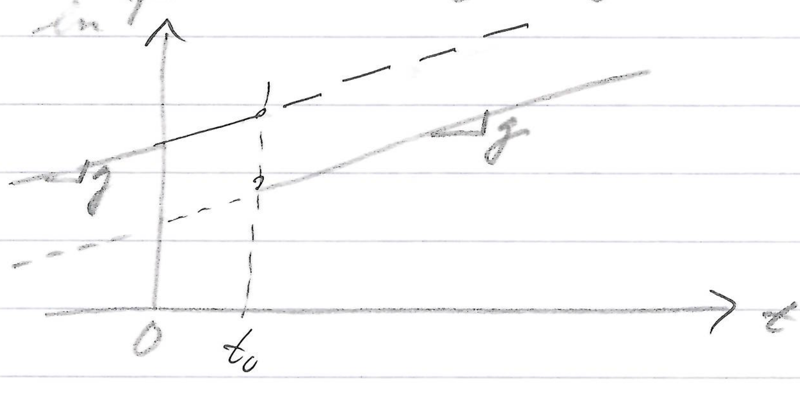
$$\rho - n > (1-\theta) g,$$

which is equivalent with

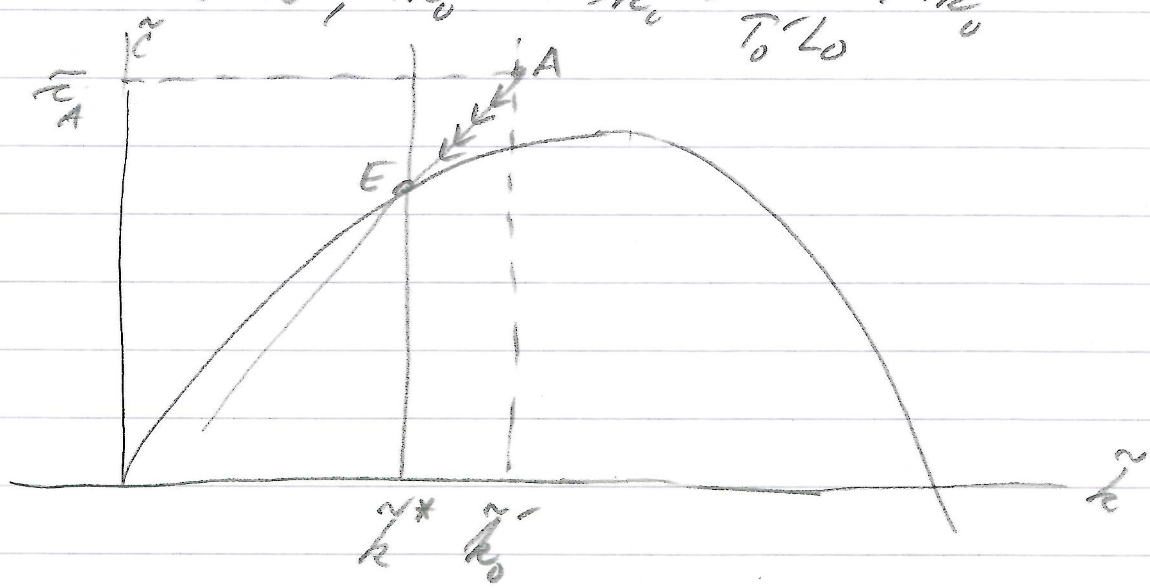
$$\rho + \theta g > g + n.$$

$\left[f'(\tilde{k}_{GR}) - \delta = g + n. \text{ so } \tilde{k}^* < \tilde{k}_{GR} \right]$

d) $T_{t_0} \rightarrow T'_{t_0} < T_{t_0}$



At $t = t_0$, $\tilde{k}_0 \rightarrow \tilde{k}_0' = \dfrac{k_0}{T_0' L_0} > \tilde{k}_0$



The economy jumps to the point A and then moves back along the saddle path for $t \to \infty$.

$\tilde{k}_t \to \tilde{k}^*$ and $\tilde{c}_t \to \tilde{c}^*$ for $t \to \infty$.

e) $r_t = f'(\tilde{k}_t) - \delta \downarrow$ at $t = t_0$

$r_t \to f'(\tilde{k}^*) - \delta \equiv r^*$ for $t \to \infty$.

f) $w_t = \frac{\partial Y_t}{\partial L_t} = F_L(K_t, T_t L_t) T_t$

$= [f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)] T_t \equiv \tilde{w}(\tilde{k}_t) T_t$

$\tilde{w}'(\tilde{k}) = f'(\tilde{k}) - (\tilde{k} f''(\tilde{k}) + \cancel{f'(\tilde{k})})$

$= -\tilde{k} f''(\tilde{k}) > 0$

so $w'_{t_0} = \tilde{w}(\tilde{k}'_{t_0}) > \tilde{w}(\tilde{k}_{t_0})$,

but $T'_0 < T_0$

So there are two opposing effects on w_t at time $t = t_0$.

g) $w_t = \tilde{w}(\tilde{k}_t) T_t \longrightarrow \tilde{w}(\tilde{k}^*) T_t'$ for $t \to \infty$

where $T_t' = T_{t_0}' e^{g(t-t_0)} < T_{t_0} e^{g(t-t_0)}$.

So in the long run w_t will necessarily be lower due to the negative technology shock.

h) $\theta = 1$. $c_t = (\rho - n)(a_t + h_t) = (\rho - n)(k_t + h_t)$

where $h_t = \int_t^\infty w_s e^{-\int_t^s (r_\tau - n) d\tau} ds$

At $t = t_0$, h_t changes:

$r_t' < r_t$ for $t > t_0$,

but also $w_t' < w_t$.

Hence impact effect on c ambiguous.

Alternative reasoning, less deep:

$c_t' \equiv \tilde{c}_t' T_t'$, where $\tilde{c}_t' > \tilde{c}_t$ for $t > t_0$

but $T_t' < T_t$ for $t > t_0$.

i) $\tau_t' \equiv \tilde{\tau}_t' T_t' \xrightarrow[t\to\infty]{} \tilde{\tau}^* T_t' = \tilde{\tau}^* T_{t_0}' e^{g(t-t_0)}$

$< \tilde{\tau}^* T_{t_0} e^{g(t-t_0)}$

j) The First Welfare Theorem requires absence of the double infinity (infinitely many agents and infinitely many periods).

Yes the Ramsey model satisfies this absence since it has a finite number of agents.

2)

a)

$$\max_{(L_t, I_t)_{t=0}^{\infty}} V_0 = \int_0^{\infty} \left[K_t^{\alpha} (T_t L_t)^{1-\alpha} - w_t L_t - I_t - \beta \frac{I_t^2}{2K_t} \right] e^{-rt} \, dt$$

s.t.

$L_t \geq 0$, I_t free

$\dot{K}_t = I_t - \delta K_t$, $K_0 > 0$

$K_t \geq 0 \quad \forall t \geq 0$

$$H = K^{\alpha} (TL)^{1-\alpha} - wL - I - \beta \frac{I^2}{2K} + q(I - \delta K)$$

b)

$$\frac{\partial H}{\partial L} = (1-\alpha) K^{\alpha} T^{1-\alpha} L^{-\alpha} - w = 0 \qquad (1)$$

$$\frac{\partial H}{\partial I} = -1 - \beta \frac{I}{K} + q = 0 \qquad (2)$$

$$\frac{\partial H}{\partial K} = \alpha K^{\alpha-1} (TL)^{1-\alpha} + \beta \frac{I^2}{2K^2} - q\delta = rq - \dot{q} \qquad (3)$$

$$\text{TVC}: \lim_{t \to \infty} K_t q_t e^{-rt} = 0 \qquad (\text{TVC})$$

$$\Rightarrow I = \frac{q-1}{\beta} K$$

c) $(2) \Rightarrow \underbrace{1+\beta\frac{I}{K}}_{} = q \Rightarrow \frac{I}{K} = \frac{1}{\beta}(q-1)$ ✓

$$MC = MB$$

So q is the value to the firm of the marginal unit of installed capital along the optimal path. Hence, q is the shadow price of installed capital.

From now, SOE

d) For $t > t_0$

$$R_t = K_t^{\alpha}(T_t L_t)^{1-\alpha} - w_t L_t - (1-\sigma) I_t - \beta \frac{I_t^2}{2K_t}$$

So (2) is changed into

$$-(1-\sigma) - \beta\frac{I}{K} + q = 0 \qquad \Rightarrow$$

$$1-\sigma+\beta\frac{I}{K} = q \qquad \Rightarrow$$

$$\beta\frac{I}{K} = q+\sigma-1 \qquad \Rightarrow$$

$$\frac{I}{K} = \frac{1}{\beta}(q+\sigma-1)$$

So $\sigma \uparrow \Rightarrow \frac{I}{K} \uparrow$ for any given q.

$$\tilde{k}_t \equiv \frac{K_t}{T_t L_t}, \quad L_t = L_0 e^{nt}$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{K}}{K} - (\delta + n) = \frac{I - \delta K}{K} - (\delta + n)$$

$$= \frac{I}{K} - (\delta + \delta + n) = \frac{1}{\beta}(q + \sigma - 1) - (\delta + \delta + n)$$

$$\Rightarrow \quad \dot{\tilde{k}} = \left[\frac{1}{\beta} q + \frac{\sigma - 1}{\beta} - (\delta + \delta + n)\right] \tilde{k}$$

$$\Rightarrow \quad \dot{\tilde{k}} = \left[\frac{1}{\beta} q - \left(\delta + \delta + n + \frac{1 - \sigma}{\beta}\right)\right] \tilde{k}$$

$$\equiv (a q - b) \tilde{k}$$

f) (3) $\Rightarrow$

$$\dot{q} = (r + \delta) q - \alpha \tilde{k}^{\alpha - 1} - \frac{\beta}{2}\left(\frac{I}{K}\right)^2$$

$$= (r + \delta) q - \alpha \tilde{k}^{\alpha - 1} - \frac{\beta}{2} \frac{1}{\beta^2} (q + \sigma - 1)^2$$

$$= (r + \delta) q - \frac{(q + \sigma - 1)^2}{2\beta} - \alpha \tilde{k}^{\alpha - 1}$$

g) $r > \delta + n$

$\dot{\tilde{k}} = 0$ for $\tilde{k} = 0$ or $q = \beta(\delta + \delta + n) + 1 - \sigma \equiv q^*$

$\dot{\tilde{k}} \gtrless 0$ for $\tilde{k} > 0$ and $q \gtrless \beta(\delta + \delta + n) + 1 - \sigma = q^*$

$$\dot{q} \gtreqless 0 \quad \text{for} \quad (r+\delta)q - \frac{(q+\sigma-1)^2}{2\beta} \gtreqless \alpha \tilde{k}^{\alpha-1} \qquad (4)$$

In st.-st. $\dot{\tilde{k}} = 0 = \dot{q}$

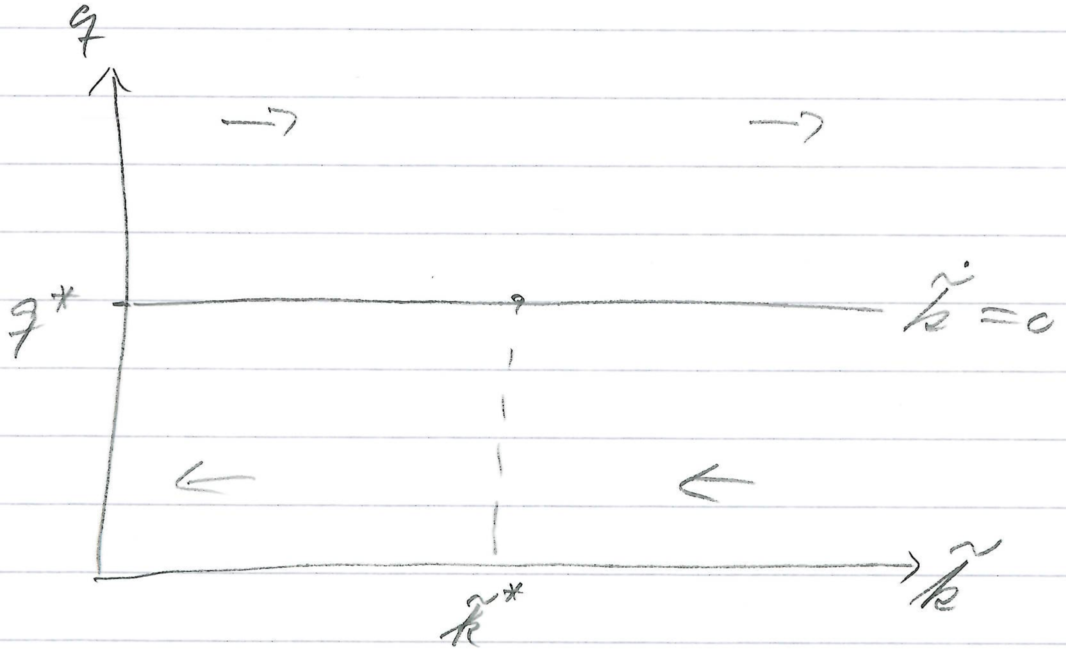
i.e., $q = q^*$ and $\tilde{k} = \tilde{k}^*$

where $\alpha \tilde{k}^{*\alpha-1} = (r+\delta)q^* - \frac{(q^*+\sigma-1)^2}{2\beta}$

$$= (r+\delta)[\beta(\delta+g+n)+1-\sigma] - \frac{[\beta(\delta+g+n)]^2}{2\beta}$$

hence

$$\tilde{k}^* = \left\{ \frac{(r+\delta)[\beta(\delta+g+n)+1-\sigma] - \frac{\beta(\delta+g+n)^2}{2}}{\alpha} \right\}^{\frac{1}{\alpha-1}}$$



To draw the $\dot{q} = 0$ curve we need information about its slope.

$$(4) \Rightarrow (r+\delta)q - \frac{(q+\sigma-1)^2}{2\beta} - \alpha \tilde{k}^{\alpha-1} = 0 \qquad (5)$$

i.e., $\varphi(\tilde{k}, q) = 0 \qquad (5')$

We have $\varphi(\tilde{k}^*, q^*) = 0$

For $(\tilde{k}, q)$ close to $(\tilde{k}^*, q^*)$, $(5')$ defines q implicitly as a function $q = \psi(\tilde{k})$ where

$$\psi'(\tilde{k}) = -\frac{\varphi_{\tilde{k}}}{\varphi_q} = +\frac{\alpha(\alpha-1)\tilde{k}^{\alpha-2}}{r+\delta - \frac{q+\sigma-1}{\beta}}$$

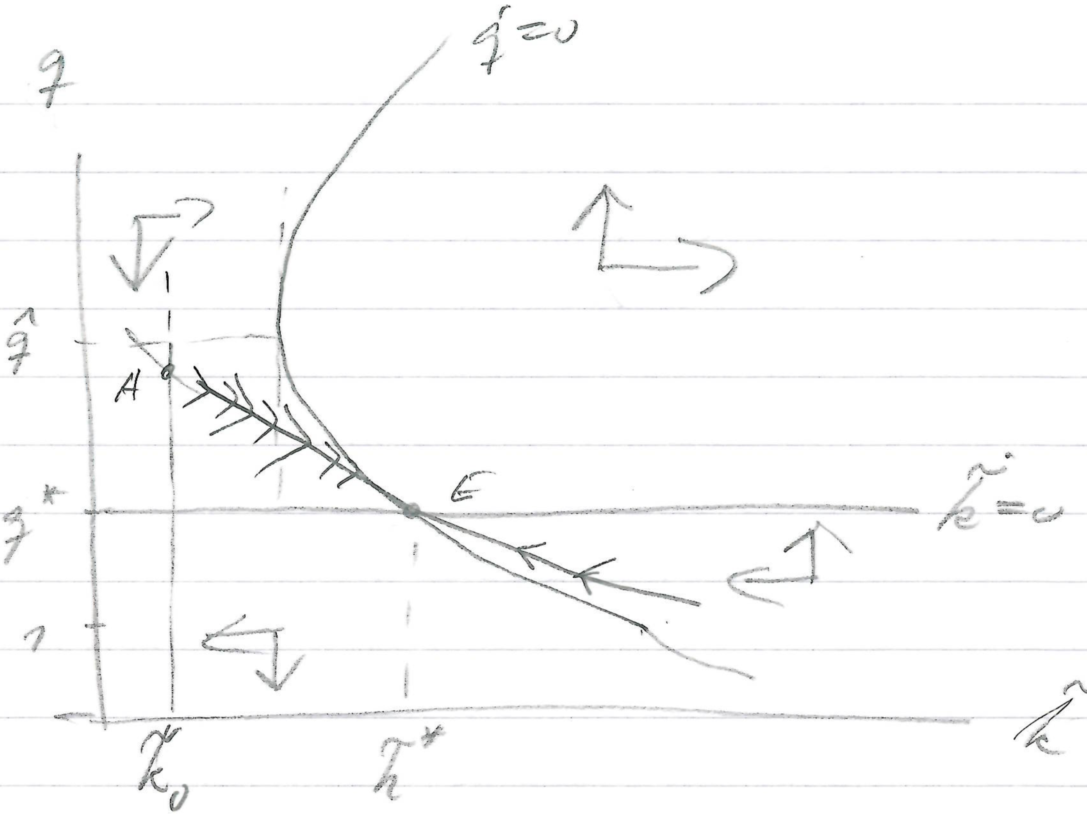
$$= \frac{\alpha(1-\alpha)\tilde{k}^{\alpha-2}}{\frac{q+\sigma-1}{\beta} - (r+\delta)}$$

$$\approx \frac{\alpha(1-\alpha)\tilde{k}^{*\,\alpha-2}}{\frac{q^*+\sigma-1}{\beta} - (r+\delta)} \qquad \left(\text{for } \tilde{k} \approx \tilde{k}^*\right)$$

$$= \frac{\alpha(1-\alpha)\tilde{k}^{*\,\alpha-2}}{\cancel{\delta}+g+n-(r+\cancel{\delta})} < 0$$

since $r > g+n$.

Phase diagram: see p. 6
The arrows indicate that the st. st. is a saddlepoint.



$$\hat{q}: \quad \frac{\hat{q}+\sigma-1}{\beta} = r+\delta \Rightarrow$$

$$\hat{q} = \beta(r+\delta) + 1 - \sigma$$

$$q > \hat{q} \Rightarrow \left.\frac{dq}{d\tilde{k}}\right|_{\dot{q}=0} > 0$$

At $t=0$ the system will be at some point on the line $\tilde{k}=\tilde{k}_0$ since $\tilde{k}_0$ is pre-determined.

The economy will over time move along the saddlepath towards the st.st. E. This evolution satisfies (TVC). The divergent paths do not.

3)

a) True. Let $T = T_0 e^{gt}$, $g > 0$

If $Y = K^{\alpha}(TL)^{1-\alpha}$, then is Harrod-neutral technical progress.

Define $B = T^{\alpha}$. Then $Y = BK^{\alpha}L^{1-\alpha}$ and so there is also Hicks-neutrality.

Define $X = T^{\frac{1-\alpha}{\alpha}}$. Then $Y = (XK)^{\alpha}L^{1-\alpha}$ and so there is also Solow-neutrality.

The ~~same~~ analogue argument goes through if we start from Hicks-neutrality or Solow neutrality.

b) The IBC of the government is the budget constraint

$$\sum_{t=0}^{\infty} \underbrace{(G_t + X_t)}_{\text{spending}}(1+r)^{-(t+1)} \leq \sum_{t=0}^{\infty} \tilde{T}_t (1+r)^{-(t+1)} - B_0$$

gross tax revenue

B_0 = government debt at the beginning of period 0.

The statement is false. The claimed

„only if" does not hold if $r \leq g_Y$

and the claimed „if" does not

hold if $1+g_Y < \lim_{t\to\infty} \frac{B_{t+1}}{B_t} < 1+r$

c) In Problem 2 there are strictly convex capital installation costs. Then the firm has to take the whole future into account in its investment decision.

In Problem 1 there are no installation costs and the firm's intertemporal value maximization problem can be reduced to a series of static profit maximization problems.