Chapter 1

Introduction

> The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive.
>
> −Robert M. Solow (1956, p. 65)

1.1 Macroeconomics

1.1.1 The field

Economics is the social science that studies the production and distribution of goods and services in society. Then, what defines the branch of economics named *macroeconomics?* There are two defining characteristics. First, *macroeconomics* is the systematic study of the economic interactions between human beings in society as a whole. This could also be said of *microeconomic* general equilibrium theory, however. The second defining characteristic of macroeconomics is that it aims at understanding the empirical regularities in the behavior of aggregate economic variables such as aggregate production, investment, unemployment, the general price level for goods and services, the inflation rate, the level of interest rates, the level of real wages, the foreign exchange rate, productivity growth etc. Thus, macroeconomics studies on the major lines of the economics of a society and does so in an intertemporal perspective − evolution over time is in focus.

The aspiration of macroeconomics is three-fold:

1. to *explain* the levels of the aggregate variables as well as their movement over time in the short run and the long run;

2. to make well-founded *forecasts* possible;

3. to provide foundations for rational *economic policy* applicable to macroeconomic problems, be they short-run distress in the form of economic recession or problems of a more long-term, structural character.

We use *economic models* to make our complex economic environment accessible for theoretical analysis. What is an economic model? It is a way of organizing one's thoughts about the economic functioning of a society. A more specific answer is to define an economic model as a conceptual structure based on a set of mathematically formulated assumptions which have an economic interpretation – a link to the economic world outside the window – and from which empirically testable predictions can be derived. In particular, a macroeconomic model is an economic model concerned with macroeconomic phenomena, i.e., the short-run fluctuations of aggregate variables as well as their long-run trend.

Any economic analysis is based upon a conceptual framework. Formulating this framework as a precisely stated economic model helps to break down the issue into assumptions about the concerns and constraints of households and firms and the character of the market environment within which these agents interact. The advantage of this approach is that it makes rigorous reasoning possible, lays bare where the underlying disagreements behind different interpretations of economic phenomena are, and makes sensitivity analysis of the conclusions amenable. By being explicit about the concerns of the agents and the technological constraints and social structures (market forms, social conventions, and legal institutions) that condition their interactions, this approach allows analysis of policy interventions, including the use of well-established tools of welfare economics. Moreover, mathematical modeling is a simple necessity to keep track of the many mutual dependencies and to provide a consistency check of the many accounting relationships involved. And mathematical modeling opens up for use of powerful mathematical theorems from the mathematical toolbox. Without these math tools it would in many cases be impossible to reach any conclusion whatsoever.

Students of economics are often perplexed or even frustrated by macroeconomics being so preoccupied with composite theoretical models. Why not study the issues each at a time? The reason is that the issues, say housing prices and changes in unemployment, are not separate, but parts of a complex system of mutually dependent variables. The economic system as a whole is more than the sum of its parts. This also brings to mind that macroeconomics has to take advantage of theoretical and empirical knowledge from other branches of economics, including microeconomics, industrial organization, game theory, political economy, behavioral economics, and even sociology and psychology.

At the same time models necessarily give a *simplified* picture of the economic reality. Ignoring secondary aspects and details is indispensable to be able to focus on the essential features of a given problem. In particular macroeconomics deliberately simplifies the description of the individual actors so as to make the analysis of the interaction between different *types* of actors manageable.

The assessment of – and choice between – *competing* simplifying frameworks should be based on how well they perform in relation to the three-fold aim of macroeconomics listed above, given the problem at hand. A necessary condition for good performance is the empirical tenability of the model's predictions. A guiding principle in the development of useful models therefore lies in confrontation of the predictions as well as the crucial assumptions with data. This can be based on a variety of methods ranging from sophisticated econometric techniques to qualitative case studies.

Three constituents make up an *economic theory:* 1) the union of connected and non-contradictory economic models, 2) the theorems derived from these, and 3) the conceptual system defining the correspondence between the variables of the models and the social reality to which they are to be applied. Being about the interaction of *human* beings in *societies*, the subject matter of economic theory is extremely complex and at the same time history dependent. The overall political, social, and economic institutions ("rules of the game" in a broad sense) evolve over time.

These circumstances explain why economic theory is far from the natural sciences with respect to precision and undisputable empirical foundation. Especially in macroeconomics, to avoid confusion, the student should be aware of the existence of differing conceptions and in several matters even conflicting theoretical schools.

1.1.2 The different "runs"

This textbook is about industrialized market economies of today. We study basic concepts, models, and analytical methods of relevance for understanding macroeconomic processes in such economies. Sometimes centripetal and sometimes centrifugal forces are dominating. A simplifying device is the distinction between "short-run", "medium-run", and "long-run" analysis. The first concentrates on the behavior of the macroeconomic variables within a time horizon of at most a few years, whereas "long-run" analysis deals with a considerably longer time horizon – indeed, long enough for changes in the capital stock, population, and technology to have a dominating influence on changes in the level of production. The "medium run" is then something in between.

To be more specific, *long-run macromodels* study the evolution of an economy's productive capacity over time. Typically a time span of at least 15 years is considered. The analytical framework is by and large *supply-dominated.* That is, variations in the employment rate for labor and capital due to demand fluctuations are abstracted away. This can to a first approximation be justified by the fact that these variations, at least in advanced economies, tend to remain within a fairly narrow band. Therefore, under "normal" circumstances the economic outcome after, say, a 20 years' interval reflects primarily the change in supply side factors such as the educational level of the labor force, the capital stock, and the technology. Within time horizon also changes in institutions (market structure, government planning and regulation, rules of the game) come into focus.

By contrast, when we speak of *short-run macromodels*, we think of models concentrating on mechanisms that determine how fully an economy uses its productive capacity at a given point in time. The focus is on the level of output and employment within a time horizon less than, say, three years. These models are typically *demand-dominated*. In this time perspective the demand side, monetary factors, and price rigidities matter significantly. Shifts in aggregate demand (induced by, e.g., changes in fiscal or monetary policy, exports, interest rates, the general state of confidence, etc.) tend to be accommodated by changes in the produced quantities rather than in the prices of manufactured goods and services. By contrast, variations in the supply of production factors and technology are diminutive and of limited importance within this time span. With Keynes' words the aim of short-run analysis is to explain "what determines the actual employment of the available resources" (Keynes 1936, p. 4).

The short and the long run make up the traditional subdivision of macroeconomics. It is convenient and fruitful, however, to include also a *medium run*, referring to a time interval of, say, three-to-fifteen years.[1] We shall call models attempting to bridge the gap between the short and the long run *medium-run macromodels*. These models deal with the regularities exhibited by *sequences* of short periods. However, in contrast to long-run models which focus on the trend of the economy, medium-run models attempt to understand the pattern characterizing the fluctuations around the trend. In this context, variations at both the demand and supply side are important. Indeed, at the centre of attention is the dynamic interaction between demand and supply factors, the correction of expectations, and the time-consuming adjustment of wages and prices. Such models are also sometimes called *business cycle models.*

Returning to the "long run", what does it embrace in this book? Well, since the surge of "new growth theory" or "endogenous growth theory" in the late 1980s and early 1990s, growth theory has developed into a specialized discipline studying the factors and mechanisms that *determine* the evolution of technology and productivity (Paul Romer 1987, 1990; Phillipe Aghion and Peter Howitt, 1992). An attempt to give a systematic account of this expanding line of work within

[1] These number-of-years figures are only a rough indication. The different "runs" are relative concepts and their appropriateness depends on the specific problem and circumstances at hand.

macroeconomics would take us too far. When we refer to "long-run macromodels", we just think of macromodels with a time horizon long enough such that changes in the capital stock, population, and technology matter. Apart from a taste of "new growth theory" in Chapter 11, we leave the *explanation* of changes in technology out of consideration, which is tantamount to regarding these changes as exogenous.[2]

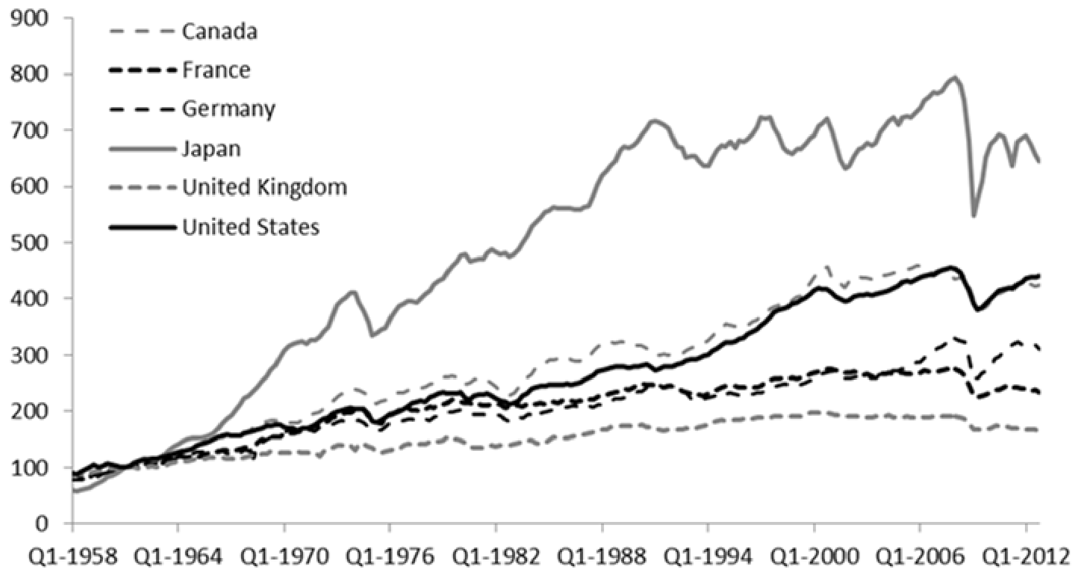

Figure 1.1: Quarterly Industrial Production Index in six major countries (Q1-1958 to Q2-2013; index Q1-1961=100). Source: OECD Industry and Service Statistics. Note: Industrial production includes manufacturing, mining and quarrying, electricity, gas, and water, and construction.

In addition to the time scale dimension, the national-international dimension is important for macroeconomics. Most industrialized economies participate in international trade of goods and financial assets. This results in considerable mutual dependency and co-movement of these economies. Downturns as well as upturns occur at about the same time, as indicated by Fig. 1.1. In particular the economic recessions triggered by the oil price shocks in 1973 and 1980 and by the disruption of credit markets in the outbreak 2007 of the Global Financial Crisis are visible across the countries, as also shown by the evolution of GDP, cf. Fig. 1.2. Many of the models and mechanisms treated in this text will therefore be considered not only in a closed economy setup, but also from the point of view of open economies.

[2]References to textbooks on economic growth are given in *Literature notes* at the end of this chapter.

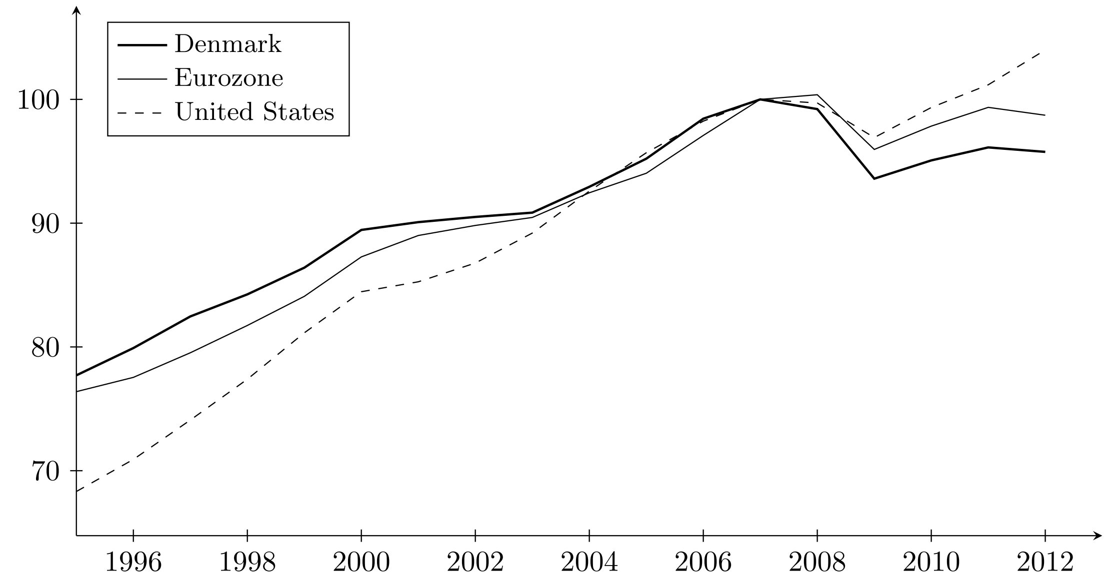

Figure 1.2: Indexed real GDP for Denmark, Eurozone and US, 1995-2012 (2007=100). Source: EcoWin and Statistics Denmark.

1.2 Elements of macroeconomic analysis

1.2.1 Model elements

Basic categories

- Agents: We use simple descriptions of the economic agents (decision makers): A *household* is an abstract entity making consumption, saving and labor supply decisions. A *firm* is an abstract entity making decisions about production and sales. The administrative staff and sales personnel are treated along with the production workers as an undifferentiated labor input.

- Households face *budget constraints*, and firms face *technological constraints,* in macroeconomics.typically described as *production functions.*

- Resources: *Physical capital* refers to stocks of *reproducible durable* means of production such as machines and structures. Reproducible *non-durable* means of production include raw materials, semi-manufacture, and energy (often lumped together as *intermediate goods). Natural resources* include land and other non-reproducible means of production. *Human capital* is the stock of productive skills embodied in an individual.

- Goods, labor, and assets *markets.*

- Market forms and other *rules* regulating the economic interactions.

Types of variables

Endogenous variable = variable whose value is determined within the particular model considered. *Exogenous* variable = variable whose value the particular model considered takes as given.

Stock = a variable measured as a quantity at a given point in time. *Flow.*= a variable measured as a quantity per time unit.

State variable = variable whose value is determined historically at any point in time. For example, the stock (quantity) of water in a bathtub at time t is historically determined as the accumulated quantity of water stemming from the previous inflow and outflow. But if y_t is a variable which is not tied down by its own past but, on the contrary, can immediately adjust if new conditions or new information emerge, then y_t is a *jump* variable. A decision about how much to consume and how much to save − or dissave − in a given month is an example of a jump variable. Returning to our bath tub example: in the moment we pull out the waste plug, the outflow of water per time unit will jump from zero to a positive value and is thus a jump variable.

A state variable may alternatively be called a *predetermined* variable. And a jump variable may alternatively be called a *non-predetermined variable* or a *control* variable.

Types of model relations

Although model relations can take different forms, in macroeconomics they often have the form of equations. A taxonomy for macroeconomic model relations is the following:

1. *Technology equations* describe relations between inputs and output (production functions and similar).

2. *Preference equations* express preferences, e.g. $U = \sum_{t=0}^{T} \frac{u(c_t)}{(1+\rho)^t}$, $\rho > 0, u' > 0, u'' < 0$.

3. *Budget constraints*, whether in the form of an equation or an inequality.

4. *Institutional equations* refer to relationships required by law (e.g., how the tax levied depends on income) and similar.

5. *Behavioral equations* describe the behavioral response to the determinants of behavior. This includes an agent's optimizing behavior written as a function of its determinants. A consumption function is an example. Whether first-order conditions in optimization problems should be considered behavioral equations or just separate first-order conditions is a matter of taste.

6. *Identity equations* are true by definition of the variables involved. National income accounting equations are an example.

7. *Equilibrium equations* define the condition for equilibrium ("state of rest") of some kind, for instance equality of Walrasian demand and Walrasian supply. No-arbitrage conditions for the asset markets also belong under the heading equilibrium condition.

8. *Initial conditions* are equations fixing the initial values of the state variables in a dynamic model

Types of analysis

Static versus dynamic models

A *static model* is a model where time does not enter or at least where all variables refer to the same point in time. A *dynamic* model is a model that establishes a link from the state of the economic system (including its recent history) to the subsequent state. A dynamic model thus allows a derivation of the evolution over time of the endogenous variables.

Macroeconomics is about studies processes in real time and the emphasis is thus on dynamic models. Occasionally we consider *quasi-static models.* The modifier "quasi-" is meant to indicate that although the model concentrates on a single period, it considers some variables as inherited from *the past* and some variables that involve expectations about the future. What we call *temporary equilibrium models* belong to this category. Their role is to serve as a prelude to a more elaborate dynamic model dealing with a sequence of states.

Dynamic analysis aims at establishing dynamic properties of an economic system: is the system stable or unstable, is it asymptotically stable, if so, is it globally or only locally asymptotically stable? Is it oscillatory? If the system is asymptotically stable, how fast is the adjustment?

A study of *dynamic effects of a parameter shift in real time* is a variety of dynamic analysis. Comparative analysis is a different thing; in *comparative dynamics* we compare solutions to a dynamic model under alternative values of the parameters and exogenous variables; in *comparative statics* we compare solutions to a static model under alternative values of the parameters and exogenous variables.

In dynamic modeling and analysis we have a choice between framing the model in period terms or in continuous time. *Period analysis*, also called discrete time analysis, is the method we generally apply up to Chapter 9, where a transition to *continuous-time analysis* is undertaken.

Partial equilibrium analysis versus general equilibrium analysis

We say that a given single market is in *partial equilibrium* at a given point in time if for given prices and quantities in the other markets, the agents' chosen actions in this market are mutually compatible. In contrast, the concept of general equilibrium takes the mutual dependencies between markets into account. We say that a given economy is in *general equilibrium* at a given point in time if in all markets, the actions chosen by the agents are mutually compatible.

An analyst trying to clarify a partial equilibrium problem is doing *partial equilibrium analysis.* Thus partial equilibrium analysis does not take into account the feedbacks from the outcome in a single market to the rest of the economy and the feedbacks from these feedbacks − and so on. In contrast, an analyst trying to clarify a general equilibrium problem is doing *general equilibrium analysis.* This requires considering the mutual dependencies in the system of markets as a whole.

Sometimes in the literature also the analysis of the constrained maximization problem of a single decision maker is called partial equilibrium analysis. Consider for instance the consumption-saving decision of a household. Then the derivation of the saving function of the household is by some authors included under the heading partial equilibrium analysis for the reason that the real wage and real interest rate appearing as arguments in the derived saving function are arbitrary. In this book, however, we shall call the analysis of a single decision maker's problem *partial analysis*, not partial equilibrium analysis. The motivation is that transparency is improved if one preserves the notion of equilibrium for a state of a *market* or a state of a *system of markets.*

1.2.2 From input to output

In macroeconomic theory the production of a firm, a sector, or the economy as a whole is often represented by a two-inputs-one-output production function,

$$Y = F(K, L), \tag{1.1}$$

where Y is output (value added in real terms), K is capital input, and L is labor input ($K \geq 0$, $L \geq 0$). The idea is that for several issues it is useful to think of output as a homogeneous good which is produced by two inputs, one of which is *capital,* by which we mean a *reproducible* durable means of production, the other being *labor,* often considered a *non-producible* human input. Of course, thinking of these variables as representing one-dimensional entities is a drastic abstraction, but may nevertheless be worthwhile in a first approach.

Simple as it looks, an equation like (1.1) may nevertheless raise several conceptual issues.

The time dimension of input and output

A key issue is: how are the variables entering (1.1) *denominated,* that is, what is the *dimension* of the variables? Or in what units are the variables measured? It is most satisfactory, from a theoretical as well as empirical point of view, to think of both outputs and inputs as *flows:* quantities per unit of time. This is generally recognized as far as Y is concerned. It is less recognized, however, concerning K and L, a circumstance which is probably related to a *tradition in macroeconomic notation*, as we will now explain.

Let the time unit be one year. Then the K appearing in the production function should be seen as the number of machine hours per year. Similarly, L should be seen as the number of labor hours per year. Unless otherwise specified, it should be understood that the rate of utilization of the production factors is constant over time. For convenience, one can then *normalize the rate of utilization of each factor to equal one.* We thus define one *machine-year* as the service of a machine in operation h hours a year. If K machines are in operation and on average deliver one machine-year per year, then the total capital input is K machine-years per year:

$$K \text{ (machine-yrs/yr)} = K \text{ (machines)} \times 1 \text{ ((machine-yrs/yr)/machine)}, \quad (1.2)$$

where the dimension of the variables is indicated in brackets. Note that to be correct, an equation should have not only the same quantity on both sides, but also the same dimension. Both conditions are satisfied by (1.2), since $K \times 1 = K$ and (machines $\times$ (machine-yrs/yr)/machine) = (machine-yrs/yr). Sometimes we consider equations where a bare number, also known as a *dimensionless quantity*, appears on both sides. Such quantities may arise as the the product or ratio of two quantities that are not dimensionless. For instance, the fraction of income saved is a bare number since both saving and income are measured in the same units, say, euros per year, whereby the dimensions cancel out. In such cases the variable in question is said to have dimension *one*.[3]

Considering the labor input, suppose similarly that the stock of laborers is L men and that on average they deliver one *man-year* (say h hours) per year. Then the total labor input is L man-years per year:

$$L\text{(man-yrs/yr)} = L\text{(men)} \times 1\text{((man-yrs/yr)/man)}. \quad (1.3)$$

Now, a reason that stocks and flows may be confused is that often the same symbol, K, appearing in the production function as a capital *input flow*, also,

[3]It is like in physics. Length, time, and speed are measured in dimensional units, such as metre, second and metre/second whereas the alcohol percentage in a beverage is a bare number.

within the same model, appears as the capital *stock* in an accumulation equation like

$$K_{t+1} = K_t + I_t - \delta K_t. \tag{1.4}$$

In this equation, I_t is gross investment in period t, and δ is the rate of physical capital depreciation due to wear and tear ($0 \leq \delta \leq 1$). So the symbol K_t must represent the capital *stock* at the beginning of period t. In (1.4) there is no role for the rate of *utilization* of the capital stock, which is, however, of key importance in (1.1). Similarly, there is a tradition in macroeconomics to denote the number of heads in the labor force by L and write, for example, $L_t = L_0(1+n)^t$, where n is a constant growth rate of the labor force. Here L_t measures a stock (number of persons) whereas in (1.1) and (1.3) L measures a flow that depends on the average rate of utilization of the stock over the year.

This text will not attempt a break with this tradition of using the same symbol for two in principle different variables. But we should ensure that our notation *is* consistent. This requires normalization of the utilization rates for capital and labor in the production function so as to equal one, as indicated in (1.2) and (1.3) above. We are then allowed to use the same symbol for a stock and the corresponding flow because the *values* of the two variables will coincide and their dimensions are the same.

As an illustration of the importance of being aware of the distinction between stock and flows, let

$$\begin{aligned} Y &= GDP \text{ per year, and} \\ P &= \text{average size of population over the year.} \end{aligned}$$

Then income per year per capita can be decomposed the following way:

$$\begin{aligned} \frac{GDP}{P} &\equiv \frac{\text{value added/yr}}{\#\text{people}} = \frac{\text{value added/yr}}{\#\text{hours of work/yr}} \\ &\times \frac{\#\text{hours of work/yr}}{\#\text{employed workers}} \times \frac{\#\text{employed workers}}{\#\text{workers}} \times \frac{\#\text{workers}}{\#\text{people}}, \end{aligned} \tag{1.5}$$

where # stand for "number of", and "employed workers" and "workers" stand for "full-time" people, thus weighting by the fraction of a standard man-year they actually work or at least want to work, respectively. That is, aggregate per capita income equals average labor productivity times average labor intensity times the employment rate times the workforce participation rate. An increase from one year to the next in per capita income thus reflects the net effect of changes in the four ratios on the right-hand side. Similarly, a fall in per capita income (a ratio between a flow and a stock) need not reflect for instance a fall in productivity, but may reflect, say, a fall in the employment rate (a rise in unemployment) or in the participation rate due to an ageing population.

Natural resources?

A *second* conceptual issue concerning the production function in (1.1) is: what about the role of land and other natural resources? As farming requires land and factories and office buildings require building sites, a third argument, a natural resource input, should in principle appear in (1.1). In theoretical macroeconomics for industrialized economies, to simplify, this third factor is often left out because it does not vary much as an input to production and tends to be of secondary importance in value terms.

Intermediate goods?

A *third* conceptual issue concerning the production function in (1.1) relates to the question: what about *intermediate goods*? By intermediate goods we mean non-durable means of production like raw materials and energy. Certainly, raw materials and energy are generally necessary inputs at the micro level. It therefore seems strange to regard output as produced by only capital and labor. Again, the motivation is that putting the engineering input-output relations involving intermediate goods aside is a convenient simplification. One imagines that at a lower stage of production, raw materials and energy are continuously produced by capital and labor, but are then immediately used up at a higher stage of production, again using capital and labor. The value of these materials are not part of value added in the sector or in the economy as a whole. Since value added is what macroeconomics usually focuses at and what the Y in (1.1) represents, materials therefore are often not explicit in the model.

On the other hand, if of interest for the problems studied, the analysis *should,* of course, take into account that at the aggregate level in real world situations, there will generally be a (minor) difference between produced and used-up raw materials which then constitute net investment in inventories of materials.

To further clarify this point as well as more general aspects of how macroeconomic models are related to national income and product accounts, the next section gives a review of national income accounting.

1.3 Macroeconomic models and national income accounting

(very incomplete)

Stylized national income and product accounts

We give here a stylized picture of national income and product accounts with emphasis on the conceptual structure. The basic point to be aware of is that national income accounting looks at output from *three sides*:

- the production side (value added),
- the use side,
- the income side.

These three "sides" refer to different approaches to the practical measurement of production and income: the "output approach", the "expenditure approach", and the "income approach".

Consider a closed economy with three production sectors. Sector 1 produces intermediate goods (including raw materials and energy) in the amount Q_1 per time unit, Sector 2 produces durable capital goods in the amount Q_2 per time unit, and the third sector produces consumption goods in the amount Q_3 per time unit.

It is common to distinguish between three basic *production factors* available ex ante a given production process. These are *land* or, more generally, non-producible means of production, *labor*, and *capital* (producible durable means of production). In practice also intermediate goods are a necessary production input. As mentioned above, in simple models this input is regarded as itself produced at an early stage within the production period and then used up during the remainder of the production process. In more rigorous dynamic analyses, however, the intermediate goods are considered produced *prior* to the production process in which they are used. To see what this looks like and what it means to abstract from it in the simpler models, we here consider intermediate goods as a fourth input type produced separately in Sector 1.

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1.4 Some terminological points

(Incomplete)

We follow the convention in macroeconomics and, unless otherwise specified, use "capital" for physical capital, that is, a (re-producible) production factor. In other branches of economics and in everyday language "capital" may mean the funds (sometimes called "financial capital") that finance purchases of physical capital.

By a household's *wealth* (sometimes denoted *net wealth*), W, we mean the value of the total stock of real as well as financial resources, possessed by the household at a given point in time. This wealth generally has two main components, the *human wealth*, which is the present value of the expected stream of future labor income,[4] and the *non-human wealth.* The latter is the sum of the value of the household's *physical assets* (also called *real* assets) and its *net financial assets.* Typically, housing wealth is the dominating component in households' physical assets. By *net financial assets* is meant the difference between the value of financial assets and the value of financial liabilities. *Financial assets* include cash as well as paper claims that entitles the owner to future transfers from the issuer of the claim, perhaps conditional on certain events. Bonds and shares of stock are examples. A *financial liability* of an economic agent is an obligation to transfer resources to others in the future. A mortgage loan is an example.

In spite of the described distinction between what is called physical assets and what is called financial assets, often in macroeconomics the household's "financial wealth" is used as synonymous with its non-human wealth. In this book, unless otherwise indicated, we follow this convention. Thereby, a household's *financial wealth* is the total value of its non-human assets, thus including not only its net financial assets, but also its physical assets like land, house, car, machines, and other equipment.

Somewhat at odds with this convention, macroeconomics (including this book) generally uses "investment" as synonymous with "physical capital investment", that is, procurement of new machines and plants by firms and new houses or apartments by households. Then, when having purchases of *financial* assets in mind, macroeconomists talk of *financial investment.*

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Saving (flow) versus savings (stock).

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1.5 Brief history of macroeconomics

Text not yet available.

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1.6 Literature notes

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[4] And is thus to be distinguished from *human capital*, which, as defined in Section 2.1, is a production factor.

The modern theory of economic growth ("new growth theory", "endogenous growth theory") is extensively covered in dedicated textbooks like **?**, **?**, **?**, **?**, and **?**. A good introduction to analytical development economics is Basu (1997).

?, **?**, and **?** present useful overviews of the history of macroeconomics. For surveys on recent developments on the research agenda within theory as well as practical policy analysis, see **?**, **?**, and **?**. Somewhat different perspectives, from opposite poles, are offered by **?** and **?**.

Chapter 2

Review of technology and firms

The aim of this chapter is threefold. First, we shall introduce this book's vocabulary concerning firms' technology and technological change. Second, we shall refresh our memory of key notions from microeconomics relating to firms' behavior and factor market equilibrium under simplifying assumptions, including perfect competition. Finally, to prepare for the many cases where perfect competition and other simplifying assumptions are not good approximations to reality, we give an introduction to firms' behavior under more realistic conditions including monopolistic competition.

The vocabulary pertaining to other aspects of the economy, for instance households' preferences and behavior, is better dealt with in close connection with the specific models to be discussed in the subsequent chapters. Regarding the distinction between discrete and continuous time analysis, most of the definitions contained in this chapter are applicable to both.

2.1 The production technology

Consider a two-input-one-output production function given by

$$Y = F(K, L), \tag{2.1}$$

where Y is output (value added) per time unit, K is capital input per time unit, and L is labor input per time unit ($K \geq 0$, $L \geq 0$). We may think of (2.1) as describing the output of a firm, a sector, or the economy as a whole. It is in any case a very simplified description, ignoring the heterogeneity of output, capital, and labor. Yet, for many macroeconomic questions it may be useful in a first approach.

Note that in (2.1) not only Y but also K and L represent *flows,* that is, quantities per unit of time. If the time unit is one year, we think of K as measured in machine hours per year. Similarly, we think of L as measured in labor hours per year. Unless otherwise specified, it is understood that the rate of utilization of the production factors is constant over time and normalized to one for each production factor. As explained in Chapter 1, we can then use the same symbol, K, for the *flow* of capital services as for the *stock* of capital. Similarly with L.

2.1.1 A neoclassical production function

By definition, Y, K and L are non-negative. It is generally understood that a production function, $Y = F(K, L)$, is *continuous* and that $F(0, 0) = 0$ (no input, no output). Sometimes, when a production function is specified by a certain formula, that formula may not be defined for $K = 0$ or $L = 0$ or both. In such a case we adopt the convention that the domain of the function is understood extended to include such boundary points whenever it is possible to assign function values to them such that continuity is maintained. For instance the function $F(K, L) = \alpha L + \beta KL/(K + L)$, where $\alpha > 0$ and $\beta > 0$, is not defined at $(K, L) = (0, 0)$. But by assigning the function value 0 to the point $(0, 0)$, we maintain both continuity and the "no input, no output" property.

We call the production function *neoclassical* if for all (K, L), with $K > 0$ and $L > 0$, the following additional conditions are satisfied:

(a) $F(K, L)$ has continuous first- and second-order partial derivatives satisfying:

$$F_K > 0, \quad F_L > 0, \tag{2.2}$$

$$F_{KK} < 0, \quad F_{LL} < 0. \tag{2.3}$$

(b) $F(K, L)$ is strictly quasiconcave (i.e., the level curves, also called isoquants, are strictly convex to the origin).

In words: (a) says that a neoclassical production function has continuous substitution possibilities between K and L and the *marginal productivities* are positive, but diminishing in own factor. Thus, for a given number of machines, adding one more unit of labor, adds to output, but less so, the higher is already the labor input. And (b) says that every isoquant, $F(K, L) = \bar{Y}$, has a strictly convex form qualitatively similar to that shown in Fig. 2.1.[1] When we speak of for example F_L as the marginal *productivity* of labor, it is because the "pure"

[1] For any fixed $\bar{Y} \geq 0$, the associated *isoquant* is the level set $\left\{(K, L) \in \mathbb{R}_+ \mid F(K, L) = \bar{Y}\right\}$. A refresher on mathematical terms such as *level set*, *boundary point*, *convex function*, etc. is contained in Math Tools.

partial derivative, $\partial Y/\partial L = F_L$, has the denomination of a productivity (output units/yr)/(man-yrs/yr). It is quite common, however, to refer to F_L as the marginal *product* of labor. Then a unit marginal increase in the labor input is understood: $\Delta Y \approx (\partial Y/\partial L)\Delta L = \partial Y/\partial L$ when $\Delta L = 1$. Similarly, F_K can be interpreted as the marginal *productivity* of capital or as the marginal *product* of capital. In the latter case it is understood that $\Delta K = 1$, so that $\Delta Y \approx (\partial Y/\partial K)\Delta K = \partial Y/\partial K$.

The definition of a neoclassical production function can be extended to the case of n inputs. Let the input quantities be $X_1, X_2, \ldots, X_n$ and consider a production function $Y = F(X_1, X_2, \ldots, X_n)$. Then F is called neoclassical if all the marginal productivities are positive, but diminishing in own factor, and F is strictly quasiconcave (i.e., the upper contour sets are strictly convex, cf. Appendix A). An example where $n = 3$ is $Y = F(K, L, J)$, where J is land, an important production factor in an agricultural economy.

Returning to the two-factor case, since $F(K, L)$ presumably depends on the level of technical knowledge and this level depends on time, t, we may want to replace (2.1) by

$$Y_t = F(K_t, L_t, t), \tag{2.4}$$

where the third argument indicates that the production function may shift over time, due to changes in technology. We then say that F is a neoclassical production function if for all t in a certain time interval it satisfies the conditions (a) and (b) w.r.t its first two arguments. *Technological progress* can then be said to occur when, for K_t and L_t held constant, output increases with t.

For convenience, to begin with we skip the explicit reference to time and level of technology.

The marginal rate of substitution Given a neoclassical production function F, we consider the isoquant defined by $F(K, L) = \bar{Y}$, where $\bar{Y}$ is a positive constant. The *marginal rate of substitution*, MRS_{KL}, of K for L at the point (K, L) is defined as the absolute slope of the isoquant $\left\{(K, L) \in \mathbb{R}^2_{++} \middle| F(K, L) = \bar{Y}\right\}$ at that point, cf. Fig. 2.1. For some reason (unknown to this author) the tradition in macroeconomics is to write $Y = F(K, L)$ and in spite of ordering the arguments of F this way, nonetheless have K on the vertical and L on the horizontal axis when considering an isoquant. At this point we follow the tradition.

The equation $F(K, L) = \bar{Y}$ defines K as an implicit function $K = \varphi(L)$ of L. By implicit differentiation we get $F_K(K, L)dK/dL + F_L(K, L) = 0$, from which follows

$$MRS_{KL} \equiv -\frac{dK}{dL}_{|Y=\bar{Y}} = -\varphi'(L) = \frac{F_L(K, L)}{F_K(K, L)} > 0. \tag{2.5}$$

So MRS_{KL} equals the ratio of the marginal productivities of labor and capital, respectively.[2] The economic interpretation of MRS_{KL} is that it indicates (approximately) how much of K can be saved by applying an extra unit of labor. Hence, a cost-minimizing firm that plans to produce $\bar{Y}$ units, will choose inputs, K and L, such that the marginal rate of substitution of K for L equals the inverse factor price ratio.

Since F is neoclassical, by definition F is strictly quasi-concave and so the marginal rate of substitution is diminishing as substitution proceeds, i.e., as the labor input is further increased along a given isoquant. Notice that this feature characterizes the marginal rate of substitution for any neoclassical production function, whatever the returns to scale (see below).

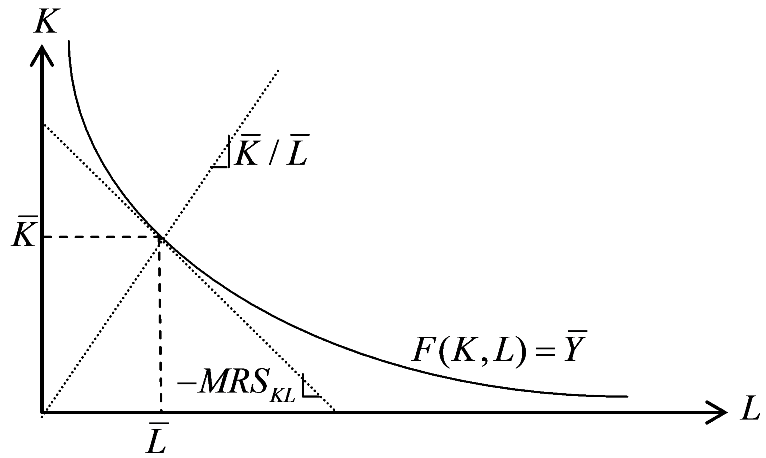


Figure 2.1: MRS_{KL} as the absolute slope of the isoquant representing $F(K,L) = \bar{Y}$.

When we want to draw attention to the dependency of the marginal rate of substitution on the factor combination considered, we write $MRS_{KL}(K,L)$. Sometimes in the literature, the marginal rate of substitution between two production factors, K and L, is called the *technical* rate of substitution (or the technical rate of transformation) in order to distinguish from a consumer's marginal rate of substitution between two consumption goods.

As is well-known from microeconomics, a firm that minimizes production costs for a given output level and given factor prices, will choose a factor combination such that MRS_{KL} equals the ratio of the factor prices. If $F(K,L)$ is homogeneous of degree q, then the marginal rate of substitution depends only on the factor proportion and is thus the same at any point on the ray $K = (\bar{K}/\bar{L})L$. In this case the expansion path is a straight line.

[2] The subscript $|Y = \bar{Y}$ in (2.5) signifies that "we are moving along a given isoquant $F(K,L) = \bar{Y}$", i.e., we are considering the relation between K and L under the restriction $F(K,L) = \bar{Y}$.

The Inada conditions A continuously differentiable production function is said to satisfy the *Inada conditions*[3] if

$$\lim_{K\to 0} F_K(K,L) = \infty, \lim_{K\to\infty} F_K(K,L) = 0, \tag{2.6}$$
$$\lim_{L\to 0} F_L(K,L) = \infty, \lim_{L\to\infty} F_L(K,L) = 0. \tag{2.7}$$

In this case, the marginal productivity of either production factor has no upper bound when the input of the factor becomes infinitely small. And the marginal productivity is gradually vanishing when the input of the factor increases without bound. Actually, (2.6) and (2.7) express *four* conditions, which it is preferable to consider separately and label one by one. In (2.6) we have two *Inada conditions for MPK* (the marginal productivity of capital), the first being a *lower*, the second an *upper* Inada condition for MPK. And in (2.7) we have two *Inada conditions for MPL* (the marginal productivity of labor), the first being a *lower*, the second an *upper* Inada condition for MPL. In the literature, when a sentence like "the Inada conditions are assumed" appears, it is sometimes not made clear which, and how many, of the four are meant. Unless it is evident from the context, it is better to be explicit about what is meant.

The definition of a neoclassical production function we have given is quite common in macroeconomic journal articles and convenient because of its flexibility. Yet there are textbooks that define a neoclassical production function more narrowly by including the Inada conditions as a requirement for calling the production function neoclassical. In contrast, in this book, when in a given context we need one or another Inada condition, we state it explicitly as an additional assumption.

2.1.2 Returns to scale

If all the inputs are multiplied by some factor, is output then multiplied by the same factor? There may be different answers to this question, depending on circumstances. We consider a production function $F(K,L)$ where $K > 0$ and $L > 0$. Then F is said to have *constant returns to scale* (CRS for short) if it is homogeneous of degree one, i.e., if for all $(K,L) \in \mathbb{R}^2_{++}$ and all $\lambda > 0$,

$$F(\lambda K, \lambda L) = \lambda F(K,L).$$

As all inputs are scaled up or down by some factor, output is scaled up or down by the same factor.[4] The assumption of CRS is often defended by the *replication*

[3] After the Japanese economist Ken-Ichi Inada, 1925-2002.

[4] In their definition of a neoclassical production function some textbooks add constant returns to scale as a requirement besides (a) and (b) above. This book follows the alternative

argument saying that "by doubling all inputs we are always able to double the output since we are essentially just replicating a viable production activity". Before discussing this argument, lets us define the two alternative "pure" cases.

The production function $F(K, L)$ is said to have *increasing returns to scale* (IRS for short) if, for all $(K, L) \in \mathbb{R}^2_{++}$ and all $\lambda > 1$,

$$F(\lambda K, \lambda L) > \lambda F(K, L).$$

That is, IRS is present if, when increasing the *scale* of operations by scaling up every input by some factor > 1, output is scaled up by *more* than this factor. One argument for the plausibility of this is the presence of equipment indivisibilities leading to high unit costs at low output levels. Another argument is that gains by specialization and division of labor, synergy effects, etc. may be present, at least up to a certain level of production. The IRS assumption is also called the *economies of scale* assumption.

Another possibility is *decreasing returns to scale* (DRS). This is said to occur when for all $(K, L) \in \mathbb{R}^2_{++}$ and all $\lambda > 1$,

$$F(\lambda K, \lambda L) < \lambda F(K, L).$$

That is, DRS is present if, when all inputs are scaled up by some factor, output is scaled up by *less* than this factor. This assumption is also called the *diseconomies of scale* assumption. The underlying hypothesis may be that control and coordination problems confine the expansion of size. Or, considering the "replication argument" below, DRS may simply reflect that behind the scene there is an additional production factor, for example land or a irreplaceable quality of management, which is tacitly held fixed, when the factors of production are varied.

EXAMPLE 1 The production function

$$Y = AK^{\alpha}L^{\beta}, \qquad A > 0, 0 < \alpha < 1, 0 < \beta < 1, \tag{2.8}$$

where A, α, and β are given parameters, is called a *Cobb-Douglas production function*. The parameter A depends on the choice of measurement units; for a given such choice it reflects efficiency, also called the "total factor productivity". Exercise 2.2 asks the reader to verify that (2.8) satisfies (a) and (b) above and is therefore a neoclassical production function. The function is homogeneous of degree $\alpha + \beta$. If $\alpha + \beta = 1$, there are CRS. If $\alpha + \beta < 1$, there are DRS, and if

terminology where, if in a given context an assumption of constant returns to scale is needed, this is stated as an additional assumption and we talk about a *CRS-neoclassical production function*.

$\alpha + \beta > 1$, there are IRS. Note that α and β must be less than 1 in order not to violate the diminishing marginal productivity condition. □

EXAMPLE 2 The production function

$$Y = A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}}, \tag{2.9}$$

where A, α, and β are parameters satisfying $A > 0$, $0 < \alpha < 1$, and $\beta < 1$, $\beta \neq 0$, is called a *CES production function* (CES for Constant Elasticity of Substitution). For a given choice of measurement units, the parameter A reflects efficiency (or "total factor productivity") and is thus called the *efficiency parameter.* The parameters α and β are called the *distribution parameter* and the *substitution parameter,* respectively. The latter name comes from the property that the higher is β, the more sensitive is the cost-minimizing capital-labor ratio to a rise in the relative factor price. Equation (2.9) gives the CES function for the case of constant returns to scale; the cases of increasing or decreasing returns to scale are presented in Chapter 4.5. A limiting case of the CES function (2.9) gives the Cobb-Douglas function with CRS. Indeed, for fixed K and L,

$$\lim_{\beta \to 0} A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} = AK^{\alpha}L^{1-\alpha}.$$

This and other properties of the CES function are shown in Chapter 4.5. The CES function has been used intensively in empirical studies. □

EXAMPLE 3 The production function

$$Y = \min(AK, BL), \qquad A > 0, B > 0, \tag{2.10}$$

where A and B are given parameters, is called a *Leontief production function*[5] (or a *fixed-coefficients production function; A* and *B* are called the *technical coefficients.* The function is not neoclassical, since the conditions (a) and (b) are not satisfied. Indeed, with this production function the production factors are not substitutable at all. This case is also known as the case of *perfect complementarity* between the production factors. The interpretation is that already installed production equipment requires a fixed number of workers to operate it. The inverse of the parameters A and B indicate the required capital input per unit of output and the required labor input per unit of output, respectively. Extended to many inputs, this type of production function is often used in multi-sector input-output models (also called Leontief models). In aggregate analysis neoclassical production functions, allowing substitution between capital and labor, are more popular

[5] After the Russian-American economist and Nobel laureate Wassily Leontief (1906-99) who used a generalized version of this type of production function in what is known as *input-output analysis.*

than Leontief functions. But sometimes the latter are preferred, in particular in short-run analysis with focus on the use of already installed equipment where the substitution possibilities tend to be limited.[6] As (2.10) reads, the function has CRS. A generalized form of the Leontief function is $Y = \min(AK^\gamma, BL^\gamma)$, where $\gamma > 0$. When $\gamma < 1$, there are DRS, and when $\gamma > 1$, there are IRS. □

The replication argument The assumption of CRS is widely used in macroeconomics. The model builder may appeal to the *replication argument.* This is the argument saying that by doubling all the inputs, we should always be able to double the output, since we are just "replicating" what we are already doing. Suppose we want to double the production of cars. We may then build another factory identical to the one we already have, man it with identical workers and deploy the same material inputs. Then it is reasonable to assume output is doubled.

In this context it is important that the CRS assumption is about *technology,* functions linking outputs to inputs. Limits to the *availability* of input resources is an entirely different matter. The fact that for example managerial talent may be in limited supply does not preclude the thought experiment that *if* a firm could double all its inputs, including the number of talented managers, then the output level could also be doubled.

The replication argument presupposes, first, that *all* the relevant inputs are explicit as arguments in the production function. Second, that these are changed equiproportionately. This exhibits a problem in defending CRS of our present production function, F, by an appeal to the replication argument. Besides capital and labor, also land is a necessary input and should in principle appear as a separate argument.[7] If an industrial firm decides to duplicate what it has been doing, it needs a piece of land to build another plant like the first. Then, on the basis of the replication argument, we should in fact expect DRS with respect to capital and labor alone. In manufacturing and services, empirically, this and other possible sources for departure from CRS with respect to capital and labor may be minor and so many macroeconomists feel comfortable enough with assuming CRS with respect to K and L alone, at least as a first approximation. This approximation is, however, less applicable to poor countries, where natural resources may be a quantitatively important production factor.

There is a further problem with the replication argument. By definition, CRS is present if and only if, by changing all the inputs equiproportionately by *any* positive factor λ (not necessarily an integer), the firm is able to get output changed

[6] Cf. Section 2.5.2.

[7] Recall from Chapter 1 that we think of "capital" as producible means of production, whereas "land" refers to non-producible natural resources, including for instance building sites.

by the same factor. Hence, the replication argument requires that indivisibilities are negligible, which is certainly not always the case. In fact, the replication argument is more an argument *against* DRS than *for* CRS in particular. The argument does not rule out IRS due to synergy effects as scale is increased.

Sometimes the replication line of reasoning is given a more subtle form. This builds on a useful *local* measure of returns to scale, named the *elasticity of scale*.

The elasticity of scale*[8] To allow for indivisibilities and mixed cases (for example IRS at low levels of production and CRS or DRS at higher levels), we need a local measure of returns to scale. One defines the *elasticity of scale*, $\eta(K, L)$, of a differentiable production function $F(K, L)$ at the point (K, L), where $F(K, L) > 0$, as

$$\eta(K, L) = \frac{\lambda}{F(K, L)} \frac{dF(\lambda K, \lambda L)}{d\lambda} \approx \frac{\Delta F(\lambda K, \lambda L)/F(K, L)}{\Delta\lambda/\lambda}, \text{ evaluated at } \lambda = 1. \tag{2.11}$$

So the elasticity of scale at a point (K, L) indicates the (approximate) percentage increase in output when both inputs are increased by 1 percent. We say that

$$\text{if } \eta(K, L) \begin{cases} > 1, & \text{then there are locally } \textit{IRS}, \\ = 1, & \text{then there are locally } \textit{CRS}, \\ < 1, & \text{then there are locally } \textit{DRS}. \end{cases} \tag{2.12}$$

The production function *may* have the same elasticity of scale everywhere. This is the case if and only if the production function is homogeneous of some degree $h > 0$. In that case $\eta(K, L) = h$ for all (K, L) for which $F(K, L) > 0$, and h indicates the *global elasticity of scale*. The Cobb-Douglas function, cf. Example 1, is homogeneous of degree $\alpha + \beta$ and has thereby global elasticity of scale equal to $\alpha + \beta$.

Note that the elasticity of scale at a point (K, L) will always equal the sum of the partial output elasticities at that point:

$$\eta(K, L) = \frac{F_K(K, L)K}{F(K, L)} + \frac{F_L(K, L)L}{F(K, L)}. \tag{2.13}$$

This follows from the definition in (2.11) by taking into account that

$$\begin{aligned} \frac{dF(\lambda K, \lambda L)}{d\lambda} &= F_K(\lambda K, \lambda L)K + F_L(\lambda K, \lambda L)L \\ &= F_K(K, L)K + F_L(K, L)L, \text{ when evaluated at } \lambda = 1. \end{aligned}$$

[8] A section headline marked by * indicates that in a first reading the section can be skipped – or at least just skimmed through.

Fig. 2.2 illustrates a popular case from introductory economics, an average cost curve which from the perspective of the individual firm is U-shaped: at low levels of output there are falling average costs (thus IRS), at higher levels rising average costs (thus DRS).[9] Given the input prices w_K and w_L and a specified output level $F(K, L) = \bar{Y}$, we know that the cost-minimizing factor combination $(\bar{K}, \bar{L})$ is such that $F_L(\bar{K}, \bar{L})/F_K(\bar{K}, \bar{L}) = w_L/w_K$. It is shown in Appendix A that the elasticity of scale at $(\bar{K}, \bar{L})$ will satisfy:

$$\eta(\bar{K}, \bar{L}) = \frac{LAC(\bar{Y})}{LMC(\bar{Y})}, \tag{2.14}$$

where $LAC(\bar{Y})$ is average costs (the minimum unit cost associated with producing $\bar{Y}$) and $LMC(\bar{Y})$ is marginal costs at the output level $\bar{Y}$. The L in LAC and LMC stands for "long-run", indicating that both capital and labor are considered variable production factors within the period considered. At the optimal plant size, Y^*, there is equality between LAC and LMC, implying a unit elasticity of scale. That is, locally we have CRS. That the long-run average costs are here portrayed as rising for $\bar{Y} > Y^*$, is not essential for the argument but may reflect either that coordination difficulties are inevitable or that some additional production factor, say the building site of the plant, is tacitly held fixed.

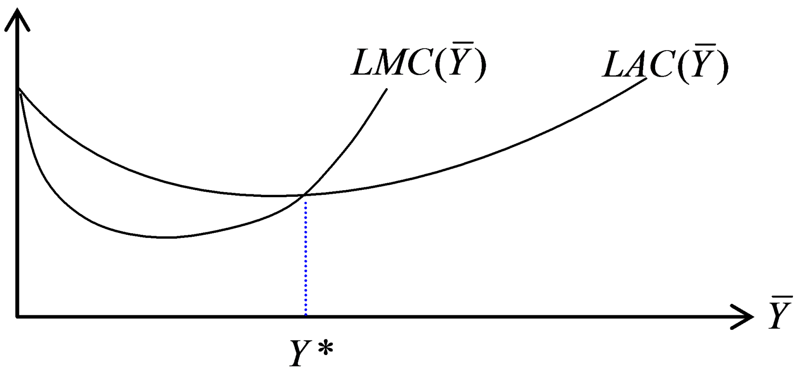


Figure 2.2: Locally CRS at optimal plant size.

Anyway, on this basis Robert Solow (1956) came up with a more subtle replication argument for CRS at the aggregate level. Even though technologies may differ across plants, the surviving plants in a competitive market will have the same average costs at the optimal plant size. In the medium and long run, changes in aggregate output will take place primarily by entry and exit of optimal-size

[9] By a "firm" is generally meant the company as a whole. A company may have several "manufacturing plants" placed at different locations.

plants. Then, with a large number of relatively small plants, each producing at approximately constant unit costs for small output variations, we can without substantial error assume constant returns to scale at the aggregate level. So the argument goes. Notice, however, that even in this form the replication argument is not entirely convincing since the question of indivisibility remains. The optimal, i.e., cost-minimizing, plant size may be large relative to the market − and is in fact so in many industries. Besides, in this case also the perfect competition premise breaks down.

2.1.3 Properties of the production function under CRS

The empirical evidence concerning returns to scale is mixed (see the literature notes at the end of the chapter). Notwithstanding the theoretical and empirical ambiguities, the assumption of CRS with respect to capital and labor has a prominent role in macroeconomics. In many contexts it is regarded as an acceptable approximation and a convenient simple background for studying the question at hand.

Expedient inferences of the CRS assumption include:

(i) marginal costs are constant and equal to average costs (so the right-hand side of (2.14) equals unity);

(ii) if production factors are paid according to their marginal productivities, factor payments exactly exhaust total output so that pure profits are neither positive nor negative (so the right-hand side of (2.13) equals unity);

(iii) a production function known to exhibit CRS and satisfy property (a) from the definition of a neoclassical production function above, will automatically satisfy also property (b) and consequently *be* neoclassical;

(iv) a neoclassical two-factor production function with CRS has, for all $(K, L) \in \mathbb{R}^2_{++}$, $F_{KL} > 0$, i.e., it exhibits *direct complementarity* between K and L.What is ruled out in the CRS case is thus that $F_{KL} < 0$ (in which case K and L are said to be *direct substitutes),* or that $F_{KL} = 0$.

(v) a two-factor production function that has CRS and is twice continuously differentiable with positive marginal productivity of each factor everywhere in such a way that all isoquants are strictly convex to the origin, *must* have *diminishing* marginal productivities everywhere and thereby be neoclassical.[10]

[10]Proofs of claim (iii), (iv), and (v) are in Appendix B.

A principal implication of the CRS assumption is that it allows a reduction of dimensionality. Considering a neoclassical production function, $Y = F(K, L)$ with $L > 0$, we can under CRS write $F(K, L) = LF(K/L, 1) \equiv Lf(k)$, where $k \equiv K/L$ is called the *capital-labor ratio* (sometimes the *capital intensity*) and $f(k)$ is the *production function in intensive form* (sometimes named the per capita production function). Thus output per unit of labor depends only on the capital intensity:

$$y \equiv \frac{Y}{L} = f(k).$$

When the original production function F is neoclassical, under CRS the expression for the marginal productivity of capital simplifies:

$$F_K(K, L) = \frac{\partial Y}{\partial K} = \frac{\partial [Lf(k)]}{\partial K} = Lf'(k)\frac{\partial k}{\partial K} = f'(k). \tag{2.15}$$

And the marginal productivity of labor can be written

$$\begin{aligned} F_L(K, L) &= \frac{\partial Y}{\partial L} = \frac{\partial [Lf(k)]}{\partial L} = f(k) + Lf'(k)\frac{\partial k}{\partial L} \\ &= f(k) + Lf'(k)K(-L^{-2}) = f(k) - kf'(k). \end{aligned} \tag{2.16}$$

A neoclassical CRS production function in intensive form always has a positive first derivative and a negative second derivative, i.e., $f' > 0$ and $f'' < 0$. The property $f' > 0$ follows from (2.15) and (2.2). And the property $f'' < 0$ follows from (2.3) combined with

$$F_{KK}(K, L) = \frac{\partial f'(k)}{\partial K} = f''(k)\frac{\partial k}{\partial K} = f''(k)\frac{1}{L}.$$

For a neoclassical production function with CRS, we also have

$$f(k) - f'(k)k > 0 \text{ for all } k > 0, \tag{2.17}$$

in view of $f(0) \geq 0$ and $f'' < 0$. Moreover,

$$\lim_{k \to 0^+} [f(k) - f'(k)k] = f(0). \tag{2.18}$$

Indeed, from the *mean value theorem*[11] we know that for any $k > 0$ there exists a number $a \in (0, 1)$ such that $f'(ak) = (f(k) - f(0))/k$. For this a we thus have $f(k) - f'(ak)k = f(0) < f(k) - kf'(k)$, where the inequality follows from $f'(ak) > f'(k)$, by $f'' < 0$. In view of $f(0) \geq 0$, this establishes (2.17). And from $f(k)$

[11] This theorem says that if f is continuous in $[\alpha, \beta]$ and differentiable in (α, β), then there exists at least one point γ in (α, β) such that $f'(\gamma) = (f(\beta) - f(\alpha))/(\beta - \alpha)$.

$> f(k) - kf'(k) > f(0)$ and continuity of f (so that $\lim_{k\to 0^+} f(k) = f(0)$) follows (2.18).

Under CRS the Inada conditions for MPK can be written

$$\lim_{k\to 0^+} f'(k) = \infty, \qquad \lim_{k\to\infty} f'(k) = 0. \tag{2.19}$$

In this case standard parlance is just to say that "f satisfies the Inada conditions".

An input which must be positive for positive output to arise is called an *essential input*; an input which is not essential is called an *inessential input.* The second part of (2.19), representing the upper Inada condition for MPK under CRS, has the implication that *labor* is an essential input; but capital need not be, as the production function $f(k) = a + bk/(1 + k)$, $a > 0, b > 0$, illustrates. Similarly, under CRS the upper Inada condition for MPL implies that *capital* is an essential input. These claims are proved in Appendix C. Combining these results, when *both* the upper Inada conditions hold and CRS obtain, then both capital and labor are essential inputs.[12]

Fig. 2.3 is drawn to provide an intuitive understanding of a neoclassical CRS production function and at the same time illustrate that the lower Inada conditions are more questionable than the upper Inada conditions. The left panel of Fig. 2.3 shows output per unit of labor for a *CRS neoclassical production function* satisfying the Inada conditions for MPK. The $f(k)$ in the diagram could for instance represent the Cobb-Douglas function in Example 1 with $\beta = 1 - \alpha$, i.e., $f(k) = Ak^\alpha$. The right panel of Fig. 2.3 shows a non-neoclassical case where only two alternative *Leontief techniques* are available, technique 1: $y = \min(A_1k, B_1)$, and technique 2: $y = \min(A_2k, B_2)$. In the exposed case it is assumed that $B_2 > B_1$ and $A_2 < A_1$ (if $A_2 \geq A_1$ at the same time as $B_2 > B_1$, technique 1 would not be efficient, because the same output could be obtained with less input of at least one of the factors by shifting to technique 2). If the available K and L are such that $k \equiv K/L < B_1/A_1$ or $k > B_2/A_2$, some of either L or K, respectively, is idle. If, however, the available K and L are such that $B_1/A_1 < k < B_2/A_2$, it is efficient to *combine* the two techniques and use the fraction μ of K and L in technique 1 and the remainder in technique 2, where $\mu = (B_2/A_2 - k)/(B_2/A_2 - B_1/A_1)$. In this way we get the "labor productivity curve" OPQR (the envelope of the two techniques) in Fig. 2.3. Note that for $k \to 0$, MPK stays equal to $A_1 < \infty$, whereas for all $k > B_2/A_2$, $MPK = 0$.

A similar feature remains true, when we consider *many,* say n, alternative efficient Leontief techniques available. Assuming these techniques cover a considerable range with respect to the B/A ratios, we get a labor productivity curve

[12] Given a Cobb-Douglas production function, both production factors are essential whether we have DRS, CRS, or IRS.

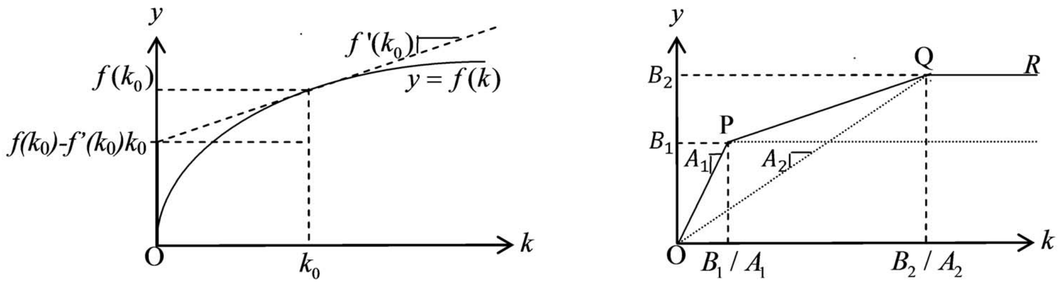


Figure 2.3: Two labor productivity curves based on CRS technologies. Left: neoclassical technology with Inada conditions for MPK satisfied; the graphical representation of MPK and MPL at $k = k_0$ as $f'(k_0)$ and $f(k_0) - f'(k_0)k_0$ are indicated. Right: the line segment PQ makes up an efficient combination of two efficient Leontief techniques.

looking more like that of a neoclassical CRS production function. On the one hand, this gives some intuition of what lies behind the assumption of a neoclassical CRS production function. On the other hand, it remains true that for all $k > B_n/A_n$, $MPK = 0$,[13] whereas for $k \to 0$, MPK stays equal to $A_1 < \infty$, thus questioning the lower Inada condition.

The implausibility of the lower Inada conditions is also underlined if we look at their implication in combination with the more reasonable upper Inada conditions. Indeed, the four Inada conditions taken *together* imply, under CRS, that output has no upper bound when either input goes towards infinity for fixed amount of the other input (see Appendix C).

2.2 Technological change

When considering the movement over time of the economy, we shall often take into account the existence of *technological change*. When technological change occurs, the production function becomes time-dependent. Over time the production factors tend to become more productive: more output for given inputs. To put it differently: the isoquants move inward. When this is the case, we say that the technological change displays *technological progress*.

Concepts of neutral technological change

A first step in taking technological change into account is to replace (2.1) by (2.4). Empirical studies often specialize (2.4) by assuming that technological

[13] Here we assume the techniques are numbered according to ranking with respect to the size of B.

change take a form known as *factor-augmenting* technological change:

$$Y_t = F(B_t K_t, A_t L_t), \tag{2.20}$$

where F is a (time-independent) neoclassical production function, Y_t, K_t, and L_t are output, capital, and labor input, respectively, at time t, while B_t and A_t are time-dependent "efficiencies" of capital and labor, respectively, reflecting technological change.

In macroeconomics an even more specific form is often assumed, namely the form of *Harrod-neutral technological change.*[14] This amounts to assuming that B_t in (2.20) is a constant (which we can then normalize to one). So only A_t, which is then conveniently denoted T_t, is changing over time, and we have

$$Y_t = F(K_t, T_t L_t). \tag{2.21}$$

The efficiency of labor, T_t, is then said to indicate the *technology level.* Although one can imagine natural disasters implying a fall in T_t, generally T_t tends to rise over time and then we say that (2.21) represents *Harrod-neutral technological progress.* An alternative name often used for this is *labor-augmenting* technological progress. The names "factor-augmenting" and, as here, "labor-augmenting" have become standard and we shall use them when convenient, although they may easily be misunderstood. To say that a change in T_t is labor-augmenting might be understood as meaning that more labor is required to reach a given output level for given capital. In fact, the opposite is the case, namely that T_t has risen so that less labor input is required. The idea is that the technological change affects the output level *as if* the labor input had been increased exactly by the factor by which T was increased, and nothing else had happened. (We might be tempted to say that (2.21) reflects "labor saving" technological change. But also this can be misunderstood. Indeed, keeping L unchanged in response to a rise in T implies that the same output level requires *less capital* and thus the technological change is "capital saving".)

If the function F in (2.21) is homogeneous of degree one (so that the technology exhibits CRS with respect to capital and labor), we may write

$$\tilde{y}_t \equiv \frac{Y_t}{T_t L_t} = F(\frac{K_t}{T_t L_t}, 1) = F(\tilde{k}_t, 1) \equiv f(\tilde{k}_t), \qquad f' > 0, f'' < 0.$$

where $\tilde{k}_t \equiv K_t/(T_t L_t) \equiv k_t/T_t$ (habitually called the "effective" capital-labor ratio or capital intensity). In rough accordance with a general trend in aggregate productivity data for industrialized countries we often assume that T grows at a constant rate, g, so that in discrete time $T_t = T_0(1+g)^t$ and in continuous

[14] After the English economist Roy F. Harrod, 1900-1978.

time $T_t = T_0e^{gt}$, where $g > 0$. The popularity in macroeconomics of the hypothesis of labor-augmenting technological progress derives from its consistency with Kaldor's "stylized facts", cf. Chapter 4.

There exists two alternative concepts of neutral technological progress. *Hicks-neutral* technological progress is said to occur if technological development is such that the production function can be written in the form

$$Y_t = T_tF(K_t, L_t), \tag{2.22}$$

where, again, F is a (time-independent) neoclassical production function, while T_t is the growing technology level.[15] The assumption of Hicks-neutrality has been used more in microeconomics and partial equilibrium analysis than in macroeconomics. If F has CRS, we can write (2.22) as $Y_t = F(T_tK_t, T_tL_t)$. Comparing with (2.20), we see that in this case Hicks-neutrality is equivalent to $B_t = A_t$ in (2.20), whereby technological change is said to be *equally factor-augmenting.*

Finally, in a symmetric analogy with (2.21), what is known as *capital-augmenting* technological progress is present when

$$Y_t = F(T_tK_t, L_t). \tag{2.23}$$

Here technological change acts as if the capital input were augmented. For some obscure reason this form became known as *Solow-neutral* technological progress.[16] This association of (2.23) to Solow's name may easily confuse people, however. In his famous growth model,[17] well-known from introductory macroeconomics, Solow assumed Harrod-neutral technological progress. And in another famous contribution, Solow generalized the concept of Harrod-neutrality to the case of *embodied* technological change and capital of *different vintages,* see below.

It is easily shown (Exercise 2.5) that if F in (2.20) is a Cobb-Douglas production function, then F satisfies all three neutrality criteria at the same time, if it satisfies one of them (which requires that technological change does not affect α and β). It can also be shown that within the class of neoclassical CRS production functions the Cobb-Douglas function is the only one with this property (see Exercise 4.??).

Note that the neutrality concepts do not say anything about the *source* of technological progress, only about the quantitative way in which it materializes. For instance, the occurrence of Harrod-neutrality should not be interpreted as if something miraculous has happened to the labor input. It only means that technological innovations predominantly are such that not only do labor and

[15] After the English economist and Nobel Prize laureate John R. Hicks, 1904-1989.

[16] After the American economist and Nobel Prize laureate Robert Solow (1924-).

[17] Solow (1956).

capital in combination become more productive, but this happens to *manifest itself* in the form (2.21), that is, *as if* an improvement in the quality of the labor input had occurred. (Even when improvement in the quality of the labor input is on the agenda, the result may be a reorganization of the production process ending up in a higher B_t along with, or instead of, a higher A_t in the expression (2.20).)

Rival versus nonrival goods

When a production function (or more generally a production possibility set) is specified, a given level of technical knowledge is presumed. As this level changes over time, the production function changes. In (2.4) this dependency on the level of knowledge was represented indirectly by the time dependency of the production function. Sometimes it is useful to let the knowledge dependency be explicit by perceiving knowledge as an additional production factor and write, for instance,

$$Y_t = F(X_t, T_t), \tag{2.24}$$

where T_t is now an index of the amount of knowledge, while X_t is a vector of ordinary inputs like raw materials, machines, labor etc. In this context the distinction between rival and nonrival inputs or more generally the distinction between rival and nonrival goods is important. A good is *rival* if its character is such that one agent's use of it inhibits other agents' use of it at the same time. A pencil is thus rival. Many production inputs like raw materials, machines, labor etc. have this property. They are elements of the vector X_t. By contrast, however, technical knowledge is a *nonrival* good. An arbitrary number of factories can simultaneously use the same piece of technical knowledge in the sense of a *list of instructions about how different inputs can be combined to produce a certain output.* An engineering principle or a pharmaceutical formula are examples. (Note that the distinction rival versus nonrival is different from the distinction excludable versus nonexcludable. A good is *excludable* if other agents, firms or households, can be excluded from using it. Other firms can thus be excluded from commercial use of a certain piece of technical knowledge if it is patented. The existence of a patent has to do with the legal status of a piece of knowledge and does not interfere with its technical character as a nonrival input. Finally, a good that is both non-rival and non-excludable is called a *pure public good.*)

What the replication argument really says is that by, conceptually, doubling all the *rival* inputs, we should always be able to double the output, since we just "replicate" what we are already doing. This is then an argument for (at least) CRS with respect to the elements of X_t in (2.24). The point is that because of its nonrivalry, we do not need to increase the stock of knowledge. Now let us imagine that the stock of knowledge *is* doubled at the same time as the rival inputs are doubled. Then *more* than a doubling of output should occur. In this sense we may speak of IRS with respect to the rival inputs and T taken together.

From the perspective of the theory of economic growth, the important distinction between a rival and a non-rival input can be exemplified this way. Adding a new tractor to the economy benefits one farmer. But adding a new idea − a new piece of technical knowledge − benefits everyone that wants to use it. In brief: the economic value of an idea is proportional to the number of users.

The perpetual inventory method

Before proceeding, a brief remark about how the capital stock K_t can be in principle measured While data on gross investment, I_t, is typically available in official national income and product accounts, data on K_t usually is not. It has been up to researchers and research institutions to make their own time-series for capital. One approach to the measurement of K_t is the *perpetual inventory method* which builds upon the accounting relationship

$$K_t = I_{t-1} + (1-\delta)K_{t-1}. \tag{2.25}$$

Assuming a constant capital depreciation rate δ, backward substitution gives

$$K_t = I_{t-1} + (1-\delta)\left[I_{t-2} + (1-\delta)K_{t-2}\right] = \ldots = \sum_{i=1}^{N}(1-\delta)^{i-1}I_{t-i} + (1-\delta)^T K_{t-N}. \tag{2.26}$$

Based on a long time series for I and an estimate of δ, one can insert these observed values in the formula and calculate K_t, starting from a rough conjecture about the initial value K_{t-N}. The result will not be very sensitive to this conjecture since for large N the last term in (2.26) becomes very small.

Embodied versus disembodied technological progress*

An additional taxonomy of technological change is the following. We say that technological change is *embodied*, if taking advantage of new technical knowledge requires construction of new investment goods. The new technology is incorporated in the design of newly produced equipment, but this equipment will not participate in subsequent technological progress. An example: only the most recent vintage of a computer series incorporates the most recent advance in information technology. Then investment goods produced later (investment goods of a later "vintage") have higher productivity than investment goods produced earlier at the same resource cost. Thus investment becomes an important driving force in productivity increases.

We may formalize embodied technological progress by writing capital accumulation in the following way:

$$K_{t+1} - K_t = Q_t I_t - \delta K_t, \tag{2.27}$$

where I_t is gross investment in period t, i.e., $I_t = Y_t - C_t$, and Q_t measures the "quality" (productivity) of newly produced investment goods. The rising level of technology implies rising Q so that a given level of investment gives rise to a greater and greater addition to the capital stock, K, measured in *efficiency units*. In aggregate models C and I are produced with the same technology, the aggregate production function. From this together with (2.27) follows that Q capital goods can be produced at the same minimum cost as one consumption good. Hence, the equilibrium price, p, of capital goods in terms of the consumption good must equal the inverse of Q, i.e., $p = 1/Q$. The output-capital ratio in value terms is $Y/(pK) = QY/K$.

Note that even if technological change does not directly appear in the production function, that is, even if for instance (2.21) is replaced by $Y_t = F(K_t, L_t)$, the economy may experience a rising standard of living when Q is growing over time.

In contrast, *disembodied technological change* occurs when new technical and organizational knowledge increases the combined productivity of the production factors independently of when they were constructed or educated. If the K_t appearing in (2.21), (2.22), and (2.23) above refers to the total, historically accumulated capital stock as calculated by (2.26), then the evolution of T in these expressions can be seen as representing disembodied technological change. All vintages of the capital equipment benefit from a rise in the technology level T_t. No new investment is needed to benefit.

Based on data for the U.S. 1950-1990, and taking quality improvements into account, Greenwood et al. (1997) estimate that embodied technological progress explains about 60% of the growth in output per man hour. So, empirically, *embodied* technological progress seems to play a dominant role. As this tends not to be fully incorporated in national income accounting at fixed prices, there is a need to adjust the investment levels in (2.26) to better take estimated quality improvements into account. Otherwise the resulting K will not indicate the capital stock measured in efficiency units.

For most issues dealt with in this book the distinction between embodied and disembodied technological progress is not of high importance. Hence, unless explicitly specified otherwise, technological change is understood to be disembodied.

2.3 The concepts of representative firm and aggregate production function

Many macroeconomic models make use of the simplifying, and not unproblematic, notion of a *representative firm.* By this is meant a fictional firm whose production "represents" the aggregate production (value added) in a sector or in society as a whole.

Suppose there are n firms in the sector considered or in society as a whole. Let F^i be the production function for firm i so that $Y_i = F^i(K_i, L_i)$, where Y_i, K_i, and L_i are output, capital input, and labor input, respectively, $i = 1, 2, \ldots, n$. Define $Y \equiv \Sigma_{i=1}^n Y_i$, $K \equiv \Sigma_{i=1}^n K_i$, and $L \equiv \Sigma_{i=1}^n L_i$. Let the firms maximize profits, taking input and output prices as given. Suppose the aggregate variables are then related through some function, F^*, such that we can write

$$Y = F^*(K, L),$$

and such that the input choices of a single fictional firm facing *this* production function coincide with the aggregate outcomes, $\Sigma_{i=1}^n Y_i$, $\Sigma_{i=1}^n K_i$, and $\Sigma_{i=1}^n L_i$, in the original economy. If this is possible, we call $F^*(K, L)$ the *aggregate production function* or the production function of the *representative* firm. It is *as if* aggregate production is the result of the behavior of this fictional single firm.

A simple example where an aggregate production function is well-defined is the following. Suppose all the firms have the *same* production function so that $Y_i = F(K_i, L_i)$, $i = 1, 2, \ldots, n$. If in addition F has CRS, we have

$$Y_i = F(K_i, L_i) = L_i F(k_i, 1) \equiv L_i f(k_i),$$

where $k_i \equiv K_i/L_i$. Hence, facing given the factor prices, profit-maximizing firms will choose the same capital intensity $k_i = k$ for all i (but not necessarily the same level of production since under CRS, this is indeterminate). From $K_i = kL_i$ then follows $\sum_i K_i = k \sum_i L_i$ so that $k = K/L$. Thence,

$$Y \equiv \sum Y_i = \sum L_i f(k_i) = f(k) \sum L_i = f(k)L = F(k, 1)L = F(K, L).$$

In this case an aggregate production function immediately appears and turns out to be exactly the same as the identical CRS production functions of the individual firms. Moreover, given F is neoclassical, the common capital-labor ratio $k_i = k$, for all i, implies that $\partial Y_i/\partial K_i = f'(k_i) = f'(k) = F_K(K, L) = \partial Y/\partial K$ for all i. So each firm's marginal productivity of capital is the same as the marginal productivity of capital calculated on the basis of the aggregate production function.

A less trivial case is the following. Let the firms have *different* concave neoclassical production functions at firm level. Define the function F by

$$\begin{aligned} F(K,L) &= \max_{(K_1,L_1,\ldots,K_n,L_n)\geq 0} F^1(K_1,L_1)+\cdots+F^n(K_n,L_n) \quad \text{s.t.} \\ \sum_i K_i &\leq K, \text{ and } \sum_i L_i \leq L. \end{aligned}$$

Then $F(K, L)$ is a "well-behaved" aggregate production function. Indeed, the n individual firms will choose inputs such that both $\partial Y_i/\partial K_i$ $(= F_K^i(K_i, L_i)$ and $\partial Y_i/\partial L_i$ $(= F_L^i(K_i, L_i)$ are the same across firms, namely equal to the cost per unit of capital and the cost per unit of labor, respectively. By the envelope theorem (see Math Tools) it can then be shown that F will be such that $\partial Y/\partial K$ $= F_K(K, L)$ and $\partial Y/\partial L$ will equal $\partial Y_i/\partial K_i$ and $\partial Y_i/\partial L_i$, respectively.

A next step is to allow also for the existence of different output goods (either within or across the single firms), different capital goods, and different types of labor. This makes the issue much more intricate, of course. Yet, if firms are price taking profit maximizers and face nonincreasing returns to scale, we at least know from microeconomics that the aggregate outcome is *as if,* for the given prices, the aggregate profit is maximized on the basis of the firms' combined production technology.[18]The problem is, however, that the conditions needed for this to imply existence of an aggregate production function which is "well-behaved" (in the sense of inheriting at least simple qualitative properties from its constituent parts) are very restrictive.

Nevertheless macroeconomics often treats aggregate output as a single homogeneous good and capital and labor as being two single and homogeneous inputs. There was in the 1960s a heated debate about the problems involved in this, with particular emphasis on the aggregation of different kinds of equipment into one variable, the capital stock "K". The debate is known as the "Cambridge controversy" because the dispute was between a group of economists from Cambridge University, UK, and a group from Massachusetts Institute of Technology (MIT), which is located in Cambridge, USA. The former group questioned the theoretical robustness of several of the neoclassical tenets, including the proposition that a lower rate of interest always induces a higher aggregate capital-labor ratio. Starting at the disaggregate level, an association of this sort is not a logical necessity because, with different production functions across the industries, the relative prices of produced inputs tend to change, when the interest rate changes. While acknowledging the possibility of "paradoxical" relationships, the MIT group maintained that in a macroeconomic context they are likely to cause

[18] See Mas-Colell (1995).

devastating problems only under exceptional circumstances. In the end this is a matter of empirical assessment.[19]

To avoid complexity and because, for many important issues in macroeconomics, there is today no well-tried alternative, this book is about models that use aggregate constructs like "Y", "K", and "L" as simplifying devices, assuming they are, for a broad class of cases, tolerable in a first rough approximation. Of course there are cases where this "as if" approach is clearly inappropriate and some disaggregation is pertinent. When for example the role of imperfect competition is in focus, we shall be ready to (modestly) disaggregate the production side of the economy into several product lines, each producing its own differentiated product. A brief example is given in Section 2.5.3.

Like the representative firm, the *representative household* and the *aggregate consumption function* are simplifying notions that should be applied only when they do not get in the way of the issue to be studied. The role of budget constraints may make it even more difficult to aggregate over households than over firms. Yet, *if* (and that is a big if) all households have the *same constant* propensity to consume out of income or wealth, aggregation is straightforward and the notion of a representative household may be a useful simplifying concept. On the other hand, if we aim at understanding, say, the *interaction* between lending and borrowing households, perhaps via financial intermediaries, the existence of *different* categories of households should be taken into account. Similarly, if the theme is conflicts of interests between firm owners and employees. And if we want to assess the welfare costs of business cycle fluctuations, we should take into account that exposure to unemployment risk tends to be very unevenly distributed in the population.

2.4 The neoclassical competitive one-sector setup

Many *long-run* macromodels, including those in the first chapters to follow, share the same abstract setup regarding the firms and the market environment in which they are placed. We give an account here which will serve as a reference point for these later chapters.

The setup is characterized by the following simplifications:

(a) There is only one produced good, an all-purpose good that can be used for consumption as well as investment. Aggregate physical capital is just the accumulated amount of what is left of the produced good after aggregate

[19] In his review of the Cambridge controversy Mas-Colell (1989) concluded that: "What the 'paradoxical' comparative statics [of disaggregate capital theory] has taught us is simply that modelling the world as having a single capital good is not *a priori* justified. So be it."

consumption. Models using this simplification are called one-sector models. One may think of "corn", a good that can be used for consumption as well as investment in the form of seed to yield corn next period.

(b) All firms are alike and maximize profit subject to the same neoclassical production function under non-increasing returns to scale.

(c) Capital goods become productive immediately upon purchase or renting (so installation costs and similar features are ignored).

(d) In all markets *perfect competition* rules. By definition this means that the economic actors are *price takers*, perceiving no constraint on how much they can sell or buy at the going market price. It is understood that market prices are flexible and adjust quickly to levels required for market clearing.

(e) Factor supplies are inelastic.

(f) There is no uncertainty. When a choice of action is made, the consequences are known.

We call this setup the *neoclassical competitive one-sector setup*. It is certainly an abstraction from the diversity and multitude of frictions in the real world. Nevertheless, the outcome under the described conditions is of theoretical interest. Think of Galilei's discovery that a falling body falls with a uniform acceleration as long as it is falling through a *perfect vacuum*.

2.4.1 Profit maximization

We consider a single firm in a single period. The firm has the neoclassical production function

$$Y = F(K, L), \tag{2.28}$$

where technological change is ignored. Although in this book often CRS will be assumed, we may throw the CRS outcome in relief by starting with a broader view.

From microeconomics we know that equilibrium with perfect competition is compatible with producers operating under the condition of locally *nonincreasing returns* to scale (cf. Fig. 2.2). In standard macroeconomics it is common to accept a lower level of generality and simply assume that F is a *concave* function.

This allows us to carry out the analysis *as if* there were non-increasing returns to scale *everywhere* (see Appendix D).[20]

Since F is neoclassical, we have $F_{KK} < 0$ and $F_{LL} < 0$ everywhere. To obtain concavity it is then necessary and sufficient to add the assumption that

$$D \equiv F_{KK}(K, L)F_{LL}(K, L) - F_{KL}(K, L)^2 \geq 0, \tag{2.29}$$

holds for all (K, L). This is a simple application of a general theorem on concave functions (see Math Tools).

Let us consider both K and L as variable production factors. Let the factor prices be denoted w_K and w_L, respectively. For the time being we assume the firm rents the machines it uses; then the price, w_K, of capital services is called the *rental price* or the *rental rate*. As *numeraire* (unit of account) we apply the output good. So all prices are measured in terms of the output good which itself has the price 1. Then *profit*, defined as revenue minus costs, is

$$\Pi = F(K, L) - w_K K - w_L L. \tag{2.30}$$

We assume both production inputs are *variable* inputs. Taking the factor prices as given from the factor markets, the firm's problem is to choose (K, L), where $K \geq 0$ and $L \geq 0$, so as to maximize Π. An interior solution will satisfy the first-order conditions

$$\frac{\partial \Pi}{\partial K} = F_K(K, L) - w_K = 0 \quad \text{or} \quad F_K(K, L) = w_K, \tag{2.31}$$

$$\frac{\partial \Pi}{\partial L} = F_L(K, L) - w_L = 0 \quad \text{or} \quad F_L(K, L) = w_L. \tag{2.32}$$

Since F is concave, so is the profit function. The first-order conditions are then *sufficient* for (K, L) to be a solution.

It is now convenient to proceed by considering the two cases, DRS and CRS, separately.

The DRS case

Suppose the production function satisfies (2.29) with strict inequality everywhere, i.e.,

$$D > 0.$$

[20] By definition, *concavity* means that by applying a weighted average of two factor combinations, (K_1, L_1) and (K_2, L_2), the obtained output is at least as large as the weighted average of the original outputs, Y_1 and Y_2. So, if $0 < \lambda < 1$ and $(K, L) = \lambda(K_1, L_1) + (1 - \lambda)(K_2, L_2)$, then $F(K, L) \geq \lambda F(K_1, L_1) + (1 - \lambda)F(K_2, L_2)$.

In combination with the neoclassical property of diminishing marginal productivities, this implies that F is *strictly concave* which in turn implies DRS everywhere. The factor demands will now be unique. Indeed, the equations (2.31) and (2.32) define the factor demands K^d and L^d ("d" for demand) as implicit functions of the factor prices:

$$K^d = K(w_K, w_L), \qquad L^d = L(w_K, w_L).$$

An easy way to find the partial derivatives of these functions is to first take the differential[21] of both sides of (2.31) and (2.32), respectively:

$$\begin{aligned} F_{KK}dK^d + F_{KL}dL^d &= dw_K, \\ F_{LK}dK^d + F_{LL}dL^d &= dw_L. \end{aligned}$$

Then we interpret these conditions as a system of two linear equations with two unknowns, the variables dK^d and dL^d. The determinant of the coefficient matrix equals D in (2.29) and is in this case positive everywhere. Using Cramer's rule (see Math Tools), we find

$$\begin{aligned} dK^d &= \frac{F_{LL}dw_K - F_{KL}dw_L}{D}, \\ dL^d &= \frac{F_{KK}dw_L - F_{LK}dw_K}{D}, \end{aligned}$$

so that

$$\frac{\partial K^d}{\partial w_K} = \frac{F_{LL}}{D} < 0, \qquad \frac{\partial K^d}{\partial w_L} = -\frac{F_{KL}}{D} < 0 \text{ if } F_{KL} > 0, \tag{2.33}$$

$$\frac{\partial L^d}{\partial w_K} = -\frac{F_{KL}}{D} < 0 \text{ if } F_{KL} > 0, \qquad \frac{\partial L^d}{\partial w_L} = \frac{F_{KK}}{D} < 0, \tag{2.34}$$

in view of $F_{LK} = F_{KL}$.[22]

In contrast to the cases of CRS and IRS (for a two-factor production function), here we cannot be sure that direct complementarity between K and L (i.e., $F_{KL} >$

[21] The *differential* of a differentiable function is a convenient tool for deriving results like (2.33) and (2.34). For a function of one variable, $y = f(x)$, the differential is denoted dy (or df) and is defined as $f'(x)dx$, where dx is some arbitrary real number (interpreted as the change in x). For a differentiable function of two variables, $z = g(x, y)$, the *differential* of the function is denoted dz (or dg) and is defined as $dz = g_x(x, y)dx + g_y(x, y)dy$, where dx and dy are arbitrary real numbers.

[22] Applying the full content of the *implicit function theorem* (see Math tools), one could directly have written down the results (2.33) and (2.34) and would not need the procedure outlined here, based on differentials. On the other hand, the present procedure is probably more intuitive and easier to remember.

0) holds everywhere; this explains the "if" in (2.33) and (2.34). In any event, the rule is that when a factor price increases, the demand for the factor in question decreases and under direct complementarity also the demand for the other factor will decrease. Although there is a substitution effect towards higher demand for the factor whose price has not been increased, this is more than offset by the negative output effect, which is due to the higher marginal costs. This is an implication of perfect competition. In a different market structure output may be determined from the demand side (think of a Keynesian short-run model) and then only the substitution effect will be operative. An increase in one factor price will then *increase* the demand for the other factor.

The CRS case

Under CRS, D in (2.29) takes the value

$$D = 0$$

everywhere, as shown in Appendix B. Then the factor prices no longer determine the factor demands uniquely. But the *relative* factor demand, $k^d \equiv K^d/L^d$, is determined uniquely by the *relative* factor price, w_L/w_K. Indeed, by (2.31) and (2.32),

$$MRS = \frac{F_L(K,L)}{F_K(K,L)} = \frac{f(k) - f'(k)k}{f'(k)} \equiv mrs(k) = \frac{w_L}{w_K}, \tag{2.35}$$

where the second equality comes from (2.15) and (2.16). By straightforward calculation,

$$mrs'(k) = -\frac{f(k)f''(k)}{f'(k)^2} = -\frac{kf''(k)/f'(k)}{\alpha(k)} > 0,$$

where $\alpha(k) \equiv kf'(k)/f(k)$ is the elasticity of f with respect to k and the numerator is the elasticity of f' with respect to k. For instance, in the Cobb-Douglas case $f(k) = Ak^\alpha$, we get $mrs'(k) = (1-\alpha)/\alpha$. Given w_L/w_K, the last equation in (2.35) gives k^d as an implicit function $k^d = k(w_L/w_K)$, where $k'(w_L/w_K) = 1/mrs'(k) > 0$. The solution is illustrated in Fig. 2.4. Under CRS (indeed, for any homogeneous neoclassical production function) the desired capital-labor ratio is an increasing function of the inverse factor price ratio and independent of the output level.

To determine K^d and L^d separately we need to know the level of output. And here we run into the general problem of indeterminacy under perfect competition combined with CRS. Saying that the output level is so as to maximize profit does not take us far. If at the going factor prices attainable profit is negative, exit from the market is profit maximizing (or rather loss minimizing), which amounts to $K^d = L^d = 0$. But if the profit is positive, there will be no upper bound to the

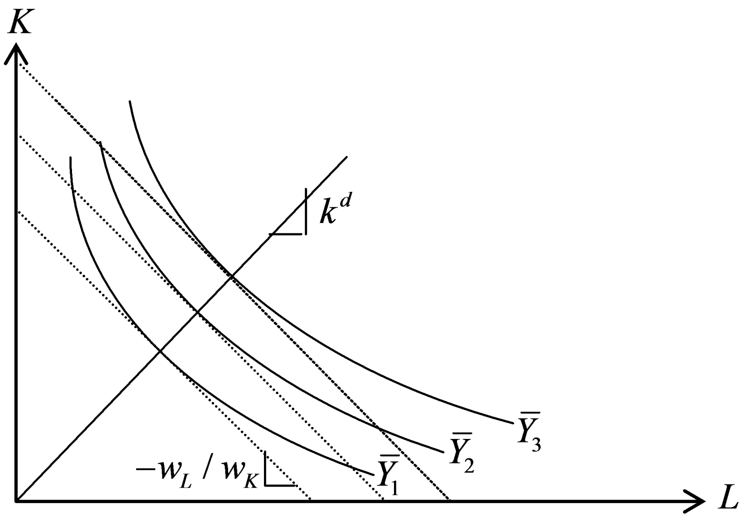


Figure 2.4: Constancy of MRS along rays when the production function is homogeneous of degree h (the cost-minimizing capital intensity is the same at all output levels).

factor demands. Owing to CRS, doubling the factor inputs will double the profits of a price taking firm. An equilibrium with positive production is only possible if profit is zero. And then the firm is indifferent with respect to the level of output. Solving the indeterminacy problem requires a look at the factor markets.

2.4.2 Clearing in factor markets

Considering a closed economy, we denote the available supplies of physical capital and labor K^s and L^s, respectively, and assume these supplies are inelastic. With respect to capital this is a "natural" assumption since in a closed economy in the short run the available amount of capital will be *predetermined,* that is, historically determined by the accumulated previous investment in the economy. With respect to labor supply it is just a simplifying assumption introduced because the question about possible responses of labor supply to changes in factor prices is a secondary issue in the present context. Since we now consider the aggregate level, we interpret K^d and L^d as factor demands by a representative firm.

The factor markets clear when

$$K^d = K^s, \tag{2.36}$$
$$L^d = L^s. \tag{2.37}$$

Achieving this equilibrium (state of "rest") requires that the factor prices adjust to their equilibrium levels, which are

$$w_K = F_K(K^s, L^s), \qquad (2.38)$$
$$w_L = F_L(K^s, L^s), \qquad (2.39)$$

by (2.31) and (2.32). This says that in equilibrium the real factor prices are determined by the *marginal productivities of the respective factors at full utilization of the given factor supplies.* This holds under DRS as well as CRS. So, under non-increasing returns to scale there is, at the macroeconomic level, a unique equilibrium (w_K, w_L, K^d, L^d) given by the above four equilibrium conditions for the factor markets.[23] It is an *equilibrium* in the sense that no agent has an incentive to "deviate".

As to *comparative statics*, since $F_{KK} < 0$, a larger capital supply implies a lower w_K, and since $F_{LL} < 0$, a larger labor supply implies a lower w_L.

The intuitive mechanism behind the *attainment* of equilibrium is that if for a short moment $w_K < F_K(K^s, L^s)$, then $K^d > K^s$ and so competition between the firms will generate an upward pressure on w_K until equality is obtained. And if for a short moment $w_K > F_K(K^s, L^s)$, then $K^d < K^s$ and so competition between the *suppliers* of capital will generate a downward pressure on w_K until equality is obtained.

Looking more carefully at the matter, however, we see that this intuitive reasoning fits at most the DRS case. In the CRS case we have $F_K(K^s, L^s) = f(k^s)$, where $k^s \equiv K^s/L^s$. Here we can only argue that for instance $w_K < F_K(K^s, L^s)$ implies $k^d > k^s$. And even if this leads to upward pressure on w_K until $k^d = k^s$ is achieved, and even if both factor prices have obtained their equilibrium levels given by (2.38) and (2.39), there is nothing to induce the representative firm (or the many firms in the actual economy taken together) to choose the "right" input *levels* so as to satisfy the clearing conditions (2.36) and (2.37). In this way the indeterminacy under CRS pops up again, this time as a problem endangering stability of the equilibrium.

Stability not guaranteed*

To substantiate the point that the indeterminacy under CRS may endanger stability of competitive equilibrium, let us consider a Walrasian *tâtonnement* adjustment process.[24] We imagine that our period is sub-divided into many short time intervals $(t, t+\Delta t)$. We still interpret K^d and L^d as factor demands per time unit by a representative firm. In the initial short time interval the factor markets

[23] At the microeconomic level, under CRS, industry structure remains indeterminate in that firms are indifferent as to their size.

[24] *Tâtonnement* is a French word meaning "groping".

may not be in equilibrium. It is assumed that no capital or labor is hired out of equilibrium. To allow an analysis in continuous time, we let $\Delta t \to 0$. A dot over a variable denotes the time derivative, i.e., $\dot{x}(t) = dx(t)/dt$. The adjustment process is the following:

$$\begin{aligned}
\dot{K}^d(t) &= \lambda_1 \left[F_K(K^d(t), L^d(t)) - w_K(t)\right], \qquad \lambda_1 > 0, \\
\dot{L}^d(t) &= \lambda_2 \left[F_L(K^d(t), L^d(t)) - w_L(t)\right], \qquad \lambda_2 > 0, \\
\dot{w}_K(t) &= K^d(t) - K^s, \\
\dot{w}_L(t) &= L^d(t) - L^s,
\end{aligned}$$

where the initial values, $K^d(0)$, $L^d(0)$, $w_K(0)$, and $w_L(0)$, are given. The parameters λ_1 and λ_2 are constant adjustment speeds. The corresponding adjustment speeds for the factor prices are set equal to one by choice of measurement units of the inputs. Of course, the four endogenous variables should be constrained to be nonnegative, but that is not important for the discussion here. The system has a unique stationary state: $K^d(t) = K^s, L^d(t) = L^s, w_K(t) = K_K(K^s, L^s)$, $w_L(t) = K_L(K^s, L^s)$.

A widespread belief, even in otherwise well-informed circles, seems to be that with such adjustment dynamics, the stationary state is at least *locally asymptotically stable*. By this is meant that there exists a (possibly only small) neighborhood, $\mathcal{N}$, of the stationary state with the property that if the initial state, $(K^d(0)$, $L^d(0)$, $w_K(0)$, $w_L(0))$, belongs to $\mathcal{N}$, then the solution $(K^d(t)$, $L^d(t)$, $w_K(t)$, $w_L(t))$ converges to the stationary state for $t \to \infty$?

Unfortunately, however, this stability property is *not* guaranteed. To bear this out, it is enough to present a counterexample. Let $F(K, L) = K^{\frac{1}{2}} L^{\frac{1}{2}}$, $\lambda_1 = \lambda_2 = K^s = L^s = 1$, and suppose $K^d(0) = L^d(0) > 0$ and $w_K(0) = w_L(0) > 0$. All this symmetry implies that $K^d(t) = L^d(t) = x(t) > 0$ and $w_K(t) = w_L(t) = w(t)$ for all $t \geq 0$. So $F_K(K^d(t), L^d(t)) = 0.5x(t)^{-0.5}x(t)^{0.5} = 0.5$, and similarly $F_L(K^d(t), L^d(t)) = 0.5$ for all $t \geq 0$. Now the system is equivalent to the two-dimensional system,

$$\dot{x}(t) = 0.5 - w(t), \tag{2.40}$$

$$\dot{w}(t) = x(t) - 1. \tag{2.41}$$

Using the theory of coupled linear differential equations, the solution is[25]

$$x(t) = 1 + (x(0) - 1)\cos t - (w(0) - 0.5)\sin t, \tag{2.42}$$

$$w(t) = 0.5 + (w(0) - 0.5)\cos t + (x(0) - 1)\sin t. \tag{2.43}$$

[25] For details, see hints in Exercise 2.6.

The solution exhibits undamped oscillations and never settles down at the stationary state, $(1, 0.5)$, if not being there from the beginning. In fact, the solution curves in the (x, w) plane will be circles around the stationary state. This is so whatever the size of the initial distance, $\sqrt{(x(0)-1)^2 + (w(0)-0.5)^2}$, to the stationary point.

The economic mechanism is as follows. Suppose for instance that $x(0) < 1$ and $w(0) < 0.5$. Then to begin with there is excess supply and so w will be falling while, with w below marginal products, x will be increasing. When x reaches its potential equilibrium value, 1, w is at its trough and so induces further increases in the factor demands, thus bringing about a phase where $x > 1$. This excess demand causes w to begin an upturn. When w reaches its potential equilibrium value, 0.5, however, excess demand, $x - 1$, is at its peak and this induces further increases in factor prices, w. This brings about a phase where $w > 0.5$ so that factor prices exceed marginal products, which leads to declining factor demands. But as x comes back to its potential equilibrium value, w is at its peak and drives x further down. Thus excess supply arises which in turn triggers a downturn of w. This continues in never ending oscillations where the overreaction of one variable carries the seed to an overreaction of the other variable soon after and so on.

This possible outcome underlines that the theoretical *existence* of equilibrium is one thing and *stability* of the equilibrium is another. In particular under CRS, where demand *functions* for inputs are absent, the issue of stability can be more intricate than one might at first glance think.

The link between capital costs and the interest rate*

Returning to the description of equilibrium, we shall comment on the relationship between the factor price w_K and the more everyday concept of an interest rate. The factor price w_K is the cost per unit of capital service. It has different names in the literature such as the *rental price,* the *rental rate,* the *unit capital cost,* or the *user cost.* It is related to the interest and depreciation costs that the owner of the capital good in question defrays. In the simple neoclassical setup considered here, it does not matter whether the firm rents the capital it uses or owns it; in the latter case, w_K, is the *imputed* capital cost, i.e., the forgone interest plus depreciation.

As to depreciation it is common in macroeconomics to apply the approximation that, due to wear and tear, a constant fraction δ (where $0 \leq \delta \leq 1$) of a given capital stock evaporates per period. If for instance the period length is one year and $\delta = 0.1$, this means that a given machine in the next year has only the fraction 0.9 of its productive capacity in the current year. Otherwise the productive characteristics of a capital good are assumed to be the same whatever its time of birth. Sometimes δ is referred to as the rate of *physical* capital depreciation or the *rate of geometric decay*.[26] When changes in relative prices can occur, the rate of decay must be distinguished from the *economic depreciation* of capital which refers to the loss in economic value of a machine after one year.

Let p_{t-1} be the price of a certain type of machine bought at the end of period $t-1$. Let prices be expressed in the same numeraire as that in which the interest rate, r, is measured. And let p_t be the price of the same type of machine one period later. Then the *economic depreciation* in period t is

$$p_{t-1} - (1-\delta)p_t = \delta p_t - (p_t - p_{t-1}).$$

The economic depreciation thus equals the value of the physical wear and tear minus the capital gain (positive or negative) on the machine. Note that if the capital good itself is the numeraire, so that $p_{t-1} = p_t = 1$, then the economic depreciation coincides with the rate of geometric decay, δ.

By holding the machine the owner faces an opportunity cost, namely the forgone interest on the value p_{t-1} placed in the machine during period t. If r_t is the interest rate on a loan from the end of period $t-1$ to the end of period t, this interest cost is $r_t p_{t-1}$. The benefit of holding the (new) machine is that it can be rented out to the representative firm and provide the return w_{Kt} at the end of the period. Since there is no uncertainty, in equilibrium we must then have $w_{Kt} = r_t p_{t-1} + \delta p_t - (p_t - p_{t-1})$, or

$$\frac{w_{Kt} - \delta p_t + p_t - p_{t-1}}{p_{t-1}} = r_t. \tag{2.44}$$

This is a *no-arbitrage* condition saying that the rate of return on holding the machine equals the rate of return obtainable in the loan market (no profitable arbitrage opportunities are available).[27]

In the simple setup considered so far, the capital good and the produced good are physically identical and thus have the same price. As the produced good is our numeraire, we have $p_{t-1} = p_t = 1$. This has two implications. *First*, the interest rate, r_t, is a real interest rate so that $1 + r_t$ measures the rate at which future units of output can be traded for current units of output. *Second,* (2.44) simplifies to

$$w_{Kt} - \delta = r_t.$$

[26] The latter name comes from the fact that if no investment occurs, then $K_t = K_{t-1} - \delta K_{t-1}$ and thus $K_t = (1-\delta)^t K_0$.

[27] In continuous time analysis the rental rate, the interest rate, and the price of the machine are considered as differentiable functions of time, $w_K(t)$, $r(t)$, and $p(t)$, respectively. In analogy with (2.44) we then get $w_K(t) = (r(t) + \delta)p(t) - \dot{p}(t)$, where $\dot{p}(t)$ denotes the time derivative of the price $p(t)$. Here δ appears as the rate of exponential decay, since, in case of no investment, $\dot{K}(t) = \delta K(t)$, hence $K(t) = K(0)e^{-\delta t}$.

Combining this with equation (2.38), we see that in the simple neoclassical setup the equilibrium real interest rate is determined as

$$r_t = F_K(K_t^s, L_t^s) - \delta, \tag{2.45}$$

where K_t^S and L_t^s are predetermined. Under CRS this takes the form $r_t = f'(k_t^s) - \delta$, where $k_t^s \equiv K_t^s/L_t^s$.

We have assumed that the firms rent capital goods from their owners, presumably the households. But as long as there is no uncertainty, no capital adjustment costs, and no taxation, it will have no consequences for the results if instead we assume that the firms own the physical capital they use and finance capital investment by issuing bonds or shares. Then such bonds and shares would constitute financial assets, owned by the households and offering a rate of return r_t as given by (2.45).

2.5 More complex model structures*

The neoclassical setup described above may be useful as a first way of organizing one's thoughts about the production side of the economy. To come closer to a model of how modern economies function, however, many modifications and extensions are needed.

2.5.1 Convex capital installation costs

In the real world the capital goods used by a production firm are usually owned by the firm itself rather than rented for single periods on rental markets. This is because inside the specific plant in which these capital goods are an integrated part, they are generally worth much more than outside. So in practice firms acquire and install fixed capital equipment with a view on maximizing discounted expected profits in the future. The cost associated with this fixed capital investment not only includes the purchase price of new equipment, but also the *installation costs* (the costs of setting up the new fixed equipment in the firm and the associated costs of reorganizing work processes).

Assuming the installation costs are strictly convex in the level of investment, the firm has to solve an *intertemporal* optimization problem. Forward-looking expectations thus become important and this has implications for how equilibrium in the output market is established and how the equilibrium interest rate is determined. Indeed, in the simple neoclassical setup above, the interest rate equilibrates the market for capital services. The value of the interest rate is simply tied down by the equilibrium condition (2.39) in this market and what happens in the output market is a trivial consequence of this. But with convex capital installation costs the firm's capital stock is given in the short run and the interest rate(s) become(s) determined elsewhere in the model, as we shall see in chapters 14 and 15.

2.5.2 Long-run versus short-run production functions

In the discussion of production functions up to now we have been silent about the distinction between "ex ante" and "ex post" substitutability between capital and labor. By ex ante is meant "when plant and machinery are to be decided upon" and by ex post is meant "after the equipment is designed and constructed". In the standard neoclassical competitive setup like in (2.35) there is a presumption that also after the construction and installation of the equipment in the firm, the ratio of the factor inputs can be fully adjusted to a change in the relative factor price. In practice, however, when some machinery has been constructed and installed, its functioning will often require a more or less fixed number of machine operators. What can be varied is just the *degree of utilization* of the machinery. That is, after construction and installation of the machinery, the choice opportunities are no longer described by the neoclassical production function but by a Leontief production function,

$$Y = \min(Au\bar{K}, BL), \qquad A > 0, B > 0, \tag{2.46}$$

where $\bar{K}$ is the size of the installed machinery (a fixed factor in the short run) measured in efficiency units, u is its utilization rate ($0 \leq u \leq 1$), and A and B are given technical coefficients measuring efficiency (cf. Section 2.1.2).

So in the short run the choice variables are u and L. In fact, essentially only u is a choice variable since efficient production trivially requires $L = Au\bar{K}/B$. Under "full capacity utilization" we have $u = 1$ (each machine is used 24 hours per day seven days per week). "Capacity" is given as $A\bar{K}$ per week. Producing efficiently at capacity requires $L = A\bar{K}/B$ and the marginal product by increasing labor input is here nil. But if demand, Y^d, is *less* than capacity, satisfying this demand efficiently requires $L = Y^d/B$ and $u = BL/(A\bar{K}) < 1$. As long as $u < 1$, the marginal productivity of labor is a *constant*, B.

The various efficient input proportions that are possible *ex ante* may be approximately described by a neoclassical CRS production function. Let this function on intensive form be denoted $y = f(k)$. When investment is decided upon and undertaken, there is thus a choice between alternative efficient pairs of the technical coefficients A and B in (2.46). These pairs satisfy

$$f(k) = Ak = B. \tag{2.47}$$

So, for an increasing sequence of k's, $k_1, k_2, \ldots, k_i, \ldots$, the corresponding pairs are $(A_i, B_i) = (f(k_i)/k_i, f(k_i))$, $i = 1, 2, \ldots$.[28] We say that ex ante, depending on the relative factor prices as they are "now" and are expected to evolve in the future, a suitable technique, (A_i, B_i), is chosen from an opportunity set described by the given neoclassical production function. But ex post, i.e., when the equipment corresponding to this technique is installed, the production opportunities are described by a Leontief production function with $(A, B) = (A_i, B_i)$.

In the picturesque language of Phelps (1963), technology is in this case *putty-clay*. Ex ante the technology involves capital which is "putty" in the sense of being in a malleable state which can be transformed into a range of various machinery requiring capital-labor ratios of different magnitude. But once the machinery is constructed, it enters a "hardened" state and becomes "clay". Then factor substitution is no longer possible; the capital-labor ratio at full capacity utilization is fixed at the level $k = B_i/A_i$, as in (2.46). Following the terminology of Johansen (1972), we say that a putty-clay technology involves a "long-run production function" which is neoclassical and a "short-run production function" which is Leontief.

Table 1. Technologies classified according to factor substitutability ex ante and ex post.

	Ex post substitution	
Ex ante substitution	possible	impossible
possible	putty-putty	putty-clay
impossible		clay-clay

In contrast, the standard neoclassical setup assumes the same range of substitutability between capital and labor ex ante and ex post. Then the technology is called *putty-putty*. This term may also be used if ex post there is at least *some* substitutability although less than ex ante. At the opposite pole of putty-putty we may consider a technology which is *clay-clay*. Here neither ex ante nor ex post is factor substitution possible. Table 1 gives an overview of the alternative cases.

The putty-clay case is generally considered the realistic case. As time proceeds, technological progress occurs. To take this into account, we may replace (2.47) and (2.46) by $f(k_t, t) = A_t k_t = B_t$ and $Y_t = \min(A_t u_t \bar{K}_t, B_t L_t)$, respectively. If a new pair of Leontief coefficients, (A_{t_2}, B_{t_2}), efficiency-dominates its predecessor (by satisfying $A_{t_2} \geq A_{t_1}$ and $B_{t_2} \geq B_{t_1}$ with at least one strict equality), it may pay the firm to invest in the new technology at the same time as

[28] The points P and Q in the right-hand panel of Fig. 2.3 can be interpreted as constructed this way from the neoclassical production function in the left-hand panel of the figure.

some old machinery is scrapped. Real wages tend to rise along with technological progress and the scrapping occurs because the revenue from using the old machinery in production no longer covers the associated labor costs.

The clay property ex-post of many technologies is important for short-run analysis. It implies that there may be non-decreasing marginal productivity of labor up to a certain point. It also implies that in its investment decision the firm will have to take expected future technologies and future factor prices into account. For many issues in long-run analysis the clay property ex-post may be less important, since over time adjustment takes place through new investment.

2.5.3 A simple portrayal of price-making firms

Another modification which is important in short- and medium-run analysis, relates to the assumed market forms. Perfect competition is not a good approximation to market conditions in manufacturing and service industries. To bring perfect competition in the output market in perspective, we give here a brief review of firms' behavior under a form of monopolistic competition that is applied in many short-run models.

Suppose there is a large number of differentiated goods, $i = 1, 2, \ldots, n$, each produced by a separate firm. In the short run n is given. Each firm has monopoly on its own good (supported, say, by a trade mark, patent protection, or simply secrecy regarding the production recipe). The goods are imperfect substitutes to each other and so indirect competition prevails. Each firm is small in relation to the "sum" of competing firms and perceives that these other firms do not respond to its actions.

In the given period let firm i face a given downward-sloping demand curve for its product,

$$Y_i \leq \left(\frac{P_i}{P}\right)^{-\varepsilon} \frac{Y}{n} \equiv \mathcal{D}(P_i), \qquad \varepsilon > 1. \tag{2.48}$$

Here Y_i is the produced quantity and the expression on the right-hand side of the inequality is the demand as a function of the price P_i chosen by the firm.[29] The "general price level" P (a kind of average across the different goods, cf. Chapter 22) and the "general demand level", given by the index Y, matter for the position of the demand curve in the (Y_i, P_i) plan, cf. Fig. 2.5. The price elasticity of demand, ε, is assumed constant and higher than one (otherwise there is no solution to the monopolist's decision problem). Variables that the monopolist perceives as exogenous are implicit in the demand function symbol $\mathcal{D}$. We imagine prices are expressed in terms of money (so they are "nominal" prices, hence denoted by capital letters whereas we generally use small letters for "real" prices).

[29] We ignore production for inventory holding.

For simplicity, factor markets are still assumed competitive. Given the nominal factor prices, W_K and W_L, firm i wants to maximize its profit

$$\Pi_i = P_i Y_i - W_K K_i - W_L L_i,$$

subject to (2.48) and the neoclassical production function $Y_i = F(K_i, L_i)$. For the purpose of simple comparison with the case of perfect competition as described in Section 2.4, we return to the case where both labor and capital are variable inputs in the short run.[30] It is no serious restriction on the problem to assume the monopolist will want to produce the amount demanded so that $Y_i = \mathcal{D}(P_i)$. It is convenient to solve the problem in two steps.

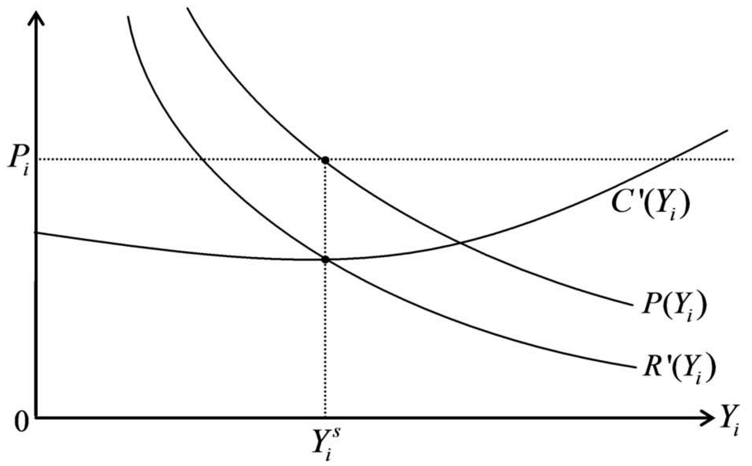


Figure 2.5: Determination of the monopolist price and output.

Step 1. Imagine the monopolist has already chosen the output level Y_i. Then the problem is to minimize cost:

$$\min_{K_i, L_i} W_K K_i + W_L L_i \text{ s.t. } F(K_i, L_i) = Y_i.$$

An interior solution (K_i, L_i) will satisfy the first-order conditions

$$\lambda F_K(K_i, L_i) = W_K, \qquad \lambda F_L(K_i, L_i) = W_L, \tag{2.49}$$

where λ is the Lagrange multiplier. Since F is neoclassical and thereby strictly quasiconcave, the first-order conditions are not only necessary but also sufficient for (K_i, L_i) to be a solution, and (K_i, L_i) will be unique so that we can write

[30] Generally, the technology would differ across the different product lines and F should thus be replaced by F^i, but for notational convenience we ignore this.

these conditional factor demands as functions, $K_i^d = K(W_K, W_L, Y_i)$ and $L_i^d = L(W_K, W_L, Y_i)$. This gives rise to the cost function $\mathcal{C}(Y_i) = W_K K(W_K, W_L, Y_i) + W_L L(W_K, W_L, Y_i)$.

Step 2. Solve

$$\max_{Y_i} \Pi(Y_i) = R(Y_i) - \mathcal{C}(Y_i) = \mathcal{P}(Y_i)Y_i - \mathcal{C}(Y_i).$$

We have here introduced "total revenue" $R(Y_i) = \mathcal{P}(Y_i)Y_i$, where $\mathcal{P}(Y_i)$ is the inverse demand function defined by $\mathcal{P}(Y_i) \equiv \mathcal{D}^{-1}(Y_i) = [Y_i/(Y/n)]^{-1/\varepsilon} P$ from (2.48). The first-order condition is

$$R'(Y_i) = \mathcal{P}(Y_i) + \mathcal{P}'(Y_i)Y_i = \mathcal{C}'(Y_i), \tag{2.50}$$

where the left-hand side is *marginal revenue* and the right-hand side is *marginal cost.*

A sufficient second-order condition is that $\Pi''(Y_i) = R''(Y_i) - \mathcal{C}''(Y_i) < 0$, i.e., the marginal revenue curve crosses the marginal cost curve from above. In the present case this is surely satisfied if we assume $\mathcal{C}''(Y_i) \geq 0$, which also ensures existence and uniqueness of a solution to (2.50). Substituting this solution, which we denote Y_i^s, cf. Fig. 2.5, into the conditional factor demand functions from Step 1, we find the factor demands, K_i^d and L_i^d. Owing to the downward-sloping demand curves the factor demands are unique whether the technology exhibits DRS, CRS, or IRS. Thus, contrary to the perfect competition case, neither CRS nor IRS pose particular problems.

From the definition $R(Y_i) = P(Y_i)Y_i$ follows

$$R'(Y_i) = P_i\left(1 + \frac{Y_i}{P_i}\mathcal{P}'(Y_i)\right) = P_i\left(1 - \frac{1}{\varepsilon}\right) = P_i\frac{\varepsilon - 1}{\varepsilon}.$$

So the pricing rule is $P_i = (1+\mu)\mathcal{C}'(Y_i)$, where Y_i is the profit maximizing output level and $\mu \equiv \varepsilon/(\varepsilon - 1) - 1 > 0$ is the mark-up on marginal cost. An analytical very convenient feature is that the markup is thus a *constant.*

In parallel with (2.31) and (2.32) the solution to firm i's decision problem is characterized by the *marginal revenue productivity* conditions

$$R'(Y_i^s)F_K(K_i^d, L_i^d) = W_K, \tag{2.51}$$

$$R'(Y_i^s)F_L(K_i^d, L_i^d) = W_L, \tag{2.52}$$

where $Y_i^s = F(K_i^d, L_i^d)$. These conditions follow from (2.49), since the Lagrange multiplier equals marginal cost (see Appendix A), which equals marginal revenue. That is, at profit maximum the marginal revenue products of capital and labor, respectively, equal the corresponding factor prices. Since $P_i > R'(Y_i^s)$, the factor prices are below the value of the marginal productivities. This reflects the market power of the firms.

In macro models a lot of symmetry is often assumed. If there is complete symmetry across product lines and if factor markets clear as in (2.36) and (2.37) with inelastic factor supplies, K^s and L^s, then $K_i^d = K^s/n$ and $L_i^d = L^s/n$. Furthermore, all firms will choose the same price so that $P_i = P$, $i = 1, 2, \ldots, n$. Then the given factor supplies, together with (2.51) and (2.52), determine the equilibrium *real* factor prices:

$$\begin{aligned} w_K &\equiv \frac{W_K}{P} = \frac{1}{1+\mu} F_K(\frac{K^s}{n}, \frac{L^s}{n}), \\ w_L &\equiv \frac{W_L}{P} = \frac{1}{1+\mu} F_L(\frac{K^s}{n}, \frac{L^s}{n}), \end{aligned}$$

where we have used that $R'(Y_i^s) = P/(1+\mu)$ under these circumstances. As under perfect competition, the real factor prices are proportional to the corresponding marginal productivities, although with a factor of proportionality less than one, namely equal to the inverse of the markup. This observation is sometimes used as a defence for applying the simpler perfect-competition framework for studying certain long-run aspects of the economy. For these aspects, the size of the proportionality factor may be immaterial, at least as long as it is relatively constant over time. Indeed, the constant markups open up for a simple transformation of many of the perfect competition results to monopolistic competition results by inserting the markup factor $1 + \mu$ the relevant places in the formulas.

If in the short term only labor is a variable production factor, then (2.51) need not hold. As claimed by Keynesian and New Keynesian thinking, also the prices chosen by the firms may be more or less fixed in the short run because the firms face price adjustment costs ("menu costs") and are reluctant to change prices too often, at least vis-a-vis changes in demand. Then in the short run only the produced quantity will adjust to changes in demand. As long as the output level is within the range where marginal cost is below the price, such adjustments are still beneficial to the firm. As a result, even (2.52) may at most hold "on average" over the business cycle. These matters are dealt with in Part V of this book.

In practice, market power and other market imperfections also play a role in the factor markets, implying that further complicating elements enter the picture. One of the tasks of theoretical and empirical macroeconomics is to clarify the aggregate implications of market imperfections and sort out which market imperfections are quantitatively important in different contexts.

2.5.4 The financing of firms' operations

We have so far talked about aspects related to production and pricing. What about the *financing* of a firm's operations? To acquire not only its fixed capital (structures and machines) but also its raw material and other intermediate inputs, a firm needs *funds* (there are expenses before the proceeds from sale arrive). These funds ultimately come from the accumulated saving of households. In long-run macromodels to be considered in the next chapters, uncertainty as well as non-neutrality of corporate taxation are ignored; in that context the capital structure (the debt-equity ratio) of firms is indeterminate and irrelevant for production outcomes.[31] In those chapters we shall therefore concentrate on the latter. Later chapters, dealing with short- and medium-run issues, touch upon cases where capital structure and bankruptcy risk matter and financial intermediaries enter the scene.

2.6 Literature notes

As to the question of the empirical validity of the constant returns to scale assumption, **?** offers an account of the econometric difficulties associated with estimating production functions. Studies by **?** and **?** suggest returns to scale are about constant or decreasing. Studies by **?**, **?**, **?**, **?**, and **?** suggest there are quantitatively significant increasing returns, either internal or external. On this background it is not surprising that the case of IRS (at least at industry level), together with market forms different from perfect competition, has received more attention in contemporary macroeconomics and in the theory of economic growth.

Macroeconomists' use of the value-laden term "technological progress" in connection with technological change may seem suspect. But the term should be interpreted as merely a label for certain types of shifts of isoquants in an abstract universe. At a more concrete and disaggregate level analysts of course make use of more refined notions about technological change, recognizing not only benefits of new technologies, but for instance also the risks, including risk of fundamental mistakes (think of the introduction and later abandonment of asbestos in the construction industry). For history of technology see, e.g., Ruttan (2001) and Smil (2003).

When referring to a Cobb-Douglas (or CES) production function some authors implicitly assume that the partial output elasticities with respect to inputs are time-independent and thereby not affected by technological change. For the case where the inputs in question are renewable and nonrenewable natural resources,

[31] In chapter 14 we return to this irrelevance proposition, called the Modigliani-Miller theorem.

Growiec and Schumacher (2008) study cases of time-dependency of the partial output elasticities.

When technical change is not "neutral" in one of the senses described, it may be systematically "biased" in alternative "directions". The reader is referred to the specialized literature on economic growth, cf. literature notes to Chapter 1.

Embodied technological progress, sometimes called investment-specific technological progress, is explored in, for instance, Solow (1960), Greenwood et al. (1997), and Groth and Wendner (2015).

Time series for different countries' aggregate and to some extent sectorial capital stocks are available from Penn World Table, ..., EU KLEMS, ..., and the AMECO database, .

The concept of Gorman preferences and conditions ensuring that a representative household is admitted are surveyed in Acemoglu (2009). Another source, also concerning the conditions for the representative firm to be a meaningful notion, is Mas-Colell et al. (1995). For general discussions of the limitations of representative agent approaches, see **?** and **?**. Reviews of the "Cambridge Controversy" are contained in Mas-Colell (1989) and **?**. The last-mentioned authors find the conditions required for the well-behavedness of these constructs so stringent that it is difficult to believe that actual economies are in any sense close to satisfy them. For less distrustful views and constructive approaches to the issues, see for instance Johansen (1972), **?**, Jorgenson et al. (2005), and **?**. For a stochastic approach to aggregation, see e.g. Gallegati et al., 2006.

Scarf (1960) provided a series of examples of lack of dynamic stability of an equilibrium price vector in an exchange economy. Mas-Colell et al. (1995) survey the later theoretical development in this field.

The counterexample to guaranteed stability of the neoclassical factor market equilibrium presented towards the end of Section 2.4 is taken from **?**, where further perspectives are discussed. It may be argued questions about stability should be studied on the basis of adjustment processes of a less mechanical nature than a Walrasian tâtonnement process. The view would be that trade out of equilibrium should be incorporated in the analysis and agents' behavior out of equilibrium should be founded on some kind of optimization or "satisficing", incorporating adjustment costs and imperfect information. This is a complicated field, and the theory seems not settled. Yet it may be fair to say that the studies of adjustment processes out of equilibrium indicate that the equilibrating force of Adam Smith's invisible hand is not without its limits. See Porter (1975), **?**, Osborne and Rubinstein (1990), **?**, and Foley (2010) for reviews and elaborate discussion of these issues.

We introduced the assumption that physical capital depreciation can be described as geometric (in continuous time exponential) evaporation of the capital stock. This formula is popular in macroeconomics, more so because of its simplicity than its realism. An introduction to more general approaches to depreciation is contained in, e.g., **?**.

2.7 Appendix

A. Strict quasiconcavity

Consider a function $f : \mathcal{A} \to \mathbb{R}$, where $\mathcal{A}$ is a convex set, $\mathcal{A} \subseteq \mathbb{R}^n$.[32] Given a real number a, if $f(x) = a$, the *upper contour set* is defined as $\{x \in \mathcal{A} | \, f(x) \geq a\}$ (the set of input bundles that can produce at least the amount a of output). The function $f(x)$ is called *quasiconcave* if its upper contour sets, for any constant a, are convex sets. If all these sets are strictly convex, $f(x)$ is called *strictly quasiconcave*.

Average and marginal costs To show that (2.14) holds with n production inputs, $n = 1, 2, \ldots$, we derive the cost function of a firm with a neoclassical production function, $Y = F(X_1, X_2, \ldots, X_n)$. Given a vector of strictly positive input prices $\mathbf{w} = (w_1, \ldots, w_n) >> 0$, the firm faces the problem of finding a cost-minimizing way to produce a given positive output level $\bar{Y}$ within the range of F. The problem is

$$\min \sum_{i=1}^{n} w_i X_i \text{ s.t. } F(X_1, \ldots, X_n) = \bar{Y} \text{ and } X_i \geq 0, \; i = 1, 2, \ldots, n.$$

An interior solution, $\mathbf{X}^* = (X_1^*, \ldots, X_n^*)$, to this problem satisfies the first-order conditions $\lambda F_i'(\mathbf{X}^*) = w_i$, where λ is the Lagrange multiplier, $i = 1, \ldots, n$.[33] Since F is neoclassical and thereby strictly quasiconcave in the interior of $\mathbb{R}_+^n$, the first-order conditions are not only necessary but also sufficient for the vector $\mathbf{X}^*$ to be a solution, and $\mathbf{X}^*$ will be unique[34] so that we can write it as a function, $\mathbf{X}^*(\bar{Y}) = (X_1^*(\bar{Y}), \ldots, X_n^*(\bar{Y}))$. This gives rise to the *cost function* $\mathcal{C}(\bar{Y}) = \sum_{i=1}^{n} w_i X_i^*(\bar{Y})$. So *average cost* is $\mathcal{C}(\bar{Y})/\bar{Y}$. We find *marginal cost* to be

$$\mathcal{C}'(\bar{Y}) = \sum_{i=1}^{n} w_i X_i^{*\prime}(\bar{Y}) = \lambda \sum_{i=1}^{n} F_i'(\mathbf{X}^*) X_i^{*\prime}(\bar{Y}) = \lambda,$$

[32] Recall that a set S is said to be *convex* if $x, y \in S$ and $\lambda \in [0, 1]$ implies $\lambda x + (1 - \lambda) y \in S$.

[33] Since in this section we use a bit of vector notation, we exceptionally mark first-order partial derivatives by a prime in order to clearly distinguish from the elements of a vector (so we write F_i' instead of our usual F_i).

[34] See Sydsaeter et al. (2008), pp. 74, 75, and 125.

where the third equality comes from the first-order conditions, and the last equality is due to the constraint $F(\mathbf{X}^*(\bar{Y})) = \bar{Y}$, which, by taking the total derivative on both sides, gives $\sum_{i=1}^n F_i'(\mathbf{X}^*)X_i^{*\prime}(\bar{Y}) = 1$. Consequently, the ratio of average to marginal costs is

$$\frac{\mathcal{C}(\bar{Y})/\bar{Y}}{\mathcal{C}'(\bar{Y})} = \frac{\sum_{i=1}^n w_i X_i^*(\bar{Y})}{\lambda \bar{Y}} = \frac{\sum_{i=1}^n F_i'(\mathbf{X}^*)X_i^*(\bar{Y})}{F(\mathbf{X}^*)},$$

which in analogy with (2.13) is the elasticity of scale at the point $\mathbf{X}^*$. This proves (2.14).

Sufficient conditions for strict quasiconcavity The claim (iii) in Section 2.1.3 was that a continuously differentiable two-factor production function $F(K, L)$ with CRS, satisfying $F_K > 0, F_L > 0$, and $F_{KK} < 0, F_{LL} < 0$, will automatically also be strictly quasi-concave in the interior of $\mathbb{R}^2$ and thus neoclassical.

To prove this, consider a function of two variables, $z = f(x, y)$, that is twice continuously differentiable with $f_1 \equiv \partial z/\partial x > 0$ and $f_2 \equiv \partial z/\partial y > 0$, everywhere. Then the equation $f(x, y) = a$, where a is a constant, defines an isoquant, $y = g(x)$, with slope $g'(x) = -f_1(x, y)/f_2(x, y)$. Substitute $g(x)$ for y in this equation and take the derivative with respect to x. By straightforward calculation we find

$$g''(x) = -\frac{f_1^2 f_{22} - 2f_1 f_2 f_{21} + f_2^2 f_{11}}{f_2^3} \tag{2.53}$$

If the numerator is negative, then $g''(x) > 0$; that is, the isoquant is strictly convex to the origin. And if this holds for all (x, y), then f is strictly quasi-concave in the interior of $\mathbb{R}^2$. A sufficient condition for a negative numerator is that $f_{11} < 0$, $f_{22} < 0$ and $f_{21} \geq 0$. All these conditions, including the last three are satisfied by the given function F. Indeed, F_K, F_L, F_{KK}, and F_{LL} have the required signs. And when F has CRS, F is homogeneous of degree 1 and thereby $F_{KL} > 0$, see Appendix B. Hereby claim (iii) in Section 2.1.3 is proved.

B. Homogeneous production functions

Claim (iv) in Section 2.1.3 is that a two-factor production function with CRS, satisfying $F_K > 0, F_L > 0$, and $F_{KK} < 0, F_{LL} < 0$, has always $F_{KL} > 0$, i.e., there is *direct complementarity* between K and L. This assertion is implied by the following observations on homogeneous functions.

Let $Y = F(K, L)$ be a twice continuously differentiable production function with $F_K > 0$ and $F_L > 0$ everywhere. Assume F is homogeneous of degree $h > 0$, that is, for all possible (K, L) and all $\lambda > 0$, $F(\lambda K, \lambda L) = \lambda^h F(K, L)$. According to Euler's theorem (see Math Tools) we then have:

CLAIM 1 For all (K, L), where $K > 0$ and $L > 0$,

$$KF_K(K,L) + LF_L(K,L) = hF(K,L). \tag{2.54}$$

Euler's theorem also implies the inverse:

CLAIM 2 If (2.54) is satisfied for all (K, L), where $K > 0$ and $L > 0$, then $F(K,L)$ is homogeneous of degree h.

Partial differentiation with respect to K and L, respectively, gives, after ordering,

$$KF_{KK} + LF_{LK} = (h-1)F_K \tag{2.55}$$
$$KF_{KL} + LF_{LL} = (h-1)F_L. \tag{2.56}$$

In (2.55) we can substitute $F_{LK} = F_{KL}$ (by Young's theorem). In view of Claim 2 this shows:

CLAIM 3 The marginal products, F_K and F_L, considered as functions of K and L, are homogeneous of degree $h-1$.

We see also that when $h \geq 1$ and K and L are positive, then

$$F_{KK} < 0 \text{ implies } F_{KL} > 0, \tag{2.57}$$
$$F_{LL} < 0 \text{ implies } F_{KL} > 0. \tag{2.58}$$

For $h = 1$ this establishes the direct complementarity result, (iv) in Section 2.1.3, to be proved. A by-product of the derivation is that also when a neoclassical production function is homogeneous of degree $h > 1$ (which implies IRS), does direct complementarity between K and L hold.

Remark. The microeconomic terminology around complementarity and substitutability may easily lead to confusion. In spite of K and L exhibiting *direct complementarity* when $F_{KL} > 0$, K and L are still *substitutes* in the sense that cost minimization for a given output level implies that a rise in the price of one factor results in higher demand for the other factor.

Claim (v) in Section 2.1.3 is the following. Suppose we face a CRS production function, $Y = F(K,L)$, that has positive marginal products, F_K and F_L, everywhere and isoquants, $K = g(L)$, satisfying the condition $g''(L) > 0$ everywhere (i.e., F is strictly quasi-concave). Then the partial second derivatives must satisfy the neoclassical conditions:

$$F_{KK} < 0, F_{LL} < 0. \tag{2.59}$$

The proof is as follows. The first inequality in (2.59) follows from (2.53) combined with (2.55). Indeed, for $h = 1$, (2.55) and (2.56) imply $F_{KK} = -F_{LK}L/K$

$= -F_{KL}L/K$ and $F_{KL} = -F_{LL}L/K$, i.e., $F_{KK} = F_{LL}(L/K)^2$ (or, in the notation of Appendix A, $f_{22} = f_{11}(x/y)^2$), which combined with (2.53) gives the conclusion $F_{KK} < 0$, when $g'' > 0$. The second inequality in (2.59) can be verified in a similar way.

Note also that for $h = 1$ the equations (2.55) and (2.56) entail

$$KF_{KK} = -LF_{LK} \text{ and } KF_{KL} = -LF_{LL}, \tag{2.60}$$

respectively. By dividing the left- and right-hand sides of the first of these equations with those of the second we conclude that $F_{KK}F_{LL} = F_{KL}^2$ in the CRS case. We see also from (2.60) that, under CRS, the implications in (2.57) and (2.58) can be turned round.

Finally, we asserted in § 2.1.1 that when the neoclassical production function $Y = F(K, L)$ is homogeneous of degree h, then the marginal rate of substitution between the production factors depends only on the factor proportion $k \equiv K/L$. Indeed,

$$MRS_{KL}(K, L) = \frac{F_L(K, L)}{F_K(K, L)} = \frac{L^{h-1}F_L(k, 1)}{L^{h-1}F_K(k, 1)} = \frac{F_L(k, 1)}{F_K(k, 1)} \equiv mrs(k), \tag{2.61}$$

where $k \equiv K/L$. The result (2.61) follows even if we only assume $F(K, L)$ is *homothetic*. When $F(K, L)$ is homothetic, by definition we can write $F(K, L) \equiv \varphi(G(K, L))$, where G is homogeneous of degree 1 and φ is an increasing function. In view of this, we get

$$MRS_{KL}(K, L) = \frac{\varphi' G_L(K, L)}{\varphi' G_K(K, L)} = \frac{G_L(k, 1)}{G_K(k, 1)},$$

where the last equality is implied by Claim 3 for $h = 1$.

C. The Inada conditions combined with CRS

We consider a neoclassical production function, $Y = F(K, L)$, exhibiting CRS. Defining $k \equiv K/L$, we can then write $Y = LF(k, 1) \equiv Lf(k)$, where $f(0) \geq 0, f' > 0$, and $f'' < 0$.

Essential inputs In Section 2.1.2 we claimed that the upper Inada condition for MPL together with CRS implies that without capital there will be no output:

$$F(0, L) = 0 \qquad \text{for any } L > 0.$$

In other words: in this case capital is an essential input. To prove this claim, let $K > 0$ be fixed and let $L \to \infty$. Then $k \to 0$, implying, by (2.16) and (2.18), that $F_L(K, L) = f(k) - f'(k)k \to f(0)$. But from the upper Inada condition for MPL we also have that $L \to \infty$ implies $F_L(K, L) \to 0$. It follows that

$$\text{the upper Inada condition for } MPL \text{ implies } f(0) = 0. \tag{2.62}$$

Since under CRS, for any $L > 0$, $F(0, L) = LF(0, 1) \equiv Lf(0)$, we have hereby shown our claim.

Similarly, we can show that the upper Inada condition for MPK together with CRS implies that labor is an essential input. Consider the output-capital ratio $x \equiv Y/K$. When F has CRS, we get $x = F(1, \ell) \equiv g(\ell)$, where $\ell \equiv L/K$, $g' > 0$, and $g'' < 0$. Thus, by symmetry with the previous argument, we find that under CRS, the upper Inada condition for MPK implies $g(0) = 0$. Since under CRS $F(K, 0) = KF(1, 0) \equiv Kg(0)$, we conclude that the upper Inada condition for MPK together with CRS implies

$$F(K, 0) = 0 \qquad \text{for any } K > 0,$$

that is, without labor, no output.

Sufficient conditions for output going to infinity when either input goes to infinity Here our first claim is that when F exhibits CRS and satisfies the upper Inada condition for MPL and the lower Inada condition for MPK, then

$$\lim_{L\to\infty} F(K, L) = \infty \qquad \text{for any } K > 0.$$

To prove this, note that Y can be written $Y = Kf(k)/k$, since $K/k = L$. Here,

$$\lim_{k\to 0} f(k) = f(0) = 0,$$

by continuity and (2.62), presupposing the upper Inada condition for MPL. Thus, for any given $K > 0$,

$$\lim_{L\to\infty} F(K, L) = K \lim_{L\to\infty} \frac{f(k)}{k} = K \lim_{k\to 0} \frac{f(k) - f(0)}{k} = K \lim_{k\to 0} f'(k) = \infty,$$

by the lower Inada condition for MPK. This verifies the claim.

Our second claim is symmetric with this and says: when F exhibits CRS and satisfies the upper Inada condition for MPK and the lower Inada condition for MPL, then

$$\lim_{K\to\infty} F(K, L) = \infty \qquad \text{for any } L > 0.$$

The proof is analogue. So, in combination, the four Inada conditions imply, under CRS, that output has no upper bound when either input goes to infinity.

D. Concave neoclassical production functions

Two claims made in Section 2.4 are proved here.

CLAIM 1 When a neoclassical production function $F(K, L)$ is concave, it has non-increasing returns to scale everywhere.

Proof. We consider a concave neoclassical production function, F. Let $\mathbf{x} = (x_1, x_2) = (K, L)$. Then we can write $F(K, L)$ as $F(\mathbf{x})$. By concavity, for all pairs $\mathbf{x}^0, \mathbf{x} \in \mathbb{R}^2_+$, we have $F(\mathbf{x}^0) - F(\mathbf{x}) \leq \sum_{i=1}^{2} F_i'(\mathbf{x})(x_i^0 - x_i)$. In particular, for $\mathbf{x}^0 = (0, 0)$, since $F(\mathbf{x}^0) = F(0, 0) = 0$, we have

$$-F(\mathbf{x}) \leq -\sum_{i=1}^{2} F_i'(\mathbf{x})x_i. \tag{2.63}$$

Suppose $\mathbf{x} \in \mathbb{R}^2_{++}$. Then $F(\mathbf{x}) > \mathbf{0}$ in view of F being neoclassical so that $F_K > 0$ and $F_L > 0$. From (2.63) we now find the elasticity of scale to be

$$\sum_{i=1}^{2} F_i'(\mathbf{x})x_i / F(\mathbf{x}) \leq 1. \tag{2.64}$$

In view of (2.13) and (2.12), this implies non-increasing returns to scale everywhere. □

CLAIM 2 When a neoclassical production function $F(K, L)$ is strictly concave, it has decreasing returns to scale everywhere.

Proof. The argument is analogue to that above, but in view of strict concavity the inequalities in (2.63) and (2.64) become strict. This implies that F has DRS everywhere. □

2.8 Exercises

2.1

Chapter 3

The basic OLG model: Diamond

There exists two main analytical frameworks for analyzing the basic intertemporal choice, consumption versus saving, and the dynamic implications of this choice: *overlapping-generations* (OLG) models and *representative agent* models. In the first type of models the focus is on (a) the interaction between different generations alive at the same time, and (b) the never-ending entrance of new generations and thereby new decision makers. In the second type of models the household sector is modelled as consisting of a finite number of infinitely-lived dynasties. One interpretation is that the parents take the utility of their descendants into account by leaving bequests and so on forward through a chain of intergenerational links. This approach, which is also called the Ramsey approach (after the British mathematician and economist Frank Ramsey, 1903-1930), will be described in Chapter 8 (discrete time) and Chapter 10 (continuous time).

In the present chapter we introduce the OLG approach which has shown its usefulness for analysis of many issues such as: public debt, taxation of capital income, financing of social security (pensions), design of educational systems, non-neutrality of money, and the possibility of speculative bubbles. We will focus on what is known as Diamond's OLG model[1] after the American economist and Nobel Prize laureate Peter A. Diamond (1940-).

Among the strengths of the model are:

- The *life-cycle* aspect of human behavior is taken into account. Although the economy is infinitely-lived, the individual agents are not. During lifetime one's educational level, working capacity, income, and needs change and this is reflected in the individual labor supply and saving behavior. The aggregate implications of the life-cycle behavior of coexisting individual agents at different stages in their life is at the centre of attention.

[1] Diamond (1965).

- The model takes elementary forms of *heterogeneity* in the population into account – there are "old" and "young", there are the currently-alive people and the future generations whose preferences are not reflected in current market transactions. Questions relating to the distribution of income and wealth across generations can be studied. For example, how does the investment in fixed capital and environmental protection by current generations affect the conditions for the succeeding generations?

3.1 Motives for saving

Before going into the specifics of Diamond's model, let us briefly consider what may in general motivate people to save:

(a) The *consumption-smoothing motive for saving.* Individuals go through a life cycle where earnings typically have a hump-shaped time pattern; by saving and dissaving the individual then attempts to obtain the desired smoothing of consumption across lifetime. This is the essence of the *life-cycle saving hypothesis* put forward by Nobel laureate Franco Modigliani (1918-2003) and associates in the 1950s. This hypothesis states that consumers plan their saving and dissaving in accordance with anticipated variations in income and needs over lifetime. Because needs vary less over lifetime than income, the time profile of saving tends to be hump-shaped with some dissaving early in life (for instance if studying), positive saving during the years of peak earnings and then dissaving after retirement.

(b) The *precautionary motive for saving.* Income as well as needs may vary due to conditions of *uncertainty*: sudden unemployment, illness, or other kinds of bad luck. By saving, the individual can obtain a buffer against such unwelcome events.

Horioka and Watanabe (1997) find that empirically, the saving motives (a) and (b) are of dominant importance (Japanese data). Yet other motives include:

(c) Saving enables the purchase of *durable consumption goods* and owner-occupied housing as well as repayment of debt.

(d) Saving may be motivated by the *desire to leave bequests* to heirs.

(e) Saving may simply be motivated by the fact that financial wealth may lead to *social prestige* and economic or political *power.*

Diamond's OLG model aims at simplicity and concentrates on motive (a). In fact only one aspect of motive (a) is considered, namely the saving for retirement. People live for two periods only, as "young" they work full-time and as "old" they retire and live by their savings. The model abstracts from a possible bequest motive.

Now to the details.

3.2 The model framework

The flow of time is divided into successive periods of equal length, taken as the time unit. Given the two-period lifetime of (adult) individuals, the period length is understood to be very long, around, say, 30 years. The main assumptions are:

1. The number of young people in period t, denoted L_t, changes over time according to $L_t = L_0(1+n)^t$, $t = 0, 1, 2, ...$, where n is a constant, $n > -1$. Indivisibility is ignored and so L_t is just considered a positive real number.

2. Only the young work. Each young supplies one unit of labor inelastically. The division of available time between work and leisure is thereby considered as exogenous.

3. Output is homogeneous and can be used for consumption as well as investment in physical capital. Physical capital is the only non-human asset in the economy; it is owned by the old and rented out to the firms. Output is the numeraire (unit of account) used in trading. Money (means of payment) is ignored.[2]

4. The economy is closed (no foreign trade).

5. Firms' technology has constant returns to scale.

6. In each period three markets are open, a market for output, a market for labor services, and a market for capital services. Perfect competition rules in all markets. Uncertainty is absent; when a decision is made, its consequences are known.

7. Agents have perfect foresight.

Assumption 7 entails the following. First, the agents are assumed to have "rational expectations" or, with a better name, "model-consistent expectations".

[2] As to the disregard of money we may imagine that agents have safe electronic accounts in a fictional central bank allowing costless transfers between accounts.

This means that forecasts made by the agents coincide with the forecasts that can be calculated on the basis of the model. Second, as there are no stochastic elements in the model (no uncertainty), the forecasts are point estimates rather than probabilistic forecasts. Thereby the model-consistent expectations take the extreme form of *perfect foresight*: the agents agree in their expectations about the future evolution of the economy and ex post this future evolution fully coincides with what was expected.

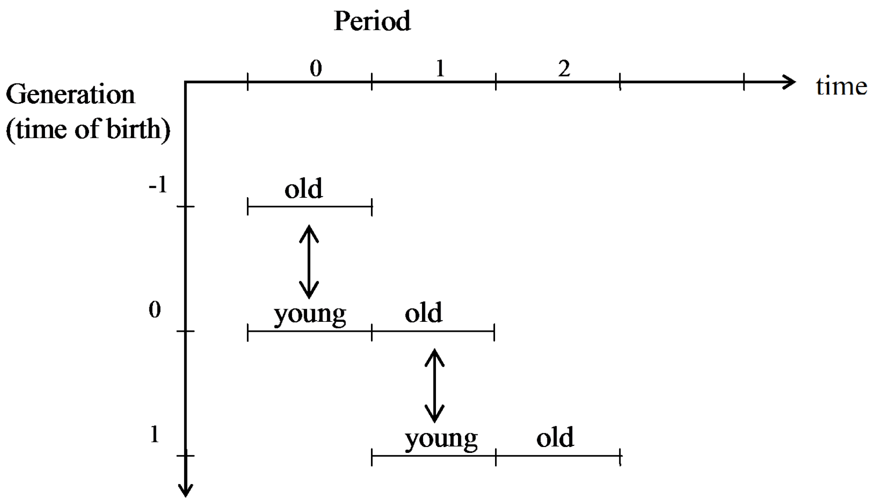


Figure 3.1: The two-period model's time structure.

Of course, this is an unrealistic assumption. The motivation is to simplify in a first approach. The results that emerge will be the outcome of economic mechanisms in isolation from expectational errors. In this sense the model constitutes a "pure" case (benchmark case).

The time structure of the model is illustrated in Fig. 3.1. In every period two generations are alive and interact with each other as indicated by the arrows. The young supply labor and earn a labor income. They consume an endogenous fraction of this income and save the remainder for retirement. Thereby the young offset the dissaving by the old, and possibly positive net investment arises in the economy. At the end of the first period the savings by the young are converted into direct ownership of capital goods. In the next period the now old owners of the capital goods rent them out to the firms. We may imagine that the firms are owned by the old, but this ownership is not visible in the equilibrium allocation because pure profits will be nil due to the combination of perfect competition and constant returns to scale.

Let the output good be the numeraire and let $\hat{r}_t$ denote the rental rate for capital in period t; that is, $\hat{r}_t$ is the real price a firm has to pay at the end of period t for the right to use one unit of someone else's physical capital through period t. So the owner of K_t units of physical capital receives a

$$\text{real (net) rate of return on capital } = \frac{\hat{r}_t K_t - \delta K_t}{K_t} = \hat{r}_t - \delta, \tag{3.1}$$

where δ is the rate of physical capital depreciation which is assumed constant, $0 \leq \delta \leq 1$.

Suppose there is also a market for loans. Assume you have lent out one unit of output from the end of period $t-1$ to the end of period t. If the *real interest rate* in the loan market is r_t, then, at the end of period t you should get back $1 + r_t$ units of output. In the absence of uncertainty, equilibrium requires that capital and loans give the same rate of return,

$$\hat{r}_t - \delta = r_t. \tag{3.2}$$

This *no-arbitrage* condition indicates how the rental rate for capital and the more everyday concept, the interest rate, would be related in an equilibrium where both the market for capital services and a loan market were active. We shall see, however, that in this model no loan market will be active in an equilibrium. Nevertheless we will follow the tradition and call the right-hand side of (3.2) the *interest rate*.

Table 3.1 provides an overview of the notation. As to our timing convention, notice that any stock variable dated t indicates the amount held at the beginning of period t. That is, the capital stock accumulated by the end of period $t-1$ and available for production in period t is denoted K_t. We therefore write $K_t = (1-\delta)K_{t-1} + I_{t-1}$ and $Y_t = F(K_t, L_t)$, where F is an aggregate production function. In this context it is useful to think of "period t" as running from date t to right before date $t+1$. So period t is the half-open time interval $[t, t+1)$ on the continuous-time axis. Whereas production and consumption take place *in* period t, we imagine that all decisions are made at discrete points in time $t = 0, 1, 2, ...$ ("dates"). We further imagine that receipts for work and lending as well as payment for the consumption in period t occur at the end of the period. These timing conventions are common in discrete-time growth and business cycle theory;[3] they are convenient because they make switching between discrete and continuous time analysis fairly easy.

[3] In contrast, in the *accounting* and *finance* literature, typically K_t would denote the *end-of-period-t* stock that begins to yield its services *next* period.

Table 3.1. List of main variable symbols

Symbol	*Meaning*
L_t	the number of young people in period t
n	generation growth rate
K_t	aggregate capital available in period t
c_{1t}	consumption as young in period t
c_{2t}	consumption as old in period t
w_t	real wage in period t
r_t	real interest rate (from end of per. $t-1$ to end of per. t)
ρ	rate of time preference (impatience)
θ	elasticity of marginal utility
s_t	saving of each young in period t
Y_t	aggregate output in period t
$C_t = c_{1t}L_t + c_{2t}L_{t-1}$	aggregate consumption in period t
$S_t = Y_t - C_t$	aggregate gross saving in period t
$\delta \in [0,1]$	capital depreciation rate
$K_{t+1} - K_t = I_t - \delta K_t$	aggregate net investment in period t

3.3 The saving by the young

Suppose the preferences of the young can be represented by a lifetime utility function as specified in (3.3). Given w_t and r_{t+1}, the decision problem of the young in period t then is:

$$\max_{c_{1t},c_{2t+1}} U(c_{1t}, c_{2t+1}) = u(c_{1t}) + (1+\rho)^{-1}u(c_{2t+1}) \quad \text{s.t.} \tag{3.3}$$

$$c_{1t} + s_t = w_t \cdot 1 \quad (w_t > 0), \tag{3.4}$$

$$c_{2t+1} = (1 + r_{t+1})s_t \quad (r_{t+1} > -1), \tag{3.5}$$

$$c_{1t} \geq 0, c_{2t+1} \geq 0. \tag{3.6}$$

The interpretation of the variables is given in Table 3.1 above. We may think of the "young" as a household consisting of one adult and $1+n$ children whose consumption is included in c_{1t}. Note that "utility" appears at two levels. There is a *lifetime utility function*, U, and a *period utility function*, u.[4] The latter is assumed to be the same in both periods of life (this has no effects on the qualitative results and simplifies the exposition). The period utility function is assumed twice continuously differentiable with $u' > 0$ and $u'' < 0$ (positive, but diminishing marginal utility of consumption). Many popular specifications of u, e.g., $u(c) = \ln c$, have the property that $\lim_{c\to 0} u(c) = -\infty$; then we *define* $u(0) = -\infty$.

[4]Other names for these two functions are the *intertemporal utility function* and the *subutility function*, respectively.

The parameter ρ is called the *rate of time preference.* It acts as a utility discount *rate,* whereas $(1+\rho)^{-1}$ is a utility discount *factor*. By definition, $\rho > -1$, but $\rho > 0$ is usually assumed. We interpret ρ as reflecting the degree of *impatience* with respect to the "arrival" of utility. When preferences can be represented in this additive way, they are called *time-separable.* In principle, as seen from period t the interest rate appearing in (3.5) should be interpreted as an *expected* real interest rate. But as long as we assume perfect foresight, there is no need that our notation distinguishes between actual and expected magnitudes.

Box 3.1. Discount rates and discount factors

By a *discount rate* is meant an interest rate applied in the construction of a discount factor. A *discount factor* is a factor by which future benefits or costs, measured in some unit of account, are converted into present equivalents. The higher the discount rate the lower the discount factor.

One should bear in mind that a discount rate depends on what is to be discounted. In (3.3) the unit of account is "utility" and ρ acts as a *utility discount rate.* In (3.7) the unit of account is the consumption good and r_{t+1} acts as a *consumption discount rate.* If people also work as old, the right-hand side of (3.7) would read $w_t + (1 + r_{t+1})^{-1}w_{t+1}$ and thus r_{t+1} would act as an *earnings discount rate.* This will be the same as the consumption discount rate if we think of real income measured in consumption units. But if we think of nominal income, that is, income measured in monetary units, there would be a *nominal earnings discount rate*, namely the *nominal* interest rate, which in an economy with inflation will exceed the consumption discount rate. Unfortunately, confusion of different discount rates is not rare.

In (3.5) the interest rate r_{t+1} acts as a (net) rate of return on saving.[5] An interest rate may also be seen as a discount rate relating to consumption over time. Indeed, by isolating s_t in (3.5) and substituting into (3.4), we may consolidate the two period budget constraints of the individual into *one* budget constraint,

$$c_{1t} + \frac{1}{1 + r_{t+1}} c_{2t+1} = w_t. \tag{3.7}$$

[5]While s_t in (3.4) appears as a *flow* (non-consumed income), in (3.5) s_t appears as a *stock* (the accumulated financial wealth at the end of period t). This notation is legitimate because the magnitude of the two is the same when the time unit is the same as the period length. Indeed, the interpretation of s_t in (3.5) $s_t = s_t \cdot \Delta t = s_t \cdot 1$ units of account.

In real life the gross payoff of individual saving may sometimes be nil (if invested in a project that completely failed). Unless otherwise indicated, it is in this book understood that an interest rate is a number exceeding -1 as indicated in (3.5). Thereby the discount factor $1/(1 + r_{t+1})$ is well-defined. In general equilibrium, the condition $1 + r_{t+1} > 0$ is always met in the present model.

In this *intertemporal budget constraint* the interest rate appears as the discount rate entering the discount factor converting future amounts of consumption into present equivalents, cf. Box 3.1.

Solving the saving problem

To avoid the possibility of corner solutions, we impose the No Fast Assumption

$$\lim_{c \to 0} u'(c) = \infty. \tag{A1}$$

In view of the sizeable period length in the model, this is definitely plausible.

Inserting the two budget constraints into the objective function in (3.3), we get $U(c_{1t}, c_{2t+1}) = u(w_t - s_t) + (1+\rho)^{-1}u((1+r_{t+1})s_t) \equiv \tilde{U}_t(s_t)$, a function of only one decision variable, s_t. According to the non-negativity constraint on consumption in both periods, (3.6), s_t must satisfy $0 \leq s_t \leq w_t$. Maximizing with respect to s_t gives the first-order condition

$$\frac{d\tilde{U}_t}{ds_t} = -u'(w_t - s_t) + (1+\rho)^{-1}u'((1+r_{t+1})s_t)(1+r_{t+1}) = 0. \tag{FOC}$$

The second derivative of $\tilde{U}_t$ is

$$\frac{d^2\tilde{U}_t}{ds_t^2} = u''(w_t - s_t) + (1+\rho)^{-1}u''((1+r_{t+1})s_t)(1+r_{t+1})^2 < 0. \tag{SOC}$$

Hence there can at most be one s_t satisfying (FOC). Moreover, for a positive wage income there always exists such an s_t. Indeed:

LEMMA 1 Let $w_t > 0$ and suppose the No Fast Assumption (A1) applies. Then the saving problem of the young has a unique solution $s_t = s(w_t, r_{t+1})$. The solution is interior, i.e., $0 < s_t < w_t$, and s_t satisfies (FOC).

Proof. Assume (A1). For any $s \in (0, w_t)$, $d\tilde{U}_t(s)/ds > -\infty$. Now consider the endpoints $s = 0$ and $s = w_t$. By (FOC) and (A1),

$$\begin{aligned}
\lim_{s \to 0} \frac{d\tilde{U}_t}{ds} &= -u'(w_t) + (1+\rho)^{-1}(1+r_{t+1}) \lim_{s \to 0} u'((1+r_{t+1})s) = \infty, \\
\lim_{s \to w} \frac{d\tilde{U}_t}{ds} &= -\lim_{s \to w_t} u'(w_t - s) + (1+\rho)^{-1}(1+r_{t+1})u'((1+r_{t+1})w_t) = -\infty.
\end{aligned}$$

By continuity of $d\tilde{U}_t/ds$ follows that there exists an $s_t \in (0, w_t)$ such that at $s = s_t$, $d\tilde{U}_t/ds = 0$. This is an application of the *intermediate value theorem*. It follows that (FOC) holds for this s_t. By (SOC), s_t is unique and can therefore be written as an implicit function, $s(w_t, r_{t+1})$, of the exogenous variables in the problem, w_t and r_{t+1}. □

Inserting the solution for s_t into the two period budget constraints, (3.4) and (3.5), we immediately get the optimal consumption levels, c_{1t} and c_{2t+1}.

The simple optimization method we have used here is called the *substitution method*: by substitution of the constraints into the objective function an unconstrained maximization problem is obtained.[6]

The consumption Euler equation

The first-order condition (FOC) can conveniently be written

$$u'(c_{1t}) = (1+\rho)^{-1}u'(c_{2t+1})(1+r_{t+1}). \tag{3.8}$$

This is known as an *Euler equation,* after the Swiss mathematician L. Euler (1707-1783) who was the first to study dynamic optimization problems. In the present context the condition is called a *consumption Euler equation.*

Intuitively, in an optimal plan the marginal utility cost of saving must equal the marginal utility benefit obtained by saving. The marginal utility cost of saving is the opportunity cost (in terms of current utility) of saving one more unit of account in the current period (approximately). This one unit of account is transferred to the next period with interest so as to result in $1 + r_{t+1}$ units of account in that period. An optimal plan requires that the utility cost equals the utility benefit of instead having $1 + r_{t+1}$ units of account in the next period. And this utility benefit is the discounted value of the extra utility that can be obtained next period through the increase in consumption by $1 + r_{t+1}$ units compared with the situation without the saving of the marginal unit.

It may seem odd to attempt an intuitive interpretation this way, that is, in terms of "utility units". The utility concept is just a convenient mathematical device used to represent the assumed *preferences.* Our interpretation is only meant as an as-if interpretation: as if utility were something concrete. An interpretation in terms of concrete *measurable quantities* goes like this. We rewrite (3.8) as

$$\frac{u'(c_{1t})}{(1+\rho)^{-1}u'(c_{2t+1})} = 1 + r_{t+1}. \tag{3.9}$$

The left-hand side measures the *marginal rate of substitution,* MRS, of consumption as old for consumption as young, evaluated at the point (c_1, c_2). MRS is defined as the increase in period-$t+1$ consumption needed to compensate for a

[6] Alternatively, one could use the Lagrange method.

marginal decrease in period-t consumption. That is,

$$MRS_{c_2c_1} = -\frac{dc_{2t+1}}{dc_{1t}}|_{U=\bar{U}} = \frac{u'(c_{1t})}{(1+\rho)^{-1}u'(c_{2t+1})}, \tag{3.10}$$

where we have used implicit differentiation in $U(c_{1t}, c_{2t+1}) = \bar{U}$. The right-hand side of (3.9) indicates the marginal rate of transformation, MRT, which is the rate at which saving allows an agent to shift consumption from period t to period $t+1$ via the market. In an optimal plan MRS must equal MRT.

Even though interpretations in terms of "MRS equals MRT" are more satisfactory, we will often use "as if" interpretations like the one before. They are a convenient short-hand for the more elaborate interpretation.

By the Euler equation (3.8),

$$\rho \lesseqqgtr r_{t+1} \text{ implies } u'(c_{1t}) \gtreqqless u'(c_{2t+1}), \text{ i.e., } c_{1t} \lesseqqgtr c_{2t+1},$$

respectively, in the optimal plan (because $u'' < 0$). That is, absent uncertainty the optimal plan entails either increasing, constant or decreasing consumption over time according to whether the rate of time preference is below, equal to, or above the market interest rate, respectively. For example, when $\rho < r_{t+1,}$ the plan is to start with relatively low consumption in order to take advantage of the relatively high rate of return on saving.

Note that there are infinitely many pairs (c_{1t}, c_{2t+1}) satisfying the Euler equation (3.8). Only when requiring the two period budget constraints, (3.4) and (3.5), satisfied, do we get the unique solution s_t and thereby the unique solution for c_{1t} and c_{2t+1}.

Properties of the saving function

The first-order condition (FOC), where the two budget constraints are inserted, determines the saving as an implicit function of the market prices faced by the young decision maker, i.e., $s_t = s(w_t, r_{t+1})$.

The partial derivatives of this function can be found by applying the *implicit function theorem* on (FOC). A practical procedure is the following. We first interpret $d\tilde{U}_t/ds_t$ in (FOC) as a function, f, of the variables involved, s_t, w_t, and r_{t+1}, i.e.,

$$\frac{d\tilde{U}_t}{ds_t} = -u'(w_t - s_t) + (1+\rho)^{-1}u'((1+r_{t+1})s_t)(1+r_{t+1}) \equiv f(s_t, w_t, r_{t+1}).$$

By (FOC),

$$f(s_t, w_t, r_{t+1}) = 0. \tag{*}$$

The implicit function theorem (see Math tools) now implies that if $\partial f/\partial s_t \neq 0$, then the equation (*) defines s_t as an implicit function of w_t and r_{t+1}, $s_t = s(w_t, r_{t+1})$, with partial derivatives

$$\frac{\partial s_t}{\partial w_t} = -\frac{\partial f/\partial w_t}{D} \quad \text{and} \quad \frac{\partial s_t}{\partial r_{t+1}} = -\frac{\partial f/\partial r_{t+1}}{D},$$

where $D \equiv \partial f/\partial s_t \equiv d^2\tilde{U}_t/ds_t^2 < 0$ by (SOC). We find

$$\begin{aligned} \frac{\partial f}{\partial w_t} &= -u''(c_{1t}) > 0, \\ \frac{\partial f}{\partial r_{t+1}} &= (1+\rho)^{-1}\left[u'(c_{2t+1}) + u''(c_{2t+1})s_t(1+r_{t+1})\right]. \end{aligned}$$

Consequently, the partial derivatives of the saving function $s_t = s(w_t, r_{t+1})$ are

$$s_w \equiv \frac{\partial s_t}{\partial w_t} = \frac{u''(c_{1t})}{D} > 0 \quad (\text{but } < 1), \tag{3.11}$$

$$s_r \equiv \frac{\partial s_t}{\partial r_{t+1}} = -\frac{(1+\rho)^{-1}[u'(c_{2t+1}) + u''(c_{2t+1})c_{2t+1}]}{D}, \tag{3.12}$$

where in the last expression we have used (3.5).[7]

The role of w_t for saving is straightforward. Indeed, (3.11) shows that $0 < s_w < 1$, which implies that $0 < \partial c_{1t}/\partial w_t < 1$ and $0 < \partial c_{2t}/\partial w_t < 1 + r_{t+1}$. The positive sign of these two derivatives indicate that consumption in each of the periods is a *normal* good (which certainly is plausible since we are talking about the total consumption by the individual in each period).[8]

The sign of s_r in (3.12) is seen to be ambiguous. This ambiguity regarding the role of r_{t+1} for saving reflects that the Slutsky substitution and income effects on consumption as young of a rise in the interest rate are of opposite signs. To

[7] A perhaps more straightforward procedure, not requiring full memory of the exact content of the implicit function theorem, is based on "implicit differentiation". First, keeping r_{t+1} fixed, one calculates the total derivative w.r.t. w_t on both sides of (FOC). Next, keeping w_t fixed, one calculates the total derivative w.r.t. r_{t+1} on both sides of (FOC).

Yet another possible procedure is based on "total differentiation" in terms of *differentials*. Taking the differential w.r.t. s_t, w_t, and r_{t+1} on both sides of (FOC) gives $-u''(c_{1t})(dw_t - ds_t) + (1+\rho)^{-1} \cdot \{u''(c_{2t+1})\left[(1+r_{t+1})ds_t + s_t dr_{t+1}\right](1+r_{t+1}) + u'(c_{2t+1})dr_{t+1}\} = 0$. By rearranging we find the ratios ds_t/dw_t and ds_t/dr_{t+1}, which will indicate the value of the partial derivatives (3.11) and (3.12).

[8] Recall, a consumption good is called *normal* for given consumer preferences if the demand for it is an increasing function of the consumer's wealth. Since in this model the consumer is born without any financial wealth, the consumer's wealth at the end of period t is simply the present value of labor earnings through life, which here, evaluated at the beginning of period t, is $w_t/(1+r_t)$ as there is no labor income in the second period of life, cf. (3.7).

understand this, it is useful to keep the intertemporal budget constraint, (3.7), in mind. The *substitution effect* on c_{1t} is negative because the higher interest rate makes future consumption cheaper in terms of current consumption. And the *income effect* on c_{1t} is positive because with a higher interest rate, a given budget can buy more consumption in both periods, cf. (3.7). Generally there would be a third Slutsky effect, a *wealth effect* of a rise in the interest rate. But such an effect is ruled out in this model. This is because there is no labor income in the second period of life. Indeed, as indicated by (3.4), the human wealth of a member of generation t, evaluated at the end of period t, is simply w_t, which is independent of r_{t+1}. (In contrast, with labor income, w_{t+1}, in the second period, the human wealth would be $w_t + w_{t+1}/(1+r_{t+1})$. This present discounted value of life-time earnings clearly depends negatively on r_{t+1}, and so a negative wealth effect on c_{1t} of a rise in the interest rate would arise.)

Rewriting (3.12) gives

$$s_r = \frac{(1+\rho)^{-1}u'(c_{2t+1})[\theta(c_{2t+1})-1]}{D} \gtreqless 0 \text{ for } \theta(c_{2t+1}) \lesseqgtr 1, \tag{3.13}$$

respectively, where $\theta(c_{2t+1})$ is the absolute *elasticity of marginal utility* of consumption in the second period, that is,

$$\theta(c_{2t+1}) \equiv -\frac{c_{2t+1}}{u'(c_{2t+1})}u''(c_{2t+1}) \approx -\frac{\Delta u'(c_{2t+1})/u'(c_{2t+1})}{\Delta c_{2t+1}/c_{2t+1}} > 0,$$

where the approximation is valid for a "small" increase, Δc_{2t+1}, in c_{2t+1}. The inequalities in (3.13) show that when the absolute elasticity of marginal utility is below one, then the substitution effect on consumption as young of an increase in the interest rate dominates the income effect and saving increases. The opposite is true if the elasticity of marginal utility is above one.

The reason that $\theta(c_{2t+1})$ has this role is that $\theta(c_{2t+1})$ reflects how sensitive marginal utility of c_{2t+1} is to a rise in c_{2t+1}. To see the intuition, consider the case where consumption as young – and thus saving – happens to be unaffected by an increase in the interest rate. Even in this case, consumption as old, c_{2t+1}, is automatically increased (in view of the higher income as old through the higher rate of return on the unchanged saving); and the marginal utility of c_{2t+1} is thus decreased in response to a higher interest rate. The point is that this outcome can only be optimal if the elasticity of marginal utility of c_{2t+1} is of "medium" size. A very high absolute elasticity of marginal utility of c_{2t+1} would result in a sharp decline in marginal utility – so sharp that not much would be lost by dampening the automatic rise in c_{2t+1} and instead increase c_{1t}, thus reducing saving. On the other hand, a very low elasticity of marginal utility of c_{2t+1} would result in only a small decline in marginal utility – so small that it is beneficial to take advantage of the higher rate of return and save *more*, thus accepting a first-period utility loss brought about by a lower c_{1t}.

We see from (3.12) that an absolute elasticity of marginal utility equal to exactly one is the case leading to the interest rate being *neutral* vis-a-vis the saving of the young. What is the intuition behind this? Neutrality vis-a-vis the saving of the young of a rise in the interest rate requires that c_{1t} remains unchanged since $c_{1t} = w_t - s_t$. In turn this requires that the marginal utility, $u'(c_{2t+1})$, on the right-hand side of (3.8) falls by the same percentage as $1 + r_{t+1}$ rises. At the same time, the budget (3.5) as old tells us that c_{2t+1} has to rise by the same percentage as $1 + r_{t+1}$ if s_t remains unchanged. Altogether we thus need that $u'(c_{2t+1})$ falls by the same percentage as c_{2t+1} rises. But this requires that the absolute elasticity of $u'(c_{2t+1})$ with respect to c_{2t+1} is exactly one.

The elasticity of marginal utility, also called the marginal utility flexibility, will generally depend on the level of consumption, as implicit in the notation $\theta(c_{2t+1})$. There exists a popular special case, however, where the elasticity of marginal utility is constant.

EXAMPLE 1 *The CRRA utility function.* If we impose the requirement that $u(c)$ should have an absolute elasticity of marginal utility of consumption equal to a constant $\theta > 0$, then one can show (see Appendix A) that the utility function must, up to a positive linear transformation, be of the CRRA form:

$$u(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta}, & \text{when } \theta \neq 1, \\ \ln c, & \text{when } \theta = 1. \end{cases}, \tag{3.14}$$

It may seem odd that when $\theta \neq 1$, we subtract the constant $1/(1-\theta)$ from $c^{1-\theta}/(1-\theta)$. Adding or subtracting a constant from a utility function does not affect the marginal rate of substitution and consequently not behavior. So we could do without this constant, but its occurrence in (3.14) has two formal advantages. One is that in contrast to $c^{1-\theta}/(1-\theta)$, the expression $(c^{1-\theta}-1)/(1-\theta)$ can be interpreted as valid even for $\theta = 1$, namely as identical to $\ln c$. This is because $(c^{1-\theta}-1)/(1-\theta) \to \ln c$ for $\theta \to 1$ (by L'Hôpital's rule for "0/0"). Another advantage is that the kinship between the different members, indexed by θ, of the CRRA family becomes more transparent. Indeed, by defining $u(c)$ as in (3.14), all graphs of $u(c)$ will go through the same point as the log function, namely $(1, 0)$, cf. Fig. 3.2. The equation (3.14) thus displays the CRRA utility function in *normalized form*.

The higher is θ, the more "curvature" does the corresponding curve in Fig. 3.2 have. In turn, more "curvature" reflects a higher incentive to smooth consumption across time. The reason is that a large curvature means that the marginal utility will drop sharply if consumption rises and will increase sharply if consumption falls. Consequently, not much utility is lost by lowering consumption when it

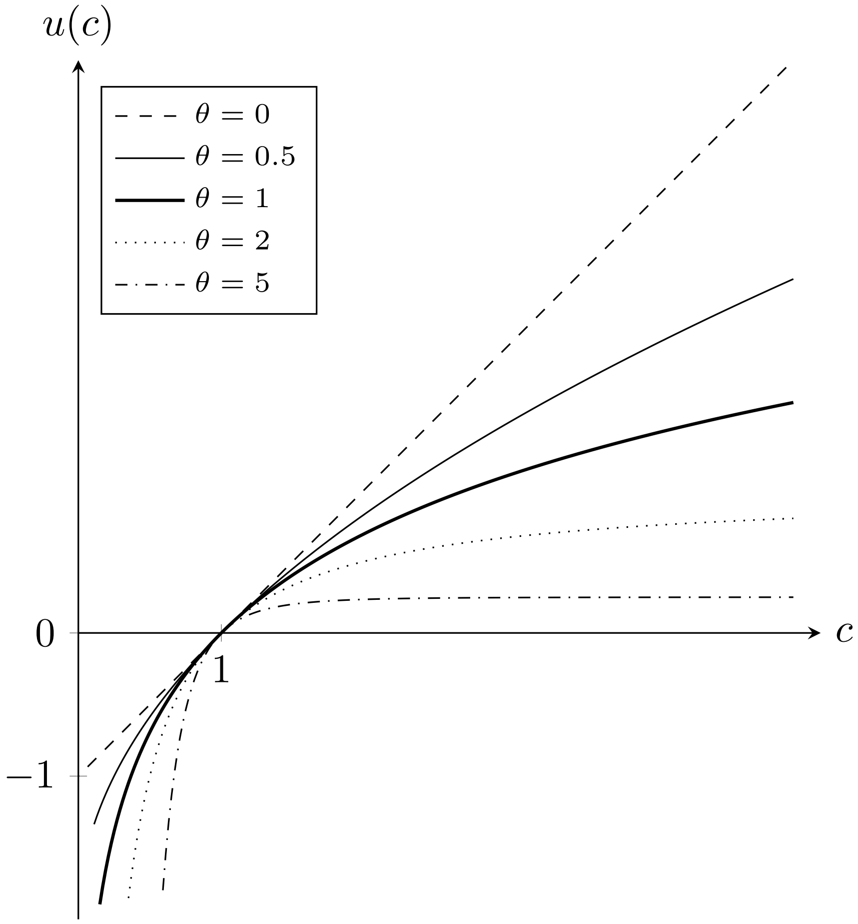


Figure 3.2: The CRRA family of utility functions.

is relatively high but there is a lot of utility to be gained by raising it when it is relatively low. So the curvature θ indicates the degree of *aversion towards variation in consumption*. Or we may say that θ indicates the strength of the *preference for consumption smoothing*.[9] □

Suppose the period utility is of CRRA form as given in (3.14). (FOC) then yields an explicit solution for the saving of the young:

$$s_t = \frac{1}{1 + (1 + \rho)(\frac{1+r_{t+1}}{1+\rho})^{\frac{\theta-1}{\theta}}} w_t. \tag{3.15}$$

We see that the signs of $\partial s_t/\partial w_t$ and $\partial s_t/\partial r_{t+1}$ shown in (3.11) and (3.13), re-

[9]The name CRRA is a shorthand for *Constant Relative Risk Aversion* and comes from the theory of behavior under uncertainty. Also in that theory does the CRRA function constitute an important benchmark case. And θ is in that context called the *degree of relative risk aversion*.

spectively, are confirmed. Moreover, the saving of the young is in this special case proportional to income with a factor of proportionality that depends on the interest rate (as long as $\theta \neq 1$). But in the general case the saving-income ratio depends also on the income level.

A major part of the attempts at empirically estimating θ suggests that $\theta > 1$. Based on U.S. data, Hall (1988) provides estimates above 5, while Attanasio and Weber (1993) suggest $1.25 \leq \theta \leq 3.33$. For Japanese data Okubo (2011) suggests $2.5 \leq \theta \leq 5.0$. As these studies relate to much shorter time intervals than the implicit time horizon of about 2×30 years in the Diamond model, we should be cautious. But if the estimates *were* valid also to that model, we should expect the income effect on current consumption of an increase in the interest rate to dominate the substitution effect, thus implying $s_r < 0$ *as long as there is no wealth effect* of a rise in the interest rate.

When the elasticity of marginal utility of consumption is a constant, θ, its inverse, $1/\theta$, equals the *elasticity of intertemporal substitution* in consumption. This concept refers to the willingness to substitute consumption over time when the interest rate changes. Under certain conditions the elasticity of intertemporal substitution reflects the elasticity of the ratio c_{2t+1}/c_{1t} with respect to $1 + r_{t+1}$ when we move along a given indifference curve. The next subsection, which can be omitted in a first reading, goes more into detail with the concept.

Digression: The elasticity of intertemporal substitution*

Consider a two-period consumption problem like the one above. Fig. 3.3 depicts a particular indifference curve, $u(c_1) + (1 + \rho)^{-1}u(c_2) = \bar{U}$. At a given point, (c_1, c_2), on the curve, the marginal rate of substitution of period-2 consumption for period-1 consumption, MRS, is given by

$$MRS = -\frac{dc_2}{dc_1}|_{U=\bar{U}} \ ,$$

that is, MRS at the point (c_1, c_2) is the absolute value of the slope of the tangent to the indifference curve at that point.[10] Under the "normal" assumption of "strictly convex preferences" (as for instance in the Diamond model), MRS is rising along the curve when c_1 decreases (and thereby c_2 increases). Conversely, we can let MRS be the independent variable and consider the corresponding point on the indifference curve, and thereby the ratio c_2/c_1, as a function of MRS. If we raise MRS along the indifference curve, the corresponding value of the ratio c_2/c_1 will also rise.

[10] When the meaning is clear from the context, to save notation we just write MRS instead of the more precise $MRS_{c_2 c_1}$.

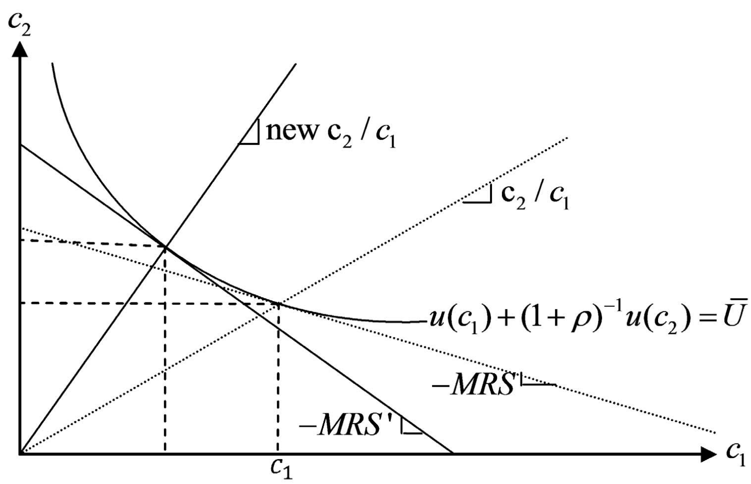


Figure 3.3: Substitution of period 2-consumption for period 1-consumption as MRS increases to MRS'.

The *elasticity of intertemporal substitution in consumption* at a given point is defined as the elasticity of the ratio c_2/c_1 with respect to the marginal rate of substitution of c_2 for c_1, when we move along the indifference curve through the point (c_1, c_2). Letting the elasticity with respect to x of a differentiable function $f(x)$ be denoted $\mathrm{E}\ell_x f(x)$, the elasticity of intertemporal substitution in consumption can be written

$$\mathrm{E}\ell_{MRS}\frac{c_2}{c_1} = \frac{MRS}{c_2/c_1}\frac{d\,(c_2/c_1)}{dMRS}\,|_{U=\bar{U}} \approx \frac{\frac{\Delta(c_2/c_1)}{c_2/c_1}}{\frac{\Delta MRS}{MRS}},$$

where the approximation is valid for a "small" increase, ΔMRS, in MRS.

A more concrete understanding is obtained when we take into account that in the consumer's optimal plan, MRS equals the ratio of the discounted prices of good 1 and good 2, that is, the ratio $1/(1/(1+r))$ given in (3.7). Indeed, from (3.10) and (3.9), omitting the time indices, we have

$$MRS = -\frac{dc_2}{dc_1}\,|_{U=\bar{U}} = \frac{u'(c_1)}{(1+\rho)^{-1}u'(c_2)} = 1 + r \equiv R. \tag{3.16}$$

Letting $\sigma(c_1, c_2)$ denote the elasticity of intertemporal substitution, evaluated at the point (c_1, c_2), we then have

$$\sigma(c_1, c_2) = \frac{R}{c_2/c_1}\frac{d\,(c_2/c_1)}{dR}\,|_{U=\bar{U}} \approx \frac{\frac{\Delta(c_2/c_1)}{c_2/c_1}}{\frac{\Delta R}{R}}. \tag{3.17}$$

Consequently, the elasticity of intertemporal substitution can here be interpreted as the approximate percentage increase in the consumption ratio, c_2/c_1, triggered by a one percentage increase in the inverse price ratio, holding the utility level unchanged.[11]

Given $u(c)$, we let $\theta(c)$ be the absolute elasticity of marginal utility of consumption, i.e., $\theta(c) \equiv -cu''(c)/u'(c)$. As shown in Appendix B, we then find the elasticity of intertemporal substitution to be

$$\sigma(c_1, c_2) = \frac{c_2 + Rc_1}{c_2\theta(c_1) + Rc_1\theta(c_2)}. \tag{3.18}$$

We see that if $u(c)$ belongs to the CRRA class and thereby $\theta(c_1) = \theta(c_2) = \theta$, then $\sigma(c_1, c_2) = 1/\theta$. In this case (as well as whenever $c_1 = c_2$) the elasticity of marginal utility and the elasticity of intertemporal substitution are simply the inverse of each other.

3.4 Production

Output is homogeneous and can be used for consumption as well as investment in physical capital. The capital stock is thereby just accumulated non-consumed output. We may imagine a "corn economy" where output is corn, part of which is eaten (flour) while the remainder is accumulated as capital (seed corn).

The specification of technology and production conditions follows the simple competitive one-sector setup discussed in Chapter 2. Although the Diamond model is a long-run model, we shall in this chapter for simplicity ignore technological change.

The representative firm

There is a representative firm with a neoclassical production function and constant returns to scale (CRS). Omitting the time argument t when not needed for clarity, we have

$$Y = F(K, L) = LF(k, 1) \equiv Lf(k), \qquad f' > 0, f'' < 0, \tag{3.19}$$

where Y is output (GNP) per period, K is capital input, L is labor input, and $k \equiv K/L$ is the capital-labor ratio. The derived function, f, is the production

[11] This characterization is equivalent to saying that the elasticity of substitution between two consumption goods indicates the approximate percentage *decrease* in the ratio of the chosen quantities of the goods (when moving along a given indifference curve) induced by a one-percentage *increase* in the *corresponding* price ratio.

function in intensive form. Capital installation and other adjustment costs are ignored. With $\hat{r}$ denoting the rental rate for capital, profit is $\Pi \equiv F(K, L) - \hat{r}K - wL$. The firm maximizes Π under perfect competition. This gives, first, $\partial\Pi/\partial K = F_K(K, L) - \hat{r} = 0$, that is,

$$F_K(K, L) = \frac{\partial [Lf(k)]}{\partial K} = f'(k) = \hat{r}. \tag{3.20}$$

Second, $\partial\Pi/\partial L = F_L(K, L) - w = 0$, that is,

$$F_L(K, L) = \frac{\partial [Lf(k)]}{\partial L} = f(k) - kf'(k) = w. \tag{3.21}$$

We may interpret these two conditions as saying that the firm will in every period use capital and labor up to the point where the marginal productivities of K and L, respectively, given the chosen input of the other factor, are equal to the respective factor prices from the market. Such an intuitive formulation does not take us far, however. Indeed, because of CRS there may be infinitely many pairs (K, L), if any, that satisfy (3.20) and (3.21). What we can definitely ascertain is that in view of $f'' < 0$, a $k > 0$ satisfying (3.20) will be *unique*.[12] Let us call it the *desired capital-labor ratio* and recognize that at this stage the separate factor inputs, K and L, are indeterminate. While (3.20) and (3.21) are just first-order conditions for the profit maximizing representative firm, to get further we have to appeal to *equilibrium* in the factor markets.

Factor prices in equilibrium

Let the aggregate demand for capital services and labor services be denoted K^d and L^d, respectively. Clearing in factor markets in period t implies

$$K_t{}^d = K_t, \tag{3.22}$$

$$L_t{}^d = L_t = L_0(1+n)^t, \tag{3.23}$$

where K_t is the aggregate supply of capital services and L_t the aggregate supply of labor services. As was called attention to in Chapter 1, unless otherwise specified it is understood that the rate of utilization of each production factor is constant over time and normalized to one. So the quantity K_t will at one and the same time measure both the capital input, a flow, and the available capital stock. Similarly,

[12] It might seem that k is overdetermined because we have two equations, (3.20) and (3.21), but only one unknown. This reminds us that for *arbitrary* factor prices, $\hat{r}$ and w, there will generally *not* exist a k satisfying both (3.20) and (3.21). But in equilibrium the factor prices faced by the firm are not arbitrary. They are equilibrium prices, i.e., they are adjusted so that (3.20) and (3.21) become consistent.

the quantity L_t will at one and the same time measure both the labor input, a flow, and the size of the labor force as a stock (= the number of young people).

The aggregate input demands, K^d and L^d, are linked through the desired capital-labor ratio, k^d. In equilibrium we have $K_t^d/L_t^d = k_t{}^d = K_t/L_t \equiv k_t$, by (3.22) and (3.23). The k in (3.20) and (3.21) can thereby be identified with the ratio of the stock supplies, $k_t \equiv K_t/L_t > 0$, which is a predetermined variable. Interpreted this way, (3.20) and (3.21) *determine* the equilibrium factor prices $\hat{r}_t$ and w_t in each period. In view of the no-arbitrage condition (3.2), the real interest rate satisfies $r_t = \hat{r}_t - \delta$, where δ is the capital depreciation rate, $0 \leq \delta \leq 1$. So in equilibrium we end up with

$$r_t = f'(k_t) - \delta \equiv r(k_t) \qquad (r'(k_t) = f''(k_t) < 0), \tag{3.24}$$

$$w_t = f(k_t) - k_t f'(k_t) \equiv w(k_t) \qquad (w'(k_t) = -k_t f''(k_t) > 0), \tag{3.25}$$

where causality is from the right to the left in the two equations. In line with our general perception of perfect competition, cf. Section 2.4 of Chapter 2, it is understood that the factor prices, $\hat{r}_t$ and w_t, adjust quickly to the market-clearing levels.

Technical Remark. In these formulas it is understood that $L > 0$, but we may allow $K = 0$, i.e., $k = 0$. In case $f'(0)$ is not immediately well-defined, we interpret $f'(0)$ as $\lim_{k\to 0^+} f'(k)$ if this limit exists. If it does not, it must be because we are in a situation where $\lim_{k\to 0^+} f'(k) = \infty$, since $f''(k) < 0$ (an example is the Cobb-Douglas function, $f(k) = Ak^\alpha$, $0 < \alpha < 1$, where $\lim_{k\to 0^+} f'(k)$ $= \lim_{k\to 0^+} A\alpha k^{\alpha-1} = +\infty$). In this situation we simply include $+\infty$ in the range of $r(k)$ and define $r(0) \cdot 0 \equiv \lim_{k\to 0^+}(f'(k) - \delta)k = 0$, where the last equality comes from the general property of a neoclassical CRS production function that $\lim_{k\to 0^+} kf'(k) = 0$, cf. (2.18) of Chapter 2. Letting $r(0) \cdot 0 = 0$ also fits well with intuition since, when $k = 0$, nobody receives capital income anyway. Note that since $\delta \in [0, 1]$, $r(k) > -1$ for all $k \geq 0$. What about $w(0)$? We interpret $w(0)$ as $\lim_{k\to 0} w(k)$. From (2.18) of Chapter 2 we have that $\lim_{k\to 0^+} w(k) = f(0)$ $\equiv F(0, 1) \geq 0$. If capital is essential, $F(0, 1) = 0$. Otherwise, $F(0, 1) > 0$. Finally, since $w' > 0$, we have, for $k > 0$, $w(k) > 0$ as also noted in Chapter 2. □

To fix ideas we have assumed that households (here the old) own the physical capital and rent it out to the firms. In view of perfect competition and constant returns to scale, pure profit is nil in equilibrium. As long as the model ignores uncertainty and capital installation costs, the results will be unaffected if instead we let the firms themselves own the physical capital and finance capital investment by issuing bonds and shares. These bonds and shares would then be accumulated by the households and constitute their financial wealth instead of the capital goods themselves. The equilibrium rate of return, r_t, would be the same.

3.5 The dynamic path of the economy

As in microeconomic general equilibrium theory, it is important to distinguish between the set of technically feasible allocations and an allocation brought about, within this set, by a specific economic institution (the rules of the game). The economic institution assumed by the Diamond model is the private-ownership perfect-competition market institution.

We shall in the next subsections introduce three different concepts concerning allocations over time in this economy. The three concepts are: *technically feasible paths*, *temporary equilibrium*, and *equilibrium path*. These concepts are mutually related in the sense that there is a whole *set* of technically feasible paths, *within which* there may exist a unique equilibrium path, which in turn is a sequence of states that have certain properties, including the temporary equilibrium property.

3.5.1 Technically feasible paths

When we speak of technically feasible paths, the focus is merely upon what is feasible from the point of view of the given technology as such and available initial resources. That is, we disregard the agents' preferences, their choices given the constraints, their interactions in markets, the market forces etc.

The technology is represented by (3.19) and there are two exogenous resources, the labor force, $L_t = L_0(1+n)^t$, and the initial capital stock, K_0. From national income accounting aggregate consumption can be written $C_t \equiv Y_t - S_t = F(K_t, L_t) - S_t$, where S_t denotes aggregate gross saving, and where we have inserted (3.19). In a closed economy aggregate gross saving equals (ex post) aggregate gross investment, $K_{t+1} - K_t + \delta K_t$. So

$$C_t = F(K_t, L_t) - (K_{t+1} - K_t + \delta K_t). \tag{3.26}$$

Let c_t denote aggregate consumption per unit of labor in period t, i.e.,

$$c_t \equiv \frac{C_t}{L_t} = \frac{c_{1t}L_t + c_{2t}L_{t-1}}{L_t} = c_{1t} + \frac{c_{2t}}{1+n}.$$

Combining this with (3.26) and using the definitions of k and $f(k)$, we obtain the dynamic resource constraint of the economy:

$$c_{1t} + \frac{c_{2t}}{1+n} = f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}. \tag{3.27}$$

DEFINITION 1 Let $\bar{k}_0 \geq 0$ be the historically given initial ratio of available capital and labor. Let the path $\{(k_t, c_{1t}, c_{2t})\}_{t=0}^{\infty}$ have nonnegative k_t, c_{1t}, and c_{2t}

for all $t = 0, 1, 2, \ldots$. The path is called *technically feasible* if it has $k_0 = \bar{k}_0$ and satisfies (3.27) for all $t = 0, 1, 2, \ldots$.

The next subsections consider how, for given household preferences, the private-ownership market institution with profit-maximizing firms under perfect competition generates a *selection* within the set of technically feasible paths. A member of this selection (which may but need not have just one member) is called an *equilibrium path.* It constitutes a sequence of states with certain properties, one of which is the temporary equilibrium property.

3.5.2 A temporary equilibrium

Standing in a given period, it is natural to think of next period's interest rate as an *expected* interest rate that provisionally can deviate from the ex post realized one. We let $r^e_{t+1} > -1$ denote the expected real interest rate of period $t+1$ as seen from period t.

Essentially, by a temporary equilibrium in period t is meant a state where for a *given* r^e_{t+1}, all markets clear in the period. There are three markets, namely two factor markets and a market for produced goods. We have already described the two factor markets. In the market for produced goods the representative firm supplies the amount $Y^s_t = F(K^d_t, L^d_t)$ in period t. The demand side in this market has two components, consumption, C_t, and *gross* investment, I_t. Equilibrium in the goods market requires that demand equals supply, i.e.,

$$C_t + I_t = c_{1t}L_t + c_{2t}L_{t-1} + I_t = Y^s_t = F(K^d_t, L^d_t), \tag{3.28}$$

where consumption by the young and old, c_{1t} and c_{2t}, respectively, were determined in Section 3.

By definition, aggregate gross investment equals aggregate net investment, I^N_t, plus capital depreciation, i.e.,

$$I_t = I^N_t + \delta K_t \equiv I^N_{1t} + I^N_{2t} + \delta K_t \equiv S^N_{1t} + S^N_{2t} + \delta K_t = s_t L_t + (-K_t) + \delta K_t. \tag{3.29}$$

The first equality follows from the definition of net investment and the assumption that capital depreciation equals δK_t. Next comes an identity reflecting that aggregate net investment is the sum of net investment by the young and net investment by the old. In turn, saving in this model is directly an act of acquiring capital goods. So the net investment by the young, I^N_{1t}, and the old, I^N_{2t}, are identical to their net saving, S^N_{1t} and S^N_{2t}, respectively. As we have shown, the net saving by the young in the model equals $s_t L_t$. And the net saving by the old is negative and equals $-K_t$. Indeed, because they have no bequest motive, the old consume all they have and leave nothing as bequests. Hence, the young in any period enter the period with no non-human wealth. Consequently, any non-human wealth existing at the beginning of a period must belong to the old in that period and be the result of their saving as young in the previous period. As K_t constitutes the aggregate non-human wealth in our closed economy at the beginning of period t, we therefore have

$$s_{t-1}L_{t-1} = K_t. \tag{3.30}$$

Recalling that the net saving of any group is by definition the same as the increase in its non-human wealth, the net saving of the old in period t is $-K_t$. Aggregate net saving in the economy is thus $s_tL_t + (-K_t)$, and (3.29).is thereby explained.

DEFINITION 2 Let a given period t have capital stock $K_t \geq 0$, labor supply $L_t > 0$, and hence capital-labor ratio $k_t = K_t/L_t$. Let the expected real interest rate be given as $r^e_{t+1} > -1$. And let the functions $s(w_t, r^e_{t+1})$, $w(k_t)$, and $r(k_t)$ be defined as in Lemma 1, (3.25), and (3.24), respectively. Then a *temporary equilibrium* in period t is a state $(k_t, c_{1t}, c_{2t}, w_t, r_t)$ of the economy such that (3.22), (3.23), (3.28), and (3.29) hold (i.e., all markets clear) for $c_{1t} = w_t - s_t$, c_{2t} $= (k_t + r(k_t)k_t)(1 + n)$, where $w_t = w(k_t) > 0$ and $s_t = s(w_t, r^e_{t+1})$.

The reason for the requirement $w_t > 0$ in the definition is that if $w_t = 0$, people would have nothing to live on as young and nothing to save from for retirement. The system would not be economically viable in this case. With regard to the equation for c_{2t} in the definition, note that (3.30) gives $s_{t-1} = K_t/L_{t-1} = (K_t/L_t)(L_t/L_{t-1}) = k_t(1 + n)$, which is the wealth of each old at the beginning of period t. Substituting into $c_{2t} = (1 + r_t)s_{t-1}$, we get $c_{2t} = (1 + r_t)k_t(1 + n)$, which can also be written $c_{2t} = (k_t + r_tk_t)(1 + n)$. This last way of writing c_{2t} has the advantage of being applicable even if $k_t = 0$, cf. Technical Remark in Section 3.4. The remaining conditions for a temporary equilibrium are self-explanatory.

PROPOSITION 1 Suppose the No Fast Assumption (A1) applies. Consider a given period t with a given $k_t \geq 0$. Then for any $r^e_{t+1} > -1$,
(i) if $k_t > 0$, there exists a temporary equilibrium, $(k_t, c_{1t}, c_{2t}, w_t, r_t)$, and c_{1t} and c_{2t} are positive;
(ii) if $k_t = 0$, a temporary equilibrium exists if and only if capital is not essential; in that case, $w_t = w(k_t) = w(0) = f(0) > 0$ and c_{1t} and s_t are positive (while $c_{2t} = 0$);
(iii) whenever a temporary equilibrium exists, it is unique.

Proof. We begin with (iii). That there is at most one temporary equilibrium is immediately obvious since w_t and r_t are functions of the given k_t : $w_t = w(k_t)$ and $r_t = r(k_t)$. And given w_t, r_t, and r^e_{t+1}, c_{1t} and c_{2t} are uniquely determined.

(i) Let $k_t > 0$. Then, by (3.25), $w(k_t) > 0$. We claim that the state $(k_t, c_{1t}, c_{2t}, w_t, r_t)$, with $w_t = w(k_t)$, $r_t = r(k_t)$, $c_{1t} = w(k_t) - s(w(k_t), r_{t+1}^e)$, and $c_{2t} = (1+r(k_t))k_t(1+n)$, is a temporary equilibrium. Indeed, Section 3.4 showed that the factor prices $w_t = w(k_t)$ and $r_t = r(k_t)$ are consistent with clearing in the factor markets in period t. Given that these markets clear (by price adjustment), it follows by Walras' law (see Appendix C) that also the third market, the goods market, clears in period t. So all criteria in Definition 2 are satisfied. That $c_{1t} > 0$ follows from $w(k_t) > 0$ and the No Fast Assumption (A1), in view of Lemma 1. That $c_{2t} > 0$ follows from $c_{2t} = (1 + r(k_t))k_t(1 + n)$ when $k_t > 0$, since $r(k_t) > -1$ always.

(ii) Let $k_t = 0$. Suppose $f(0) > 0$. Then, by Technical Remark in Section 3.4, $w_t = w(0) = f(0) > 0$ and $c_{1t} = w_t - s(w_t, r_{t+1}^e)$ is well-defined, positive, and less than w_t, in view of Lemma 1; so $s_t = s(w_t, r_{t+1}^e) > 0$. The old in period 0 will starve since $c_{2t} = (0 + 0)(1 + n)$, in view of $r(0) \cdot 0 = 0$, cf. Technical Remark in Section 3.4. Even though this is a bad situation for the old, it is consistent with the criteria in Definition 2. On the other hand, if $f(0) = 0$, we get $w_t = f(0) = 0$, which violates one of the criteria in Definition 2. □

Point (ii) of the proposition says that a temporary equilibrium *may* exist even in a period where $k = 0$. The old in this period will starve and not survive. But if capital is not essential, the young get positive labor income out of which they will save a part for their old age and be able to maintain life also next period which will be endowed with positive capital. Then, by our assumptions the economy is viable forever.[13]

Generally, the term "equilibrium" is used to denote a state of "rest", often just "temporary rest". The temporary equilibrium in the present model is an example of a state of "temporary rest" in the following sense: (a) the agents optimize, given their expectations and the constraints they face; and (b) the aggregate demands and supplies in the given period are mutually consistent, i.e., markets clear. The qualification "temporary" is motivated by two features. First, in the next period the conditioning circumstances may be different, possibly as a direct consequence of the currently chosen actions. Second, the given expectations may turn out wrong.

3.5.3 An equilibrium path

The concept of an equilibrium path, also called an intertemporal equilibrium, requires more conditions satisfied. The concept refers to a sequence of temporary equilibria such that *expectations* of the agents are *fulfilled* in every period:

[13] For simplicity, the model ignores that in practice a certain minimum per capita consumption level (the subsistence minimum) is needed for viability.

DEFINITION 3 An *equilibrium path* is a technically feasible path $\{(k_t, c_{1t}, c_{2t})\}_{t=0}^{\infty}$ such that for $t = 0, 1, 2, \ldots$, the state $(k_t, c_{1t}, c_{2t}, w_t, r_t)$ is a temporary equilibrium with $r_{t+1}^e = r(k_{t+1})$.

To characterize such a path, we forward (3.30) one period and rearrange so as to get

$$K_{t+1} = s_t L_t. \tag{3.31}$$

Since $K_{t+1} \equiv k_{t+1} L_{t+1} = k_{t+1} L_t (1 + n)$, this can be written

$$k_{t+1} = \frac{s(w(k_t), r(k_{t+1}))}{1 + n}, \tag{3.32}$$

using that $s_t = s(w_t, r_{t+1}^e)$, $w_t = w(k_t)$, and $r_{t+1}^e = r_{t+1} = r(k_{t+1})$ in a sequence of temporary equilibria with fulfilled expectations. Equation (3.32) is a first-order difference equation, known as the *fundamental difference equation* or the *law of motion* of the Diamond model.

PROPOSITION 2 Suppose the No Fast Assumption (A1) applies. Then,
(i) for any $k_0 > 0$ there exists at least one equilibrium path;
(ii) if $k_0 = 0$, an equilibrium path exists if and only if $f(0) > 0$ (i.e., capital not essential);
(iii) in any case, an equilibrium path has a positive real wage in all periods and positive capital in all periods except possibly the first;
(iv) an equilibrium path satisfies the first-order difference equation (3.32).

Proof. (i) and (ii): see Appendix D. (iii) For a given t, let $k_t \geq 0$. Then, since an equilibrium path is a sequence of temporary equilibria, we have, from Proposition 1, $w_t = w(k_t) > 0$ and $s_t = s(w(k_t), r_{t+1}^e)$, where $r_{t+1}^e = r(k_{t+1})$. Hence, by Lemma 1, $s(w(k_t), r_{t+1}^e) > 0$, which implies $k_{t+1} > 0$, in view of (3.32). This shows that only for $t = 0$ is $k_t = 0$ possible along an equilibrium path. (iv) This was shown in the text above. □

The formal proofs of point (i) and (ii) of the proposition are quite technical and placed in the appendix. But the graphs in the ensuing figures 3.4-3.7 provide an intuitive verification. The "only if" part of point (ii) reflects the not very surprising fact that *if* capital were an essential production factor, no capital "now" would imply no income "now", hence no saving and investment and thus no capital in the next period and so on. On the other hand, the "if" part of point (ii) says that when capital is not essential, an equilibrium path can set off even from an initial period with no capital. Then point (iii) adds that an equilibrium path will have positive capital in all subsequent periods. Finally, as to point (iv), note that the fundamental difference equation, (3.32), rests on equation (3.31). Recall from the previous subsection that the economic logic behind this key equation is that since capital is the only non-human asset in the economy and the young are born without any inheritance, the aggregate capital stock at the beginning of period $t + 1$ *must* be owned by the old generation in that period. It must thereby equal the aggregate saving these people had in the previous period where they were young.

The transition diagram

To be able to further characterize equilibrium paths, we construct a transition diagram in the (k_t, k_{t+1}) plane. The *transition curve* is defined as the set of points (k_t, k_{t+1}) satisfying (3.32). Its form and position depends on the households' preferences and the firms' technology. Fig. 3.4 shows one *possible*, but far from necessary configuration of this curve. A complicating circumstance is that the equation (3.32) has k_{t+1} on both sides. Sometimes we are able to solve the equation explicitly for k_{t+1} as a function of k_t, but sometimes we can do so only implicitly. What is even worse is that there are cases where k_{t+1} is not unique for a given k_t. We will proceed step by step.

First, what can we say about the *slope* of the transition curve? In general, a point on the transition curve has the property that at least in a small neighborhood of this point, the equation (3.32) will define k_{t+1} as an implicit function of k_t.[14] Taking the total derivative with respect to k_t on both sides of (3.32), we get

$$\frac{dk_{t+1}}{dk_t} = \frac{1}{1+n}\left(s_w w'(k_t) + s_r r'(k_{t+1})\frac{dk_{t+1}}{dk_t}\right). \tag{3.33}$$

By ordering, the slope of the transition curve within this small neighborhood can be written

$$\frac{dk_{t+1}}{dk_t} = \frac{s_w\left(w\left(k_t\right), r\left(k_{t+1}\right)\right) w'\left(k_t\right)}{1+n-s_r\left(w\left(k_t\right), r\left(k_{t+1}\right)\right) r'\left(k_{t+1}\right)}, \tag{3.34}$$

when the denominator,

$$\mathcal{D}(k_t, k_{t+1}) \equiv 1 + n - s_r(w(k_t), r(k_{t+1}))r'\left(k_{t+1}\right),$$

differs from nil.

In view of $s_w > 0$ and $w'(k_t) = -k_t\ f''(k_t) > 0$, the numerator in (3.34) is always positive and we have

$$\frac{dk_{t+1}}{dk_t} \gtrless 0 \text{ for } s_r(w(k_t), r(k_{t+1})) \gtrless \frac{1+n}{r'\left(k_{t+1}\right)},$$

respectively, since $r'\left(k_{t+1}\right) = f''(k_{t+1}) < 0$.

[14] An exception occurs if the denominator in (3.34) below vanishes.

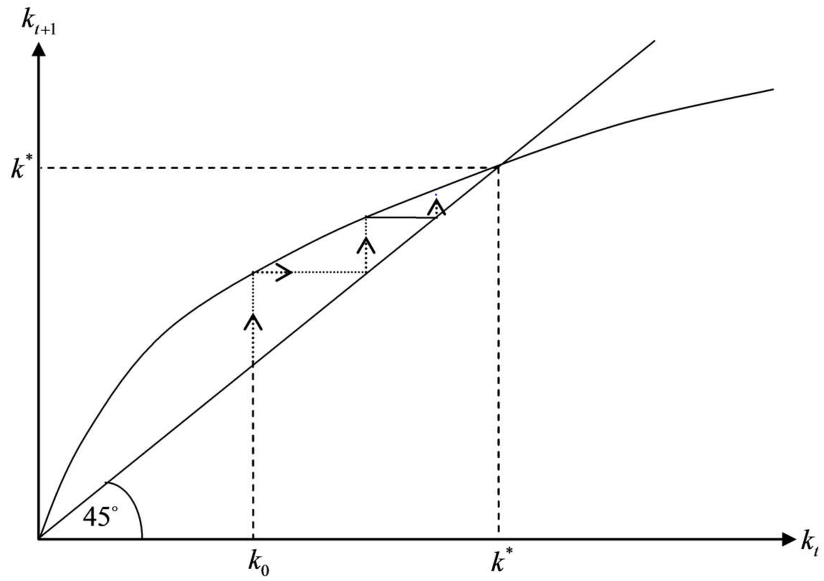


Figure 3.4: Transition curve and the resulting dynamics in the log-utility Cobb-Douglas case, cf. Example 2.

It follows that the transition curve is universally upward-sloping if and only if $s_r(w(k_t), r(k_{t+1})) > (1+n)/r'(k_{t+1})$ everywhere along the transition curve. The intuition behind this becomes visible by rewriting (3.34) in terms of small changes in k_t and k_{t+1}. Since $\Delta k_{t+1}/\Delta k_t \approx dk_{t+1}/dk_t$ for Δk_t "small", (3.34) implies

$$[1 + n - s_r(\cdot)\, r'(k_{t+1})]\, \Delta k_{t+1} \approx s_w(\cdot)\ \ w'(k_t)\Delta k_t. \tag{*}$$

Let $\Delta k_t > 0$. This rise in k_t will always raise wage income and, via the resulting rise in s_t, raise k_{t+1}, everything else equal. Everything else is *not* equal, however, since a rise in k_{t+1} implies a fall in the rate of interest. There are four cases to consider:

Case 1: $s_r(\cdot) = 0$. Then there is no feedback effect from the fall in the rate of interest. So the tendency to a rise in k_{t+1} is neither offset nor fortified.

Case 2: $s_r(\cdot) > 0$. Then the tendency to a rise in k_{t+1} will be partly offset through the *dampening* effect on saving resulting from the fall in the interest rate. This negative feedback can not fully offset the tendency to a rise in k_{t+1}. The reason is that the negative feedback on the saving of the young will only be there *if* the interest rate falls in the first place. We cannot in a period have both a *fall* in the interest rate triggering lower saving *and* a *rise* in the interest rate (via a lower k_{t+1}) *because* of the lower saving. So a *sufficient* condition for a universally upward-sloping transition curve is that the saving of the young is a non-decreasing function of the interest rate.

Case 3: $(1+n)/r'(k_{t+1}) < s_r(\cdot) < 0$. Then the tendency to a rise in k_{t+1} will be fortified through the *stimulating* effect on saving resulting from the fall in the interest rate.

Case 4: $s_r(\cdot) < (1+n)/r'(k_{t+1}) < 0$. Then the expression in brackets on the left-hand side of (*) is negative and requires therefore that $\Delta k_{t+1} < 0$ in order to comply with the positive right-hand side. This is a situation where self-fulfilling expectations operate, a case to which we return. We shall explore this case in the next sub-section.

Another feature of the transition curve is the following:

LEMMA 2 *(the transition curve is nowhere flat)* For all $k_t > 0$ such that the denominator, $\mathcal{D}(k_t, k_{t+1})$, in (3.34) differs from nil, we have $dk_{t+1}/dk_t \neq 0$.

Proof. Since $s_w > 0$ and $w'(k_t) > 0$ always, the numerator in (3.34) is always positive. □

The implication is that no part of the transition curve can be horizontal.[15]

When the transition curve crosses the 45° degree line for some $k_t > 0$, as in the example in Fig. 3.4, we have a steady state at this k_t. Formally:

DEFINITION 4 An equilibrium path $\{(k_t, c_{1t}, c_{2t})\}_{t=0}^{\infty}$ is in a *steady state* with capital-labor ratio $k^* > 0$ if the fundamental difference equation, (3.32), is satisfied with k_t as well as k_{t+1} replaced by k^*.

This exemplifies the notion of a steady state as a stationary point in a dynamic process. Some economists use the term "dynamic equilibrium" instead of "steady state". As in this book the term "equilibrium" refers to situations where the constraints and decided actions of the market participants are mutually compatible, an economy can be in "equilibrium" without being in a steady state. A steady state is seen as a *special* sequence of temporary equilibria with fulfilled expectations, namely one with the property that the endogenous variable, here k, entering the fundamental difference equation does not change over time.

EXAMPLE 2 (the log utility Cobb-Douglas case) Let $u(c) = \ln c$ and $Y = AK^{\alpha}L^{1-\alpha}$, where $A > 0$ and $0 < \alpha < 1$. Since $u(c) = \ln c$ is the case $\theta = 1$ in Example 1, by (3.15) we have $s_r = 0$. Indeed, with logarithmic utility the substitution and income effects on s_t of a rise in the interest rate offset each other; and, as discussed above, in the Diamond model there can be no wealth effect of a rise in r_{t+1}. Further, the equation (3.32) reduces to a transition *function,*

$$k_{t+1} = \frac{(1-\alpha)Ak_t^{\alpha}}{(1+n)(2+\rho)}. \tag{3.35}$$

[15] This would not generally hold if the utility function were not time-separable.

The associated transition curve is shown in Fig. 3.4 and there is for $k_0 > 0$ both a unique equilibrium path and a unique steady state with capital-labor ratio

$$k^* = \left(\frac{(1-\alpha)A}{(2+\rho)(1+n)} \right)^{1/(1-\alpha)} > 0.$$

At $k_t = k^*$, the slope of the transition curve is necessarily less than one. The dynamics therefore lead to convergence to the steady state as illustrated in the figure.[16] In the steady state the interest rate is $r^* = f'(k^*) - \delta = \alpha(1+n)(2+\rho)/(1-\alpha) - \delta$. Note that a higher n results in a lower k^*, hence a higher r^*. □

Because the Cobb-Douglas production function implies that capital is essential, (3.35) implies $k_{t+1} = 0$ if $k_t = 0$. The state $k_{t+1} = k_t = 0$ is thus a stationary point of the difference equation (3.35) considered in isolation. This state is not, however, an equilibrium path as defined above (not a steady state of an *economic* system since there is no production). We may call it a *trivial* steady state in contrast to the economically viable steady state $k_{t+1} = k_t = k^* > 0$ which is then called a *non-trivial* steady state.

Theoretically, there may be more than one (non-trivial) steady state. Nonexistence of a steady state is also possible. But before considering these possibilities, the next subsection (which may be skipped in a first reading) addresses an even more defiant feature which is that for a given k_0 there may exist more than one equilibrium path.

The possibility of multiple equilibrium paths*

It turns out that a *backward-bending* transition curve like that in Fig. 3.5 is possible within the model. Not only are there two steady states but for $k_t \in (\underline{k}, \bar{k})$ there are *three temporary equilibria* with self-fulfilling expectations. That is, for a given k_t in this interval, there are three different values of k_{t+1} that are consistent with self-fulfilling expectations. Exercise 3.3 at the end of the chapter documents this possibility by way of a numerical example.

The theoretical possibility of multiple equilibria with self-fulfilling expectations requires that there is at least one interval on the horizontal axis where a section of the transition curve has negative slope. Let us see if we can get an intuitive understanding of why in this situation multiple equilibria can arise. Consider the specific configuration in Fig. 3.5 where k', k'', and k''' are the possible values for the capital-labor ratio next period when $k_t \in (\underline{k}, \bar{k})$. In a small neighborhood of the point P associated with the intermediate value, k'', the slope of the transition curve is negative. In Fig. 3.5 a relevant neighborhood is indicated

[16] A formal proof can be based on the mean value theorem.

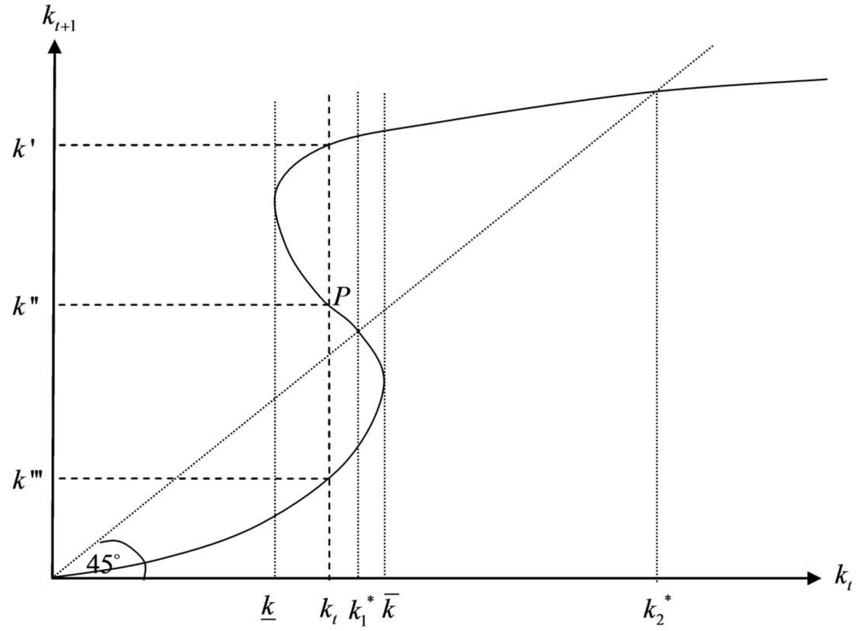


Figure 3.5: A backward-bending transition curve leads to multiple temporary equilibria with self-fulfilling expectations.

by the rectangle R. Within this rectangle the fundamental difference equation (3.32) does indeed define k_{t+1} as an implicit function of k_t, the graph of which goes through the point P and has negative slope. The only points in Fig. 3.5 that have no such neighborhood are the two points where the transition curve has vertical tangent, that is, the two points with abscissas $\underline{k}$ and $\bar{k}$, respectively.

Now, as we saw above, the negative slope requires not only that in this neighborhood $s_r(w_t, r(k_{t+1})) < 0$, but that the stricter condition $s_r(w_t, r(k_{t+1})) < (1+n)/f''(k'')$ holds (we take w_t as given since k_t is given and $w_t = w(k_t)$). That the point P with coordinates (k_t, k'') is on the transition curve indicates that, given $w_t = w(k_t)$ and an expected interest rate $r^e_{t+1} = r(k'')$, the induced saving by the young, $s(w_t, r(k''))$, will be such that $k_{t+1} = k''$, that is, the expectation is fulfilled. The fact that also the point (k_t, k'), where $k' > k''$, is on transition curve indicates that also a lower interest rate, $r(k')$, can be self-fulfilling. By this is meant that *if* an interest rate at the level $r(k')$ is expected, then this expectation induces *more* saving by the young, just enough more to make $k_{t+1} = k' > k''$, thus confirming the expectation of the lower interest rate level $r(k')$. What makes this possible is exactly the negative dependency of s_t on r^e_{t+1}. The fact that also the point (k_t, k'''), where $k''' < k''$, is on the transition curve can be similarly interpreted. It is exactly $s_r < 0$ that makes it possible that *less* saving by the young than at P can be induced by an expected *higher* interest rate, $r(k''')$, than at P.

Recognizing the ambiguity arising from the possibility of multiple equilibrium paths, we face an additional ambiguity, known as the "expectational coordination problem". The model presupposes that all the young *agree* in their expectations. Only then will one of the three mentioned temporary equilibria appear. But the model is silent about how the needed coordination of expectations is brought about, and if it is, why this coordination ends up in one rather than another of the three possible equilibria with self-fulfilling expectations. Each single young is isolated in the market and will not know what the others will expect. The market mechanism by itself provides no coordination of expectations.

As it stands, the model consequently cannot determine how the economy will evolve in the present situation with a backward-bending transition curve. Since the topic is complicated, we will here take an ad-hoc approach − we will circumvent the indeterminacy problem.[17] There are at least three ways to try to rule out the possibility of multiple equilibrium paths. One way is to discard the assumption of perfect foresight. Instead, some kind of adaptive expectations may be assumed, for example in the form of *myopic foresight,* also called *static expectations.* This means that the expectation formed by the agents in the current period about the value of a variable next period is that it will stay the same as in the current period. So here the assumption would be that the young have the expectation $r^e_{t+1} = r_t$. Then, given $k_0 > 0$, a *unique* sequence of temporary equilibria $\{(k_t, c_{1t}, c_{2t}, w_t, r_t)\}_{t=0}^{\infty}$ is generated by the model. *Oscillations* in the sense of repetitive movements up and down of k_t are possible. Even *chaotic* trajectories are possible (see Exercise 3.6).

Outside steady state the agents will experience that their expectations are systematically wrong. And the assumption of myopic foresight rules out that learning occurs. This may be too simplistic, although it *can* be argued that human beings to a certain extent have a psychological disposition to myopic foresight.

Another approach to the indeterminacy problem could be motivated by the general observation that sometimes the possibility of multiple equilibria in a model arises because of a "rough" time structure imposed on the model in question. In the present case, each period in the Diamond model corresponds to half of an adult person's lifetime. And in the first period of life there is no capital income, in the second there is no labor income. This coarse notion of time may

[17] Yet the fact that multiple self-fulfilling equilibrium paths are in several contexts theoretically possible is certainly of interest and plays an important role in certain business cycle theories of booms and busts. We shall have a little more to say about this in part VI of this book.

artificially generate a multiplicity of equilibria or, with myopic foresight, oscillations. An expanded model where people live many periods may "smooth" the responses of the system to the events impinging on it. Indeed, with working life stretching over more than one period, wealth effects of changes in the interest rate arise, thereby reducing the likelihood of a backward-bending transition curve. In Chapter 12 we shall see an example of an overlapping-generations model in continuous time where the indeterminacy problem never arises.

For now, our approach will be to stay with the rough time structure of the Diamond model because of its analytical convenience and then make the best of it by imposing conditions on the utility function, the production function, and/or parameter values so as to rule out multiple equilibria. We stay with the assumption of perfect foresight, but assume that circumstances are such that multiple equilibria with self-fulfilling expectations do not arise. Fortunately, the "circumstances" needed for this in the present model are not defying empirical plausibility.

Conditions for uniqueness of the equilibrium path

Sufficient for the equilibrium path to be unique is that preferences and technology in combination are such that the slope of the transition curve is everywhere positive. Hence we impose the Positive Slope Assumption that

$$s_r(w(k_t), r(k_{t+1})) > \frac{1+n}{f''(k_{t+1})} \tag{A2}$$

everywhere along an equilibrium path. This condition is of course always satisfied when $s_r \geq 0$ (reflecting an elasticity of marginal utility of consumption not above one) and *can* be satisfied even if $s_r < 0$ (as long as s_r is "small" in absolute value). Essentially, (A2) is an assumption that the income effect on consumption as young of a rise in the interest rate does not dominate the substitution effect "too much".

Unfortunately, when stated as in (A2), this condition is not as informative as we might wish. a condition like (A2) is not in itself very informative. This is because it is expressed in terms of an *endogenous* variable, k_{t+1}, for given k_t. A model assumption should preferably be stated in terms of what is *given,* also called the "primitives" of the model; in this model the "primitives" comprise the given preferences, demography, technology, and market form. We can state *sufficient* conditions, however, in terms of the "primitives", such that (A2) is ensured. Here we state two such sufficient conditions, both involving a CRRA period utility function with parameter θ as defined in (3.14):

(a) If $0 < \theta \leq 1$, then (A2) holds for all $k_t > 0$ along an equilibrium path.

(b) If the production function is of CES-type,[18] i.e., $f(k) = A(\alpha k^{\gamma} + 1 - \alpha)^{1/\gamma}$, $A > 0$, $0 < \alpha < 1$, $-\infty < \gamma < 1$, then (A2) holds along an equilibrium path even for $\theta > 1$, if the elasticity of substitution between capital and labor, $1/(1-\gamma)$, is not too small, i.e., if

$$\frac{1}{1-\gamma} > \frac{1 - 1/\theta}{1 + (1+\rho)^{-1/\theta}(1 + f'(k) - \delta)^{(1-\theta)/\theta}} \tag{3.36}$$

for all $k > 0$. In turn, sufficient for this is that $(1-\gamma)^{-1} > 1 - \theta^{-1}$.

That (a) is sufficient for (A2) is immediately visible in (3.15). The sufficiency of (b) is proved in Appendix D. The elasticity of substitution between capital and labor is a concept analogue to the elasticity of intertemporal substitution in consumption. It is a measure of the sensitivity of the chosen $k = K/L$ with respect to the relative factor price. The next chapter goes more into detail with the concept and shows, among other things, that the Cobb-Douglas production function corresponds to $\gamma = 0$. So the Cobb-Douglas production function will satisfy the inequality $(1-\gamma)^{-1} > 1 - \theta^{-1}$ (since $\theta > 0$), hence also the inequality (3.36).

With these or other sufficient conditions in the back of our mind we shall now proceed imposing the Positive Slope Assumption (A2). To summarize:

PROPOSITION 3 *(uniqueness of an equilibrium path)* Suppose the No Fast and Positive Slope assumptions, (A1) and (A2), apply. Then:
(i) if $k_0 > 0$, there exists a unique equilibrium path;
(ii) if $k_0 = 0$, an equilibrium path exists if and only if $f(0) > 0$ (i.e., capital not essential).

When the conditions of Proposition 3 hold, the fundamental difference equation, (3.32), of the model defines k_{t+1} as an implicit function of k_t,

$$k_{t+1} = \varphi(k_t),$$

for all $k_t > 0$, where $\varphi(k_t)$ is called a *transition function.* The derivative of this implicit function is given by (3.34) with k_{t+1} on the right-hand side replaced by $\varphi(k_t)$, i.e.,

$$\varphi'(k_t) = \frac{s_w\left(w\left(k_t\right), r\left(\varphi(k_t)\right)\right) w'(k_t)}{1 + n - s_r\left(w\left(k_t\right), r\left(\varphi(k_t)\right)\right) r'(\varphi(k_t))} > 0. \tag{3.37}$$

The positivity for all $k_t > 0$ is due to (A2). Example 2 above leads to a transition function.

[18] CES stands for Constant Elasticity of Substitution. The CES production function was briefly considered in Section 2.1 and is considered in detail in Chapter 4.

Having determined the evolution of k_t, we have in fact determined the evolution of "everything" in the economy: the factor prices $w(k_t)$ and $r(k_t)$, the saving of the young $s_t = s(w(k_t), r(k_{t+1}))$, and the consumption by both the young and the old. The mechanism behind the evolution of the economy is the Walrasian (or Classical) mechanism where prices, here w_t and r_t, always adjust so as to generate market clearing as if there were a Walrasian auctioneer and where expectations always adjust so as to be model consistent.

Existence and stability of a steady state?

Possibly the equilibrium path converges to a steady state. To address this issue, we examine the possible configurations of the transition curve in more detail. In addition to being positively sloped everywhere, the transition curve will always, for $k_t > 0$, be situated strictly below the solid curve, $k_{t+1} = w(k_t)/(1+n)$, shown in Fig. 3.6. In turn, the latter curve is always, for $k_t > 0$, strictly below the stippled curve, $k_{t+1} = f(k_t)/(1+n)$, in the figure. To be precise:

LEMMA 3 *(ceiling)* Suppose the No Fast Assumption (A1) applies. Along an equilibrium path, whenever $k_t > 0$,

$$0 < k_{t+1} < \frac{w(k_t)}{1+n} < \frac{f(k_t)}{1+n}, \qquad t = 0, 1, \dots. \tag{*}$$

Proof. From (iii) of Proposition 2, an equilibrium path has $w_t = w(k_t) > 0$ and $k_{t+1} > 0$ for $t = 0, 1, 2, \dots$. Thus,

$$0 < k_{t+1} = \frac{s_t}{1+n} < \frac{w_t}{1+n} = \frac{w(k_t)}{1+n} = \frac{f(k_t) - f'(k_t)k_t}{1+n} < \frac{f(k_t)}{1+n},$$

where the first equality comes from (3.32), the second inequality from Lemma 1 in Section 3.3, and the last inequality from the fact that $f'(k_t)k_t > 0$ when $k_t > 0$. This proves (*). □

We will call the graph $(k_t, w(k_t)/(1+n))$ in Fig. 3.6 a *ceiling*. It acts as a ceiling on k_{t+1} simply because the saving of the young cannot exceed the income of the young, $w(k_t)$. The stippled graph, $(k_t, f(k_t)/(1+n))$, in Fig. 3.6 may be called the *roof* ("everything of interest" occurs below it). While the ceiling is the key concept in the proof of Proposition 4 below, the roof is a more straightforward construct since it is directly given by the production function and is always strictly concave. The roof is always above the ceiling and so it appears as a convenient first "enclosure" of the transition curve. Let us therefore start with a characterization of the roof:

LEMMA 4 The roof, $\mathcal{R}(k) \equiv f(k)/(1+n)$, has positive slope everywhere, crosses the 45° line for at most one $k > 0$ and can only do that from above. A necessary and sufficient condition for the *roof* to be above the 45° line for small k is that either $\lim_{k\to 0} f'(k) > 1+n$ or $f(0) > 0$ (capital not essential).

Proof. Since $f' > 0$, the roof has positive slope. Since $f'' < 0$, it can only cross the 45° line once and only from above. If and only if $\lim_{k\to 0} f'(k) > 1+n$, then for small k_t, the roof is steeper than the 45° line. Obviously, if $f(0) > 0$, then close to the origin, the roof will be above the 45° line. □

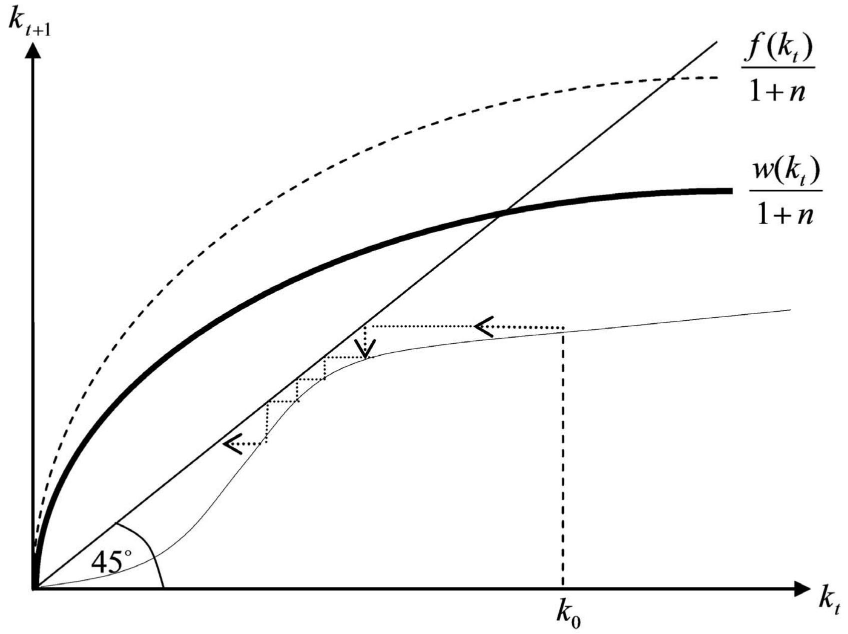


Figure 3.6: A case where both the roof and the ceiling cross the 45° line, but the transition curve does not (no steady state exists).

The ceiling is generally a more complex construct. It can have convex sections and for instance cross the 45° line at more than one point if at all. While the roof can be above the 45° line for *all* $k_t > 0$, the ceiling cannot. Indeed, (ii) of the next lemma implies that if for small k_t the ceiling is above the 45° line, the ceiling will necessarily cross the 45° line at least once for larger k_t.

LEMMA 5 Given $w(k) = f(k) - f'(k)k$ for all $k \geq 0$, where $f(k)$ satisfies $f(0) \geq 0$, $f' > 0$, $f'' < 0$, the following holds:
(i) $\lim_{k\to\infty} w(k)/k = 0$;
(ii) the ceiling, $\mathcal{C}(k) \equiv w(k)/(1+n)$, is positive and has positive slope for all $k > 0$; moreover, there exists $\bar{k} > 0$ such that $\mathcal{C}(k) < k$ for all $k > \bar{k}$.

Proof. (i) In view of $f(0) \geq 0$ combined with $f'' < 0$, we have $w(k) > 0$ for all $k > 0$. Hence, $\lim_{k\to\infty} w(k)/k \geq 0$ if this limit exists. Consider an arbitrary

$k_1 > 0$. We have $f'(k_1) > 0$. For all $k > k_1$, it holds that $0 < f'(k) < f'(k_1)$, in view of $f' > 0$ and $f'' < 0$, respectively. Hence, $\lim_{k\to\infty} f'(k)$ exists and

$$0 \leq \lim_{k\to\infty} f'(k) < f'(k_1). \tag{3.38}$$

We have

$$\lim_{k\to\infty} \frac{w(k)}{k} = \lim_{k\to\infty} \frac{f(k)}{k} - \lim_{k\to\infty} f'(k). \tag{3.39}$$

There are two cases to consider. *Case 1:* $f(k)$ has an upper bound. Then, $\lim_{k\to\infty} f(k)/k = 0$ so that $\lim_{k\to\infty} w(k)/k = -\lim_{k\to\infty} f'(k) = 0$, by (3.39) and (3.38), as $w(k)/k > 0$ for all $k > 0$. *Case 2:* $\lim_{k\to\infty} f(k) = \infty$. Then, by L'Hôpital's rule for "∞/∞", $\lim_{k\to\infty}(f(k)/k) = \lim_{k\to\infty} f'(k)$ so that (3.39) implies $\lim_{k\to\infty} w(k)/k = 0$.

(ii) As $n > -1$ and $w(k) > 0$ for all $k > 0$, $\mathcal{C}(k) > 0$ for all $k > 0$. From $w'(k) = -kf''(k) > 0$ follows $\mathcal{C}'(k) = -kf''(k)/(1+n) > 0$ for all $k > 0$; that is, the ceiling has positive slope everywhere. For $k > 0$, the inequality $\mathcal{C}(k) < k$ is equivalent to $w(k)/k < 1+n$. By (i) follows that for all $\varepsilon > 0$, there exists $k_\varepsilon > 0$ such that $w(k)/k < \varepsilon$ for all $k > k_\varepsilon$. Now, letting $\varepsilon = 1+n$ and $\bar{k} = k_\varepsilon$ proves that there exists $\bar{k} > 0$ such that $w(k)/k < 1+n$ for all $k > \bar{k}$. □

A necessary condition for existence of a (non-trivial) steady state is that the roof is above the 45^0 line for small k_t. But this is not sufficient for also the transition curve to be above the 45^0 line for small k_t. Fig. 3.6 illustrates this. Here the transition curve is in fact everywhere below the 45^0 line. In this case no steady state exists and the dynamics imply convergence towards the "catastrophic" point $(0,0)$. Given the rate of population growth, the saving of the young is not sufficient to avoid famine in the long run. This outcome will occur if the technology implies so low productivity that even when all income of the young were saved, we would have $k_{t+1} < k_t$ for all $k_t > 0$, cf. Exercise 3.2. The Malthusian mechanism will be at work and bring down n (outside the model). This exemplifies that even a trivial steady state (the point (0,0)) may be of interest in so far as it may be the point the economy is heading to (though never reaching it).

To help existence of a steady state we will impose the condition that either capital is not essential or preferences and technology fit together in such a way that the slope of the transition curve is larger than one for small k_t. That is, we assume that either

$$\begin{array}{lrcl} \text{(i)} & f(0) & > & 0 \quad \text{or} \\ \text{(ii)} & \lim_{k\to 0} \varphi'(k) & > & 1, \end{array} \tag{A3}$$

where $\varphi'(k)$ is implicitly given in (3.37). Whether condition (i) of (A3) holds in a given situation can be directly checked from the production function. If it does not, we should check condition (ii). But this condition is less amenable because the transition function φ is not one of the "primitives" of the model. There exist cases, though, where we can find an explicit transition function and try out whether (ii) holds (like in Example 2 above). But generally we can not. Then we have to resort to *sufficient* conditions for (ii) of (A3), expressed in terms of the "primitives". For example, if the period utility function belongs to the CRRA class and the production function is Cobb-Douglas at least for small k, then (ii) of (A3) holds (see Appendix E). Anyway, as (i) and (ii) of (A3) can be interpreted as reflecting two different kinds of "early steepness" of the transition curve, we shall call (A3) the Early Steepness Assumption.[19]

Before stating the proposition aimed at, we need a definition of the concept of asymptotic stability.

DEFINITION 5 Consider a first-order autonomous difference equation $x_{t+1} = g(x_t)$, $t = 0, 1, 2, \ldots$. A steady state $x^* > 0$ is (locally) *asymptotically stable* if there exists $\varepsilon > 0$ such that $x_0 \in (x^* - \varepsilon, x^* + \varepsilon)$ implies that $x_t \to x^*$ for $t \to \infty$. A steady state $x^* > 0$ is *globally asymptotically stable* if for all feasible $x_0 > 0$, it holds that $x_t \to x^*$ for $t \to \infty$.

Applying this definition on our difference equation $k_{t+1} = \varphi(k_t)$, we have:

PROPOSITION 4 *(existence and stability of a steady state)* Assume that the No Fast Assumption (A1) and the Positive Slope assumption (A2) apply as well as the Early Steepness Assumption (A3). Then there exists at least one steady state $k_1^* > 0$ that is locally asymptotically stable. If k_t does not converge to k_1^*, k_t converges to another steady state. Oscillations do not occur.

Proof. By (A1), Lemma 3 applies. From Proposition 2 we know that if (i) of (A3) holds, then $k_{t+1} = s_t/(1+n) > 0$ even for $k_t = 0$. Alternatively, (ii) of (A3) is enough to ensure that the transition curve lies above the 45° line for small k_t. According to (ii) of Lemma 5, for large k_t the ceiling is below the 45° line. Being below the ceiling, cf. Lemma 3, the transition curve must therefore cross the 45° line at least once. Let k_1^* denote the smallest k_t at which it crosses. Then $k_1^* > 0$ is a steady state with the property $0 < \varphi'(k_1^*) < 1$. By graphical inspection we see that this steady state is asymptotically stable. If it is the only (non-trivial) steady state, it is globally asymptotically stable. Otherwise, if k_t does not converge to k_1^*, k_t converges to one of the other steady states. Indeed, divergence is ruled out since, by Lemma 5, there exists $\bar{k} > 0$ such that $w(k)/(1+n) < k$ for all $k > \bar{k}$ (Fig. 3.7 illustrates). For oscillations to come about there must exist a steady state, k^{**}, with $\varphi'(k^{**}) < 0$, but this is impossible in view of (A2). □

[19] In (i) of (A3), the "steepness" is rather a "hop" at $k = 0$ if we imagine k approaching nil from below.

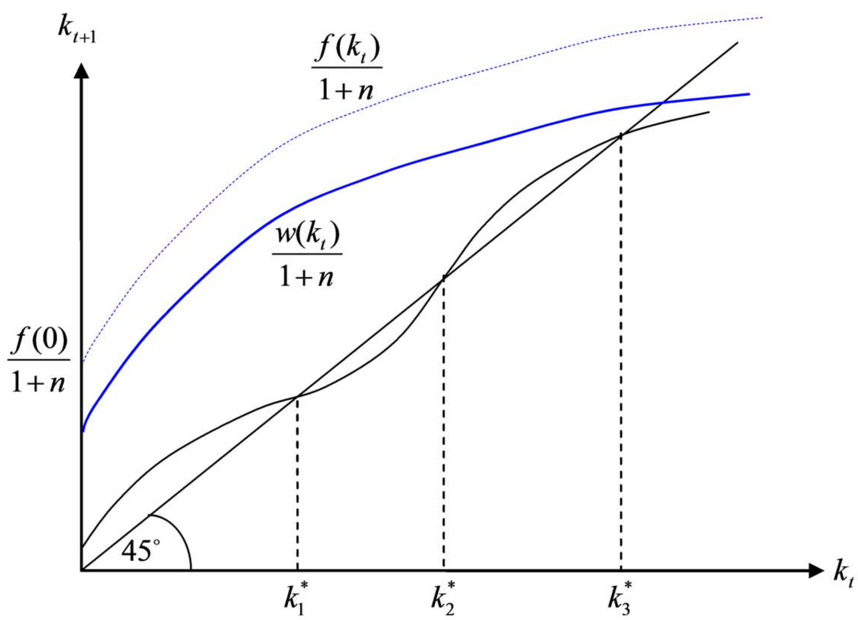


Figure 3.7: A case of multiple steady states (and capital being not essential).

From Proposition 4 we conclude that, given k_0, the assumptions (A1) - (A3) ensure existence and uniqueness of an equilibrium path; moreover, the equilibrium path converges towards *some* steady state. Thus with these assumptions, for any $k_0 > 0$, sooner or later the system settles down at some steady state $k^* > 0$. For the factor prices we therefore have

$$\begin{aligned} r_t &= f'(k_t) - \delta \rightarrow f'(k^*) - \delta \equiv r^*, \quad \text{and} \\ w_t &= f(k_t) - k_t f'(k_t) \rightarrow f(k^*) - k^* f'(k^*) \equiv w^*, \end{aligned}$$

for $t \rightarrow \infty$. But there may be more than one steady state and therefore only *local* stability is guaranteed. This can be shown by examples, where the utility function, the production function, and parameters are specified in accordance with the assumptions (A1) - (A3) (see Exercise 3.5 and ...).

Fig. 3.7 illustrates such a case (with $f(0) > 0$ so that capital is not essential). Moving West-East in the figure, the first steady state, k_1^*, is stable, the second, k_2^*, unstable, and the third, k_3^*, stable. In which of the two stable steady states the economy ends up depends on the initial capital-labor ratio, k_0. The lower steady state, k_1^*, is known as a *poverty trap*. If $0 < k_0 < k_2^*$, the economy is caught in the trap and converges to the low steady state. But with high enough k_0 ($k_0 > k_2^*$), perhaps obtained by foreign aid, the economy avoids the trap and converges to the high steady state. Looking back at Fig. 3.6, we can interpret that figure's scenario as exhibiting an *inescapable* poverty trap.

It turns out that CRRA utility combined with a Cobb-Douglas production function ensures both that (A1) - (A3) hold and that a *unique* (non-trivial) steady state exists. So in this case *global* asymptotic stability of the steady state is ensured.[20] Example 2 and Fig. 3.4 above display a special case of this, the case $\theta = 1$.

This is of course a convenient case for the analyst. A Diamond model satisfying assumptions (A1) - (A3) *and* featuring a unique steady state is called a *well-behaved* Diamond model.

We end this section with the question: Is it possible that aggregate consumption, along an equilibrium path, for some periods exceeds aggregate income? We shall see that this is indeed the case in this model if K_0 (wealth of the old in the initial period) is large enough. Indeed, from national accounting we have:

$$\begin{aligned} C_{10} + C_{20} &= F(K_0, L_0) - I_0 > F(K_0, L_0) \Leftrightarrow I_0 < 0 \\ &\Leftrightarrow K_1 < (1-\delta)K_0 \Leftrightarrow K_0 - K_1 > \delta K_0. \end{aligned}$$

So aggregate consumption in period 0 being greater than aggregate income is equivalent to a fall in the capital stock from period 0 to period 1 greater than the capital depreciation in period 0. Consider the log utility Cobb-Douglas case in Fig. 3.4 and suppose $\delta < 1$ and $L_t = L_0 = 1$, i.e., $n = 0$. Then $k_t = K_t$ for all t and by (3.35), $K_{t+1} = \frac{(1-\alpha)A}{2+\rho} K_t^\alpha$. Thus $K_1 < (1-\delta)K_0$ for

$$K_0 > \left(\frac{(1-\alpha)A}{(2+\rho)(1-\delta)} \right)^{1/(1-\alpha)}.$$

As initial K is arbitrary, this situation is possible. When it occurs, it reflects that the financial wealth of the old is so large that their consumption (recall they consume all their financial wealth as well as the interest on this wealth) exceeds what is left of current aggregate production after subtracting the amount consumed by the young. So aggregate gross investment in the economy will be negative. Of course this is only feasible if capital goods can be "eaten" or at least be immediately (without further resources) converted into consumption goods. As it stands, the model has implicitly assumed this to be the case. And this is in line with the general setup since the output good is assumed homogeneous and can either be consumed or piled up as capital.

We now turn to efficiency problems.

[20] See last section of Appendix E.

3.6 The golden rule and dynamic inefficiency

An economy described by the Diamond model has the property that even though there is perfect competition and no externalities, the outcome brought about by the market mechanism may not be Pareto optimal.[21] Indeed, the economy may *overaccumulate* forever and thus suffer from a distinctive form of production inefficiency.

A key element in understanding the concept of overaccumulation is the concept of a *golden-rule capital-labor ratio*. Overaccumulation occurs when aggregate saving maintains a capital-labor ratio above the golden-rule value forever. Let us consider these concepts in detail.

In the present section generally the period length is arbitrary except when we relate to the Diamond model and the period length therefore is half of adult lifetime.

The golden-rule capital-labor ratio

The golden rule is a principle that relates to technically feasible paths. The principle does not depend on the market form.

Consider the economy-wide resource constraint $C_t = Y_t - S_t = F(K_t, L_t) - (K_{t+1} - K_t + \delta K_t)$, where we assume that F is neoclassical with CRS. Accordingly, aggregate consumption per unit of labor can be written

$$c_t \equiv \frac{C_t}{L_t} = \frac{F(K_t, L_t) - (K_{t+1} - K_t + \delta K_t)}{L_t} = f(k_t) + (1 - \delta)k_t - (1 + n)k_{t+1}, \tag{3.40}$$

where k is the capital-labor ratio K/L. Note that C_t will generally be greater than the workers' consumption. One should simply think of C_t as the flow of produced consumption goods in the economy and c_t as this flow divided by aggregate employment, including the labor that in period t produces investment goods. How the consumption goods are distributed to different members of society is not our concern here.

DEFINITION 6 By the *golden-rule capital-labor ratio*, k_{GR}, is meant that value of the capital-labor ratio k, which results in the highest possible sustainable level of consumption per unit of labor.

Sustainability requires replicability forever. We therefore consider a steady

[21] Recall that a *Pareto optimal* path is a technically feasible path with the property that no other technically feasible path will make at least one individual better off without making someone else worse off. A technically feasible path which is not Pareto optimal is called *Pareto inferior*.

state. In a steady state $k_{t+1} = k_t = k$ so that (3.40) simplifies to

$$c = f(k) - (\delta + n)k \equiv c(k). \tag{3.41}$$

Maximizing gives the first-order condition

$$c'(k) = f'(k) - (\delta + n) = 0. \tag{3.42}$$

In view of $c''(k) = f''(k) < 0$, the condition (3.42) is both necessary and sufficient for an interior maximum. Let us assume that $\delta + n > 0$ and that f satisfies the condition

$$\lim_{k\to\infty} f'(k) < \delta + n < \lim_{k\to 0} f'(k).$$

Then (3.42) has a solution in k, and it is unique because $c''(k) < 0$. The solution is called k_{GR} so that

$$f'(k_{GR}) - \delta = n.$$

That is:

PROPOSITION 5 *(the golden rule)* The highest sustainable consumption level per unit of labor in society is obtained when in steady state the net marginal productivity of capital equals the growth rate of the economy.

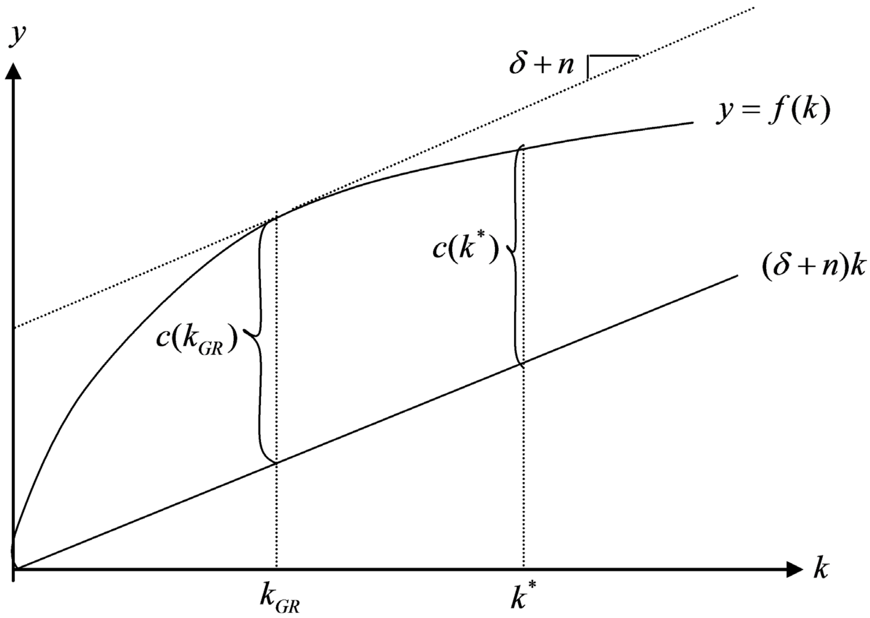


Figure 3.8: A steady state with overaccumulation.

It follows that if a society aims at the highest sustainable level of consumption and initially has $k_0 < k_{GR}$, society should increase its capital-labor ratio up to the point where the extra output obtainable by a further small increase is exactly offset by the extra gross investment needed to maintain the capital-labor ratio at that level. The intuition is visible from (3.41). The golden-rule capital-labor ratio, k_{GR}, strikes the right balance in the trade-off between high output per unit of labor and a not too high investment requirement. Although a steady state with $k > k_{GR}$ would imply higher output per unit of labor, it would also imply that a large part of that output is set aside for investment (namely the amount $(\delta + n)k$ per unit of labor) to counterbalance capital depreciation and growth in the labor force; without this investment the high capital-labor ratio k^* would not be maintained. With $k > k_{GR}$ this feature would dominate the first effect so that consumption per unit of labor ends up low. Fig. 3.8 illustrates.

The name golden rule hints at the golden rule from the Bible: "Do unto others as you would have them to do unto you." We imagine that God asks the newly born generation: "What capital-labor ratio would you prefer to be presented with, given that you must hand over the same capital-labor ratio to the next generation?" The appropriate answer is: the golden-rule capital-labor ratio.

The possibility of overaccumulation in a competitive market economy

The equilibrium path in the Diamond model with perfect competition implies an interest rate $r^* = f'(k^*) - \delta$ in a steady state. As an implication,

$$r^* \gtreqqless n \Leftrightarrow f'(k^*) - \delta \gtreqqless n \Leftrightarrow k^* \lesseqqgtr k_{GR}, \text{ respectively,}$$

in view of $f'' < 0$. Hence, a long-run interest rate below the growth rate of the economy indicates that $k^* > k_{GR}$. This amounts to a Pareto-inferior state of affairs. Indeed, everyone can be made better off if by a coordinated reduction of saving and investment, k is reduced. A formal demonstration of this is given in connection with Proposition 6 in the next subsection. Here we give an account in more intuitive terms.

Consider Fig. 3.8. Let k be gradually reduced to the level k_{GR} by refraining from investment in period t_0 and forward until this level is reached. When this happens, let k be maintained at the level k_{GR} forever by providing for the needed investment per young, $(\delta + n)k_{GR}$. Then there would be higher aggregate consumption in period t_0 and every future period. Both the immediate reduction of saving and a resulting lower capital-labor ratio to be maintained contribute to this result. There is thus scope for both young and old to consume more in every future period.

In the Diamond model a simple policy implementing such a Pareto improvement in the case where $k^* > k_{GR}$ (i.e., $r^* < n$) is to incur a lump-sum tax on the young, the revenue of which is immediately transferred lump sum to the old, hence, fully consumed. Suppose this amounts to a transfer of one good from each young to the old. Since there are $1 + n$ young people for each old person, every old receives in this way $1 + n$ goods in the same period. Let this transfer be repeated every future period. By decreasing their saving by one unit, the young can maintain unchanged consumption in their youth, and when becoming old, they receive $1 + n$ goods from the next period's young and so on. In effect, the "return" on the tax payment by the young is $1 + n$ next period. This is more than the $1 + r^*$ that could be obtained via the market through own saving.[22]

A proof that $k^* > k_{GR}$ is indeed theoretically possible in the Diamond model can be based on the log utility-Cobb-Douglas case from Example 2 in Section 3.5.3. As indicated by the formula for r^* in that example, the outcome $r^* < n$, which is equivalent to $k^* > k_{GR}$, can always be obtained by making the parameter $\alpha \in (0, 1)$ in the Cobb-Douglas function small enough. The intuition is that a small α implies a high $1 - \alpha$, that is, a high wage income $wL = (1 - \alpha)K^{\alpha}L^{-\alpha} \cdot L$ $= (1 - \alpha)Y$; this leads to high saving by the young, since $s_w > 0$. The result is a high k_{t+1} which generates a high real wage also next period and may in this manner be sustained forever.

An intuitive understanding of the fact that the perfectly competitive market mechanism may thus lead to overaccumulation, can be based on the following argument. Assume, first, that $s_r < 0$. In this case, if the young in period t expects the rate of return on their saving to end up small (less than n), the decided saving will be large in order to provide for consumption after retirement. But the aggregate result of this behavior is a high k_{t+1} and therefore a low $f'(k_{t+1})$. In this way the expectation of a low r_{t+1} is confirmed by the actual events. The young persons each do the best they can as atomistic individuals, taking the market conditions as given. Yet the aggregate outcome is an equilibrium with overaccumulation, hence a Pareto-inferior outcome.

Looking at the issue more closely, we see that $s_r < 0$ is not crucial for this outcome. Suppose $s_r = 0$ (the log utility case) and that in the current period, k_t is, for some historical reason, at least temporarily considerably above k_{GR}. Thus, current wages are high, hence, s_t is relatively high (there is in this case no offsetting effect on s_t from the relatively low expected r_{t+1}). Again, the aggregate result is a high k_{t+1} and thus the expectation is confirmed. Consequently, the situation in the next period is the same and so on. By continuity, even if $s_r > 0$, the argument goes through as long as s_r is not too large.

[22] In this model with no utility of leisure, a tax on wage income, or a mandatory pay-as-you-go pension contribution (see Chapter 5), would act like a lump-sum tax on the young.

The described tax-transfers policy will affect the equilibrium interest rate negatively. By choosing an appropriate size of the tax this policy, combined with competitive markets, will under certain conditions (see Chapter 5.1) bring the economy to the golden-rule steady state where overaccumulation has ceased and $r^* = n$.

Dynamic inefficiency and the double infinity

Another name for the overaccumulation phenomenon is *dynamic inefficiency.*

DEFINITION 7 A technically feasible path $\{(c_t, k_t)\}_{t=0}^{\infty}$ with the property that there does not exist another technically feasible path with higher c_t in some periods without smaller c_t in other periods is called *dynamically efficient.* A technically feasible path $\{(c_t, k_t)\}_{t=0}^{\infty}$ which is not dynamically efficient is called *dynamically inefficient.*

PROPOSITION 6 A technically feasible path $\{(c_t, k_t)\}_{t=0}^{\infty}$ with the property that for $t \to \infty$, $k_t \to k^* > k_{GR}$, is dynamically inefficient.

Proof. Let $k^* > k_{GR}$. Then there exists an $\varepsilon > 0$ such that $k \in (k^* - 2\varepsilon, k^* + 2\varepsilon)$ implies $f'(k) - \delta < n$ since $f'' < 0$. By concavity of f,

$$f(k) - f(k - \varepsilon) \leq f'(k - \varepsilon)\varepsilon. \tag{3.43}$$

Consider a technically feasible path $\{(c_t, k_t)\}_{t=0}^{\infty}$ with $k_t \to k^*$ for $t \to \infty$ (the reference path). Then there exists a t_0 such that for $t \geq t_0$, $k_t \in (k^* - \varepsilon, k^* + \varepsilon)$, $f'(k_t) - \delta < n$ and $f'(k_t - \varepsilon) - \delta < n$. Consider an alternative feasible path $\left\{(\hat{c}_t, \hat{k}_t)\right\}_{t=0}^{\infty}$, where a) for $t = t_0$ consumption is increased relative to the reference path such that $\hat{k}_{t_0+1} = k_{t_0} - \varepsilon$; and b) for all $t > t_0$, consumption is such that $\hat{k}_{t+1} = k_t - \varepsilon$. We now show that after period t_0, $\hat{c}_t > c_t$. Indeed, for all $t > t_0$, by (3.40),

$$\begin{aligned}
\hat{c}_t &= f(\hat{k}_t) + (1-\delta)\hat{k}_t - (1+n)\hat{k}_{t+1} \\
&= f(k_t - \varepsilon) + (1-\delta)(k_t - \varepsilon) - (1+n)(k_{t+1} - \varepsilon) \\
&\geq f(k_t) - f'(k_t - \varepsilon)\varepsilon + (1-\delta)(k_t - \varepsilon) - (1+n)(k_{t+1} - \varepsilon) \quad \text{(by (3.43))} \\
&> f(k_t) - (\delta + n)\varepsilon + (1-\delta)k_t - (1+n)k_{t+1} + (\delta + n)\varepsilon \\
&= f(k_t) + (1-\delta)k_t - (1+n)k_{t+1} = c_t,
\end{aligned}$$

by (3.40). □

Moreover, it can be shown[23] that:

PROPOSITION 7 A technically feasible path $\{(c_t, k_t)\}_{t=0}^{\infty}$ such that for $t \to \infty$, $k_t \to k^* \leq k_{GR}$, is dynamically efficient.

Accordingly, a steady state with $k^* < k_{GR}$ is never dynamically inefficient. This is because increasing k from this level always has its price in terms of a decrease in *current* consumption; and at the same time decreasing k from this

[23] See Cass (1972).

level always has its price in terms of lost *future* consumption. But a steady state with $k^* > k_{GR}$ is always dynamically inefficient. Intuitively, staying forever with $k = k^* > k_{GR}$, implies that society *never* enjoys its great capacity for producing consumption goods.

The fact that $k^* > k_{GR}$ − and therefore dynamic inefficiency − cannot be ruled out might seem to contradict the First Welfare Theorem from the microeconomic theory of general equilibrium. This is the theorem saying that under certain conditions (essentially that increasing returns to scale are absent, markets are competitive, no goods are of public good character, and there are no externalities, then market equilibria are Pareto optimal. In fact, however, the First Welfare Theorem also presupposes a finite number of periods or, if the number of periods is infinite, then a finite number of agents. In contrast, in the OLG model there is a *double infinity*: an infinite number of periods *and* agents. Hence, the First Welfare Theorem breaks down. Indeed, the case $r^* < n$, i.e., $k^* > k_{GR}$, can arise under *laissez-faire.* Then, as we have seen, everyone can be made better off by a coordinated intervention by some social arrangement (a government for instance) such that k is reduced.

The essence of the matter is that the double infinity opens up for technically feasible reallocations which are definitely beneficial when $r^* < n$ and which a central authority can accomplish but the market can not. That *nobody* need loose by the described kind of redistribution is due to the double infinity: the economy goes on forever and there is no last generation. Nonetheless, some kind of centralized *coordination* is required to accomplish a solution.

There is an analogy in "Gamow's bed problem": There are an infinite number of inns along the road, each with one bed. On a certain rainy night all innkeepers have committed their beds. A late guest comes to the first inn and asks for a bed. "Sorry, full up!" But the minister of welfare hears about it and suggests that from each inn one incumbent guest moves down the road one inn.[24]

Whether the theoretical possibility of overaccumulation should be a matter of practical concern is an empirical question about the relative size of rates of return and economic growth. To answer the question meaningfully, we need an extension of the criterion for overaccumulation so that the presence of technological progress and rising per capita consumption in the long run can be taken into account. This is one of the topics of the next chapter. At any rate, we can already here reveal that there exists no indication that overaccumulation has ever been an actual problem in industrialized market economies.

A final remark before concluding. Proposition 5 about the golden rule can be generalized to the case where instead of one there are n different capital goods in

[24] George Gamow (1904-1968) was a Russian physicist. The problem is also known as *Hilbert's hotel problem*, after the German mathematician David Hilbert (1862-1943).

the economy. Essentially the generalization says that assuming CRS-neoclassical production functions with n different capital goods as inputs, one consumption good, no technological change, and perfectly competitive markets, a steady state in which per-unit-of labor consumption is maximized has interest rate equal to the growth rate of the labor force when technological progress is ignored (see, e.g., Mas-Colell, 1989).

3.7 Concluding remarks

(Unfinished)

In several respects the conclusions we get from OLG models are different than those from other neoclassical models, in particular representative agent models (to be studied later). In OLG models the aggregate quantities are the outcome of the interplay of finite-lived agents at different stages in their life cycle. The turnover in the population plays a crucial role. In this way the OLG approach lays bare the possibility of coordination failure on a grand scale. In contrast, in a representative agent model, aggregate quantities are just a multiple of the actions of the representative household.

Regarding analytical tractability, the complexity implied by having in every period two different coexisting generations is in some respects more than compensated by the fact that the finite time horizon of the households make the *dynamics* of the model *one-dimensional*: we end up with a first-order difference equation in the capital-labor ratio, k_t, in the economy. In contrast, the dynamics of the basic representative agent model (Chapter 8 and 10) is two-dimensional (owing to the assumed infinite horizon of the households considered as dynasties).

Miscellaneous notes:

OLG gives theoretical insights concerning macroeconomic implications of life cycle behavior, allows heterogeneity, provides training in seeing the economy as consisting of a heterogeneous population where the *distribution* of agent characteristics matters for the aggregate outcome.

Farmer (1993), p. 125, notes that OLG models are difficult to apply and for this reason much empirical work in applied general equilibrium theory has regrettably instead taken the representative agent approach.

Outlook: Rational speculative bubbles in general equilibrium, cf. Chapter ?.

3.8 Literature notes

1. The Nobel Laureate Paul A. Samuelson (1915-2009) is one of the pioneers of OLG models. Building on the French economist and Nobel laureate Maurice

Allais (1911-2010), a famous article by Samuelson, from 1958, is concerned with a missing market problem. Imagine a two-period OLG economy where, as in the Diamond model, only the young have an income (by Samuelson simplifying considered an exogenous endowment of consumption goods from heaven). Contrary to the Diamond model, however, there is no capital. Also other potential stores of value are absent. Then, in the laissez-faire market economy the old have to starve because they can no longer work and had no possibility of saving - transferring income - as young.

The allocation of resources in the economy is Pareto-inferior. Indeed, if each member of the young generation hands over to the old generation one unit of consumption, and this is next period repeated by the new young generation and so on in the future, everyone will be better off. Since for every old there are $1+n$ young, the implied rate of return would be n, the population growth rate. Such transfers do not arise under laissez-faire. A kind of social contract is required. As Samuelson pointed out, a government could in period 0 issue paper notes, "money", and transfer these notes to the members of the old generation who would then use them to buy goods from the young. Provided the young believed the notes to be valuable in the next period, they would accept them in exchange for some of their goods in order to use them in the next period for buying from the new young generation etc.

We have here an example of how a social institution can solve a coordination problem.[25]

2. Diamond (1965) extended Samuelson's contribution by adding capital accumulation. Because of its antecedents Diamonds OLG model is sometimes called the Samuelson-Diamond model or the Allais-Samuelson-Diamond model. In our exposition we have drawn upon clarifications by Galor and Ryder (1989) and de la Croix and Michel (2002). The last mentioned contribution is an extensive exploration of discrete-time OLG models and their applications. An advanced and thorough treatment from a microeconomic general equilibrium perspective is contained in Bewley (2007).

3. The *life-cycle saving hypothesis* was put forward by Franco Modigliani (1918-2003) and associates in the 1950s. See for example Modigliani and Brumberg (1954). Numerous extensions of the framework, relating to the motives (b) - (e) in the list of Section 3.1, see for instance de la Croix and Michel (2002).

4. A review of the empirics of life-cycle behavior and attempts at refining life-cycle models are given in Browning and Crossley (2001).

5. Regarding the dynamic efficiency issue, both the propositions 6 and 7 were shown in a stronger form by the American economist David Cass (1937-2008).

[25] To just give a flavor of Samuelson's contribution we have here ignored several aspects, including that Samuelson assumed three periods of life.

Cass established the *general* necessary and sufficient condition for a feasible path $\{(c_t, k_t)\}_{t=0}^{\infty}$ to be dynamically efficient (Cass 1972). Our propositions 6 and 7 are more restrictive in that they are limited to paths that converge. Partly intuitive expositions of the deeper aspects of the theory are given by Shell (1971) and Burmeister (1980).

6. Diamond has also contributed to other fields of economics, including search theory for labor markets. In 2010 Diamond, together with Dale Mortensen and Christopher Pissarides, was awarded the Nobel price in economics.

From here very incomplete:

The two-period structure of Diamonds OLG model leaves little room for considering, e.g., education and dissaving in the early years of life. This kind of issues is taken up in three-period extensions of the Diamond model, see de la Croix and Michell (2002).

Multiple equilibria, self-fulfilling expectations, optimism and pessimism..

Dynamic inefficiency, see also Burmeister (1980).

Bewley 1977, 1980.

Two-sector OLG: Galor (1992). Galor's book on difference equations.

On the golden rule in a general setup, see Mas-Colell (1989).

3.9 Appendix

A. On CRRA utility

Derivation of the CRRA function Consider a utility function $u(c)$, defined for all $c > 0$ and satisfying $u'(c) > 0$, $u''(c) < 0$. Let the absolute value of the elasticity of marginal utility be denoted $\theta(c)$, that is, $\theta(c) \equiv -cu''(c)/u'(c) > 0$. We claim that if $\theta(c)$ is a positive constant, θ, then, up to a positive linear transformation, $u(c)$ must be of the form

$$u(c) = \begin{cases} \frac{c^{1-\theta}}{1-\theta}, & \text{when } \theta \neq 1, \\ \ln c, & \text{when } \theta = 1, \end{cases} \tag{*}$$

i.e., of CRRA form.

Proof. Suppose $\theta(c) = \theta > 0$. Then, $u''(c)/u'(c) = -\theta/c$. By integration, $\ln u'(c) = -\theta \ln c + A$, where A is an arbitrary constant. Take the antilogarithm function on both sides to get $u'(c) = e^A e^{-\theta \ln c} = e^A c^{-\theta}$. By integration we get

$$u(c) = \begin{cases} e^A \frac{c^{1-\theta}}{1-\theta} + B, & \text{when } \theta \neq 1, \\ e^A \ln c + B, & \text{when } \theta = 1, \end{cases}$$

where B is an arbitrary constant. This proves the claim. Letting $A = B = 0$, we get (*). □

When we want to make the kinship between the members of the "CRRA family" transparent, we maintain $A = 0$ and for $\theta = 1$ also $B = 0$, whereas for $\theta \neq 1$ we set $B = -1/(1-\theta)$. In this way we achieve that all members of the CRRA family will be represented by curves going through the same point as the log function, namely the point $(1, 0)$, cf. Fig. 3.2. For a particular $\theta > 0, \theta \neq 1$, we have $u(c) = (c^{1-\theta} - 1)/(1-\theta)$, which makes up the CRRA utility function in normalized form. Given θ, the transformation to normalized form is of no consequence for the economic behavior since adding or subtracting a constant does not affect marginal rates of substitution.

The domain of the CRRA function We want to extend the domain to include $c = 0$. If $\theta \geq 1$, the CRRA function, whether in the form $u(c) = (c^{1-\theta} - 1)/(1-\theta)$ or in the form (*), is defined only for $c > 0$, not for $c = 0$. This is because for $c \to 0$ we get $u(c) \to -\infty$. In this case we simply define $u(0) = -\infty$. This will create no problems since the CRRA function anyway has the property that $u'(c) \to \infty$, when $c \to 0$ (whether θ is larger or smaller than one). The marginal utility thus becomes very large as c becomes very small, that is, the No Fast Assumption is satisfied. This will ensure that the chosen c is strictly positive whenever there is a positive budget. So throughout this book we define the domain of the CRRA function to be $[0, \infty)$.

The range of the CRRA function Considering the CRRA function $u(c) \equiv \left(c^{1-\theta} - 1\right)(1-\theta)^{-1}$ for $c \in [0, \infty)$, we have:

$$
\begin{aligned}
\text{for } 0 &< \theta < 1, \text{ the range of } u(c) \text{ is } \left[-(1-\theta)^{-1}, \infty\right), \\
\text{for } \theta &= 1, \text{ the range of } u(c) \text{ is } [-\infty, \infty), \\
\text{for } \theta &> 1, \text{ the range of } u(c) \text{ is } [-\infty, -(1-\theta)^{-1}).
\end{aligned}
$$

Thus, in the latter case $u(c)$ is bounded from above and so allows asymptotic "saturation" to occur.

B. Deriving the elasticity of intertemporal substitution in consumption

Referring to Section 3.3, we here show that the definition of $\sigma(c_1, c_2)$ in (3.17) gives the result (3.18). Let $x \equiv c_2/c_1$ and $\beta \equiv (1+\rho)^{-1}$. Then the first-order condition (3.16) and the equation describing the considered indifference curve constitute a system of two equations

$$
\begin{aligned}
u'(c_1) &= \beta u'(xc_1)R, \\
u(c_1) + \beta u(xc_1) &= \bar{U}.
\end{aligned}
$$

For a fixed utility level $U = \bar{U}$ these equations define c_1 and x as implicit functions of R, $c_1 = c(R)$ and $x = x(R)$. We calculate the total derivative with respect to R in both equations and get, after ordering,

$$\begin{aligned} & [u''(c_1) - \beta R u''(xc_1)x]\, c'(R) - \beta R u''(xc_1)c_1 x'(R) \\ = \; & \beta u'(xc_1), \end{aligned} \tag{3.44}$$

$$[u'(c_1) + \beta u'(xc_1)x]\, c'(R) = -\beta u'(xc_1)c_1 x'(R). \tag{3.45}$$

Substituting $c'(R)$ from (3.45) into (3.44) and ordering now yields

$$-\left[x\frac{c_1 u''(c_1)}{u'(c_1)} + R\frac{xc_1 u''(xc_1)}{u'(xc_1)}\right]\frac{R}{x}x'(R) = x + R.$$

Since $-cu''(c)/u'(c) \equiv \theta(c)$, this can be written

$$\frac{R}{x}x'(R) = \frac{x + R}{x\theta(c_1) + R\theta(xc_1)}.$$

Finally, in view of $xc_1 = c_2$ and the definition of $\sigma(c_1, c_2)$, this gives (3.18).

C. Walras' law

In the proof of Proposition 1 we referred to Walras' law. Here is how Walras' law works in each period in a model like this. We consider period t, but for simplicity we skip the time index t on the variables. There are three markets, a market for capital services, a market for labor services, and a market for output goods. Suppose a "Walrasian auctioneer" calls out the price vector $(\hat{r}, w, 1)$, where $\hat{r} > 0$ and $w > 0$, and asks all agents, i.e., the young, the old, and the representative firm, to declare their supplies and demands.

The supplies of capital and labor are by assumption inelastic and equal to K units of capital services and L units of labor services. But the demand for capital and labor services depends on the announced $\hat{r}$ and w. Let the potential pure profit of the representative firm be denoted Π. If $\hat{r}$ and w are so that $\Pi < 0$, the firm declares $K^d = 0$ and $L^d = 0$. If on the other hand at the announced $\hat{r}$ and w, $\Pi = 0$ (as when $\hat{r} = r(k) + \delta$ and $w = w(k)$), the desired capital-labor ratio is given as $k^d = f'^{-1}(\hat{r})$ from (3.20), but the firm is indifferent with respect to the absolute level of the factor inputs. In this situation the auctioneer tells the firm to declare $L^d = L$ (recall L is the given labor supply) and $K^d = k^d L^d$ which is certainly acceptable for the firm. Finally, if $\Pi > 0$, the firm is tempted to declare infinite factor demands, but to avoid that, the auctioneer imposes the rule that the maximum allowed demands for capital and labor are $2K$ and $2L$, respectively.

Within these constraints the factor demands will be uniquely determined by $\hat{r}$ and w and we have

$$\Pi = \Pi(\hat{r}, w, 1) = F(K^d, L^d) - \hat{r}K^d - wL^d. \tag{3.46}$$

The owners of both the capital stock K and the representative firm must be those who saved in the previous period, namely the currently old. These elderly will together declare the consumption $c_2 L_{-1} = (1 + \hat{r} - \delta)K + \Pi$ and the net investment $-K$ (which amounts to disinvestment). The young will declare the consumption $c_1 L = wL - s(w, r^e_{+1})L$ and the net investment $sL = s(w, r^e_{+1})L$. So aggregate declared consumption will be $C = (1 + \hat{r} - \delta)K + \Pi + wL - s(w, r^e_{+1})L$ and aggregate net investment $I - \delta K = s(w, r^e_{+1})L - K$. It follows that $C + I = wL + \hat{r}K + \Pi$. The aggregate declared supply of output is $Y^s = F(K^d, L^d)$. The values of excess demands in the three markets now add to

$$\begin{aligned} Z(\hat{r}, w, 1) &\equiv w(L^d - L) + \hat{r}(K^d - K) + C + I - Y^s \\ &= wL^d - wL + \hat{r}K^d - \hat{r}K + wL + \hat{r}K + \Pi - F(K^d, L^d) \\ &= wL^d + \hat{r}K^d + \Pi - F(K^d, L^d) = 0, \end{aligned}$$

by (3.46).

This is a manifestation of Walras' law for each period: *whatever the announced price vector for the period is, the aggregate value of excess demands in the period is zero*. The reason is the following. When each household satisfies its budget constraint and each firm pays out its ex ante profit,[26] then the economy as a whole has to satisfy an aggregate budget constraint for the period considered.

The budget constraints, demands, and supplies operating in this thought experiment (and in Walras' law in general) are the *Walrasian* budget constraints, demands, and supplies. Outside equilibrium these are somewhat artificial constructs. A Walrasian budget constraint is based on the assumption that the desired actions can be realized. This assumption will be wrong unless $\hat{r}$ and w are already at their equilibrium levels. But the assumption that desired actions can be realized is never falsified because the thought experiment does not allow trades to take place outside Walrasian equilibrium. Similarly, the Walrasian consumption demand by the worker is rather hypothetical outside equilibrium. This demand is based on the income the worker *would* get if fully employed at the announced real wage, not on the actual employment (or unemployment) at that real wage.

These ambiguities notwithstanding, the important message of Walras' law goes through, namely that when two of the three markets clear (in the sense of the Walrasian excess demands being nil), so does the third.

[26] By ex ante profit is meant the hypothetical profit calculated on the basis of firms' desired supply evaluated at the announced price vector, $(\hat{r}, w, 1)$.

D. Proof of (i) and (ii) of Proposition 2

For convenience we repeat the fundamental difference equation characterizing an equilibrium path:

$$k_{t+1} = \frac{s\left(w\left(k_t\right), r\left(k_{t+1}\right)\right)}{1+n},$$

where $w(k) \equiv f(k) - f'(k)k > 0$ for all $k > 0$ and $r(k) \equiv f'(k) - \delta > -1$ for all $k \geq 0$. The key to the proof of Proposition 2 about existence of an equilibrium path is the following lemma.

LEMMA D1 Suppose the No Fast Assumption (A1) applies and let $w > 0$ and $n > -1$ be given. Then the equation

$$\frac{s\left(w, r\left(k\right)\right)}{k} = 1+n. \tag{3.47}$$

has at least one solution $k > 0$.

Proof. Note that $1+n > 0$. From Lemma 1 in Section 3.3 follows that for all possible values of $r(k)$, $0 < s(w, r(k)) < w$. Hence, for any $k > 0$,

$$0 < \frac{s\left(w, r\left(k\right)\right)}{k} < \frac{w}{k}.$$

Letting $k \to \infty$ we then have $s\left(w, r\left(k\right)\right)/k \to 0$ since $s\left(w, r\left(k\right)\right)/k$ is squeezed between 0 and 0 (as indicated in the two graphs in Fig. 3.9).

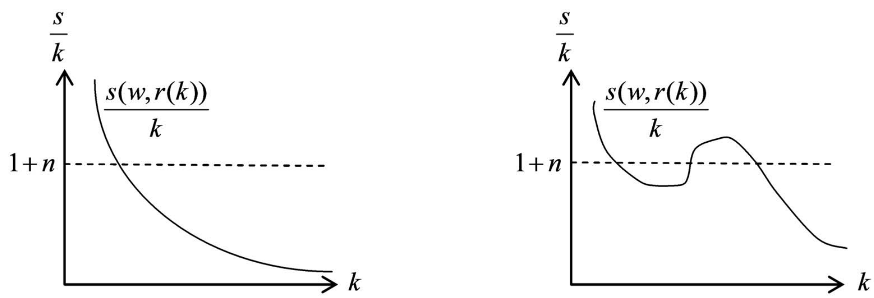


Figure 3.9: Existence of a solution to equation (3.47).

Next we consider $k \to 0$. There are two cases.

Case 1: $\lim_{k\to 0} s\left(w, r\left(k\right)\right) > 0$.[27] Then obviously $\lim_{k\to 0} s\left(w, r\left(k\right)\right)/k = \infty$.

[27] If the limit does not exist, the proof applies to the *limit inferior* of $s\left(w, r\left(k\right)\right)$ for $k \to 0$. The limit inferior for $i \to \infty$ of a sequence $\{x_i\}_{i=0}^{\infty}$ is defined as $\lim_{i\to\infty} \inf \{x_j | \, j = i, i+1, \ldots\}$, where inf of a set $S_i = \{x_j | \, j = i, i+1, \ldots\}$ is defined as the greatest lower bound for S_i.

Case 2: $\lim_{k\to 0} s(w, r(k)) = 0$.[28] In this case we have

$$\lim_{k\to 0} r(k) = \infty. \tag{3.48}$$

Indeed, since $f'(k)$ rises monotonically as $k \to 0$, the only alternative would be that $\lim_{k\to 0} r(k)$ exists and is $< \infty$; then, by Lemma 1 in Section 3.3, we would be in case 1 rather than case 2. By the second-period budget constraint, with $r = r(k)$, consumption as old is $c_2 = s(w, r(k))(1 + r(k)) \equiv c(w, k) > 0$ so that

$$\frac{s(w, r(k))}{k} = \frac{c(w, k)}{[1 + r(k)]\, k}.$$

The right-hand side of this equation goes to ∞ for $k \to 0$ since $\lim_{k\to 0} [1 + r(k)]\, k = 0$ by Technical Remark in Section 3.4 and $\lim_{k\to 0} c(w, k) = \infty$; this latter fact follows from the first-order condition (3.8), which can be written

$$0 \leq u'(c(w, k)) = (1 + \rho)\frac{u'(w - s(w, r(k))}{1 + r(k)} \leq (1 + \rho)\frac{u'(w)}{1 + r(k)}.$$

Taking limits on both sides gives

$$\lim_{k\to 0} u'(c(w, k)) = (1 + \rho) \lim_{k\to 0} \frac{u'(w - s(w, r(k)))}{1 + r(k)} = (1 + \rho) \lim_{k\to 0} \frac{u'(w)}{1 + r(k)} = 0,$$

where the second equality comes from the fact that we are in case 2 and the third comes from (3.48). But since $u'(c) > 0$ and $u''(c) < 0$ for all $c > 0$, $\lim_{k\to 0} u'(c(w, k)) = 0$ requires $\lim_{k\to 0} c(w, k) = \infty$, as was to be shown.

In both Case 1 and Case 2 we thus have that $k \to 0$ implies $s(w, r(k))/k \to \infty$. Since $s(w, r(k))/k$ is a continuous function of k, there must be at least one $k > 0$ such that (3.47) holds (as illustrated by the two graphs in Fig. 3.14). □

Now, to prove (i) of Proposition 2, consider an arbitrary $k_t > 0$. We have $w(k_t) > 0$. In (3.47), let $w = w(k_t)$. By Lemma C1, (3.47) has a solution $k > 0$. Set $k_{t+1} = k$. Starting with $t = 0$, from a given $k_0 > 0$ we thus find a $k_1 > 0$ and letting $t = 1$, from the now given k_1 we find a k_2 and so on. The resulting infinite sequence $\{k_t\}_{t=0}^{\infty}$ is an equilibrium path. In this way we have proved existence of an equilibrium path if $k_0 > 0$. Thereby (i) of Proposition 2 is proved.

But what if $k_0 = 0$? Then, if $f(0) = 0$, no temporary equilibrium is possible in period 0, in view of (ii) of Proposition 1; hence there can be no equilibrium path. Suppose $f(0) > 0$. Then $w(k_0) = w(0) = f(0) > 0$, as explained in Technical Remark in Section 3.4. Let w in equation (3.47) be equal to $f(0)$. By Lemma C1 this equation has a solution $k > 0$. Set $k_1 = k$. Letting period 1 be the new initial period, we are back in the case with initial capital positive. This proves (ii) of Proposition 2.

[28] If the limit does not exist, the proof applies to the *limit inferior* of $s(w, r(k))$ for $k \to 0$.

E. Sufficient conditions for certain properties of the transition curve

Positive slope everywhere For convenience we repeat here the condition (3.36):

$$\frac{1}{1-\gamma} > \frac{1-\sigma}{1+(1+\rho)^{-\sigma}(1+f'(k)-\delta)^{\sigma-1}}, \tag{*}$$

where we have substituted $\sigma \equiv 1/\theta$. In Section 3.5.3 we claimed that in the CRRA-CES case this condition is sufficient for the transition curve to be positively sloped everywhere. We here prove the claim.

Consider an arbitrary $k_t > 0$ and let $w \equiv w(k_t) > 0$. Knowing that $w'(k_t) > 0$ for all $k_t > 0$, we can regard k_{t+1} as directly linked to w. With k representing k_{t+1}, k must satisfy the equation $k = s(w, r(k))/(1+n)$. A sufficient condition for this equation to implicitly define k as an increasing function of w is also a sufficient condition for the transition curve to be positively sloped for all $k_t > 0$.

When $u(c)$ belongs to the CRRA class, by (3.15) with $\sigma \equiv 1/\theta$, we have $s(w, r(k)) = \left[1+(1+\rho)^{\sigma}(1+r(k))^{1-\sigma}\right]^{-1} w$. The equation $k = s(w, r(k))/(1+n)$ then implies

$$\frac{w}{1+n} = k\left[1+(1+\rho)^{\sigma}R(k)^{1-\sigma}\right] \equiv h(k), \tag{3.49}$$

where $R(k) \equiv 1 + r(k) \equiv 1 + f'(k) - \delta > 0$ for all $k > 0$. It remains to provide a sufficient condition for obtaining $h'(k) > 0$ for all $k > 0$. We have

$$h'(k) = 1 + (1+\rho)^{\sigma}R(k)^{1-\sigma}\left[1-(1-\sigma)\eta(k)\right], \tag{3.50}$$

since $\eta(k) \equiv -kR'(k)/R(k) > 0$, the sign being due to $R'(k) = f''(k) < 0$. So $h'(k) > 0$ if and only if $1-(1-\sigma)\eta(k) > -(1+\rho)^{-\sigma}R(k)^{\sigma-1}$, a condition equivalent to

$$\frac{1}{\eta(k)} > \frac{1-\sigma}{1+(1+\rho)^{-\sigma}R(k)^{\sigma-1}}. \tag{3.51}$$

To make this condition more concrete, consider the CES production function

$$f(k) = A(\alpha k^{\gamma} + 1 - \alpha), \qquad A > 0, 0 < \alpha < 1, \gamma < 1. \tag{3.52}$$

Then $f'(k) = \alpha A^{\gamma}(f(k)/k)^{1-\gamma}$ and defining $\pi(k) \equiv f'(k)k/f(k)$ we find

$$\eta(k) = (1-\gamma)\frac{(1-\pi(k))f'(k)}{1-\delta+f'(k)} \leq (1-\gamma)(1-\pi(k)) < 1-\gamma, \tag{3.53}$$

where the first inequality is due to $0 \leq \delta \leq 1$ and the second to $0 < \pi(k) < 1$, which is an implication of strict concavity of f combined with $f(0) \geq 0$. Thus, $\eta(k)^{-1} > (1-\gamma)^{-1}$ so that if (*) holds for all $k > 0$, then so does (3.51), i.e., $h'(k) > 0$ for all $k > 0$. We have hereby shown that (*) is sufficient for the transition curve to be positively sloped everywhere.

Transition curve steep for k small Here we specialize further and consider the CRRA-Cobb-Douglas case: $u(c) = (c^{1-\theta} - 1)/(1-\theta)$, $\theta > 0$, and $f(k) = Ak^{\alpha}$, $A > 0$, $0 < \alpha < 1$. In the prelude to Proposition 4 in Section 3.5 it was claimed that if this combined utility and technology condition holds at least for small k, then (ii) of (A3) is satisfied. We now show this.

Letting $\gamma \to 0$ in (3.52) gives the Cobb-Douglas function $f(k) = Ak^{\alpha}$ (this is proved in the appendix to Chapter 4). With $\gamma = 0$, clearly $(1-\gamma)^{-1} = 1 > 1 - \sigma$, where $\sigma \equiv \theta^{-1} > 0$. This inequality implies that (*) above holds and so the transition curve is positively sloped everywhere. As an implication there is a transition function, φ, such that $k_{t+1} = \varphi(k_t)$, $\varphi'(k_t) > 0$. Moreover, since $f(0) = 0$, we have, by Lemma 5, $\lim_{k_t \to 0} \varphi(k_t) = 0$.

Given the imposed CRRA utility, the fundamental difference equation of the model is

$$k_{t+1} = \frac{w(k_t)}{(1+n)\left[1 + (1+\rho)^{\sigma} R(k_{t+1})^{1-\sigma}\right]} \tag{3.54}$$

or, equivalently,

$$h(k_{t+1}) = \frac{w(k_t)}{1+n},$$

where $h(k_{t+t})$ is defined as in (3.49). By implicit differentiation we find $h'(k_{t+1})\varphi'(k_t) = w'(k_t)/(1+n)$, i.e.,

$$\varphi'(k_t) = \frac{w'(k_t)}{(1+n)h'(k_{t+1})} > 0.$$

If $k^* > 0$ is a steady-state value of k_t, (3.54) implies

$$1 + (1+\rho)^{\sigma} R(k^*)^{1-\sigma} = \frac{w(k^*)}{(1+n)k^*}, \tag{3.55}$$

and the slope of the transition curve at the steady state will be

$$\varphi'(k^*) = \frac{w'(k^*)}{(1+n)h'(k^*)} > 0. \tag{3.56}$$

If we can show that such a $k^* > 0$ exists, is unique, and implies $\varphi'(k^*) < 1$, then the transition curve crosses the 45° line from above, and so (ii) of (A3) follows in view of $\lim_{k_t \to 0} = 0$.

Defining $x(k) \equiv f(k)/k = Ak^{\alpha-1}$, where $x'(k) = (\alpha - 1)Ak^{\alpha-2} < 0$, and using that $f(k) = Ak^{\alpha}$, we have $R(k) = 1 + \alpha x(k) - \delta$ and $w(k)/k = (1-\alpha)x(k)$. Hence, (3.55) can be written

$$1 + (1+\rho)^{\sigma}(1 + \alpha x^* - \delta)^{1-\sigma} = \frac{1-\alpha}{1+n} x^*, \tag{3.57}$$

where $x^* = x(k^*)$. It is easy to show graphically that this equation has a unique solution $x^* > 0$ whether $\sigma < 1$, $\sigma = 1$, or $\sigma > 1$. Then $k^* = (x^*/A)^{1/(\alpha-1)} > 0$ is also unique.

By (3.50) and (3.57),

$$\begin{aligned} h'(k^*) &= 1 + (\frac{1-\alpha}{1+n}x^* - 1)\,[1 - (1-\sigma)\eta(k^*)] > 1 + (\frac{1-\alpha}{1+n}x^* - 1)(1 - \eta(k^*)) \\ &\geq 1 + (\frac{1-\alpha}{1+n}x^* - 1)\alpha, \end{aligned}$$

where the first inequality is due to $\sigma > 0$ and the second to the fact that $\eta(k) \leq 1 - \alpha$ in view of (3.53) with $\gamma = 0$ and $\pi(k) = \alpha$. Substituting this together with $w'(k^*) = (1-\alpha)\alpha x^*$ into (3.56) gives

$$0 < \varphi'(k^*) < \frac{\alpha x^*}{1 + n + \alpha x^*} < 1, \tag{3.58}$$

as was to be shown.

The CRRA-Cobb-Douglas case is well-behaved For the case of CRRA utility and Cobb-Douglas technology with CRS, existence and uniqueness of a steady state has just been proved. Asymptotic stability follows from (3.58). So the CRRA-Cobb-Douglas case is well-behaved.

3.10 Exercises

3.1 The dynamic accounting relation for a closed economy is

$$K_{t+1} = K_t + S^N \qquad (*)$$

where K_t is the aggregate capital stock and S_t^N is aggregate net saving. In the Diamond model, let S_{1t} be aggregate net saving of the young in period t and S_{2t} aggregate net saving of the old in the same period. On the basis of (*) give a direct proof that the link between two successive periods takes the form $k_{t+1} = s_t/(1+n)$, where s_t is the saving of each young, n is the population growth rate, and k_{t+1} is the capital/labor ratio at the beginning of period $t+1$. *Hint:* by definition, the increase in financial wealth is the same as net saving (ignoring gifts).

3.2 Suppose the production function in Diamond's OLG model is $Y = A(\alpha K^\gamma + (1-\alpha)L^\gamma)^{1/\gamma}$, $A > 0, 0 < \alpha < 1, \gamma < 0$, and $A\alpha^{1/\gamma} < 1+n$. a) Given $k \equiv K/L$, find the equilibrium real wage, $w(k)$. b) Show that $w(k) < (1+n)k$ for all $k > 0$. *Hint:*

consider the roof. c) Comment on the implication for the long-run evolution of the economy. *Hint:* consider the ceiling.

3.3 *(multiple temporary equilibria with self-fulfilling expectations)* Fig. 3.10 shows the transition curve for a Diamond OLG model with $u(c) = c^{1-\theta}/(1-\theta)$, $\theta = 8$, $\rho = 0.4$, $n = 0.2$, $\delta = 0.6$, $f(k) = A(bk^p + 1 - b)^{1/p}$, $A = 7$, $b = 0.33$, $p = -0.4$.

a) Let $t = 0$. For a given k_0 slightly below 1, how many temporary equilibria with self-fulfilling expectations are there?

b) Suppose the young in period 0 expect the real interest rate on their saving to be relatively low. Describe by words the resulting equilibrium path in this case. Comment (what is the economic intuition behind the path?).

c) In the first sentence under b), replace "low" by "high". How is the answer to b) affected? What kind of difficulty arises?

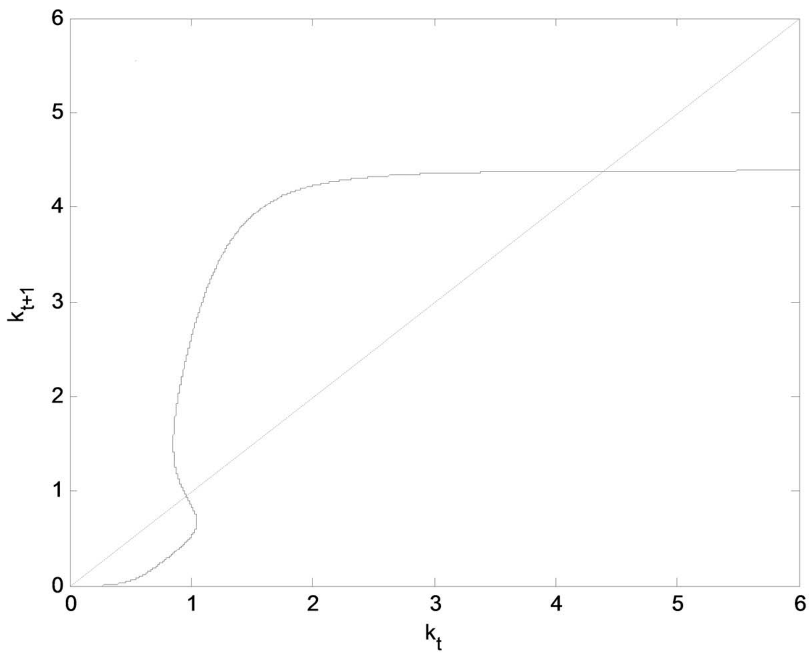


Figure 3.10: Transition curve for Diamond's OLG model in the case described in Exercise 3.3.

3.4 *(plotting the transition curve by MATLAB)* This exercise requires computation on a computer. You may use *MATLAB OLG program.*[29]

[29]Made by Marc P. B. Klemp and available at the address:

a) Enter the model specification from Exercise 3.3 and plot the transition curve.

b) Plot examples for two other values of the substitution parameter: $p = -1.0$ and $p = 0.5$. Comment.

c) Find the approximate largest lower bound for p such that higher values of p eliminates multiple equilibria.

d) In continuation of c), what is the corresponding elasticity of factor substitution, ψ? *Hint:* as shown in §4.4, the formula is $\psi = 1/(1-p)$.

e) The empirical evidence for industrialized countries suggests that $0.4 < \psi < 1.0$. Is your ψ from d) empirically realistic? Comment.

3.5 *(one stable and one unstable steady state)* Consider the following Diamond model: $u(c) = \ln c$, $\rho = 2.3$, $n = 2.097$, $\delta = 1.0$, $f(k) = A(bk^p + 1 - b)^{1/p}$, $A = 20$, $b = 0.5$, $p = -1.0$.

a) Plot the transition curve of the model. *Hint:* you may use either a program like *MATLAB OLG Program* (available on the course website) or first a little algebra and then Excel (or similar simple software).

b) Comment on the result you get. Will there exist a poverty trap? Why or why not?

c) At the stable steady state calculate numerically the output-capital ratio, the aggregate saving-income ratio, the real interest rate, and the capital income share of gross national income.

d) Briefly discuss how your results in c) comply with your knowledge of corresponding empirical magnitudes in industrialized Western countries?

e) There is one feature which this model, as a long-run model, ought to incorporate, but does not. Extend the model, taking this feature into account, and write down the fundamental difference equation for the extended model in algebraic form.

f) Plot the new transition curve. *Hint:* given the model specification, this should be straightforward if you use Excel (or similar); and if you use MATLAB OLG Program, note that by a simple "trick" you can transform your new model into the "old" form.

http://www.econ.ku.dk/okocg/Computation/.

g) The current version of the MATLAB OLG Program is not adapted to this question. So at least here you need another approach, for instance based on a little algebra and then Excel (or similar simple software). Given $k_0 = 10$, calculate numerically the time path of k_t and plot the *time profile* of k_t, i.e., the graph (t, k_t) in the tk-plane. Next, do the same for $k_0 = 1$. Comment.

3.6 *(dynamics under myopic foresight)*
(incomplete) Show the possibility of a chaotic trajectory.

3.7 Given the period utility function is CRRA, derive the saving function of the young in Diamond's OLG model. *Hint:* substitute the period budget constraints into the Euler equation.

3.8 *Short questions* a) A steady-state capital-labor ratio can be in the "dynamically efficient" region or in the "dynamically inefficient" region. How are the two mentioned regions defined? b) Give a simple characterization of the two regions. c) The First Welfare Theorem states that, given certain conditions, any competitive equilibrium ($\equiv$ Walrasian equilibrium) is Pareto optimal. Give a list of circumstances that each tend to obstruct Pareto optimality of a competitive equilibrium.

3.9 Consider a Diamond OLG model for a closed economy. Let the utility discount rate be denoted ρ and let the period utility function be specified as $u(c) = \ln c$.

a) Derive the saving function of the young. Comment.

b) Let the aggregate production function be a neoclassical production function with CRS and ignore technological progress. Let L_t denote the number of young in period t. Derive the fundamental difference equation of the model.

From now, assume that the production function is $Y = \alpha L + \beta KL/(K + L)$, where $\alpha > 0$ and $\beta > 0$ (as in Problem 2.4).

c) Draw a transition diagram illustrating the dynamics of the economy. Make sure that you draw the diagram so as to exhibit consistency with the production function.

d) Given the above information, can we be sure that there exists a unique and globally asymptotically stable steady state? Why or why not?

e) Suppose the economy is in a steady state up to and including period $t_0 > 0$. Then, at the shift from period t_0 to period $t_0 + 1$, a negative technology shock occurs such that the technology level in period $t_0 + 1$ is below that of period t_0. Illustrate by a transition diagram the evolution of the economy from period t_0 onward. Comment.

f) Let $k \equiv K/L$. In the $(t, \ln k)$ plane, draw a graph of $\ln k_t$ such that the qualitative features of the time path of $\ln k$ before and after the shock, including the long run, are exhibited.

g) How, if at all, is the real interest rate in the long run affected by the shock?

h) How, if at all, is the real wage in the long run affected by the shock?

i) How, if at all, is the labor income share of national income in the long run affected by the shock?

j) Explain by words the economic intuition behind your results in h) and i).

3.10

Chapter 4

A growing economy

In the previous chapter we ignored technological progress. An incontestable fact of real life in industrialized countries is, however, the presence of a persistent rise in GDP per capita – on average between 1.5 and 2.5 percent per year since 1870 in many developed economies. In regard to UK, USA, and Japan, see Fig. 4.1; and in regard to Denmark, see Fig. 4.2. In spite of the somewhat dubious quality of the data from before the Second World War, this observation should be taken into account in a model which, like the Diamond model, aims at dealing with long-run issues. For example, in relation to the question of dynamic inefficiency, cf. Chapter 3, the cut-off value of the steady-state interest rate is the steady-state GDP growth rate of the economy and this growth rate increases one-to-one with the rate of technological progress. We shall therefore now introduce technological progress.

On the basis of a summary of "stylized facts" about growth, Section 4.1 motivates the assumption that technological progress at the aggregate level takes the Harrod-neutral form. In Section 4.2 we extend the Diamond OLG model by incorporating this form of technological progress. Section 4.3 extends the concept of the golden rule to allow for the existence of technological progress. In Section 4.4 we address what is known as the marginal productivity theory of the functional income distribution and apply an expedient analytical tool, the elasticity of factor substitution. The next section defines the concept of elasticity of factor substitution at the general level. Section 4.6 then . goes into detail with the special case of a constant elasticity of factor substitution (the CES production function). Finally, Section 4.7 concludes with some general considerations regarding the concept of economic growth.

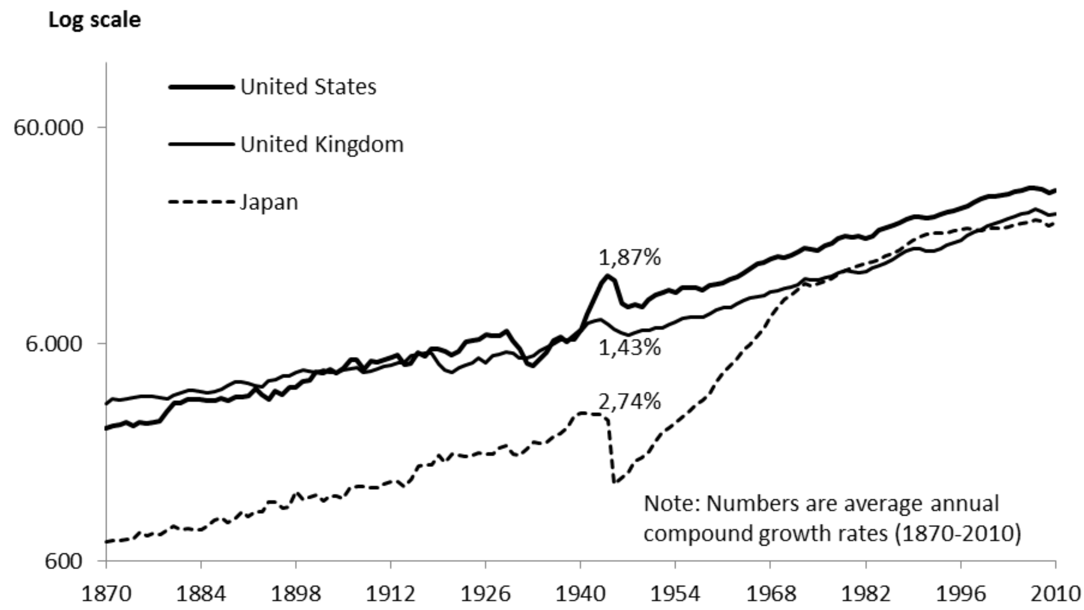


Figure 4.1: GDP per capita in USA, UK, and Japan 1870-2010. Source: Bolt and van Zanden (2013).

4.1 Harrod-neutrality and Kaldor's stylized facts

To allow for technological change, we may write aggregate production this way

$$Y_t = \tilde{F}(K_t, L_t, t), \tag{4.1}$$

where Y_t, K_t, and L_t stand for output, capital input, and labor input, respectively. Changes in technology are here represented by the dependency of the production function $\tilde{F}$ on time, t. For fixed t, the production function may still be for instance neoclassical with respect to the role of the factor inputs, the first two arguments. Often we assume that $\tilde{F}$ depends in a smooth way on time such that the partial derivative, $\partial\tilde{F}_t/\partial t$, exists and is a continuous function of (K_t, L_t, t). When $\partial\tilde{F}_t/\partial t > 0$, technological change amounts to technological *progress:* for K_t and L_t held constant, output increases with t.

A particular form of the time-dependency of the production function has attracted the attention of macroeconomists. This is known as *Harrod-neutral technological progress* and is present when we can rewrite $\tilde{F}$ such that

$$Y_t = F(K_t, T_t L_t), \tag{4.2}$$

where the "level of technology" is represented by a coefficient, T_t, on the labor input, and this coefficient is rising over time. An alternative name for this is *labor-augmenting* technological progress. The name "labor-augmenting" may sound as

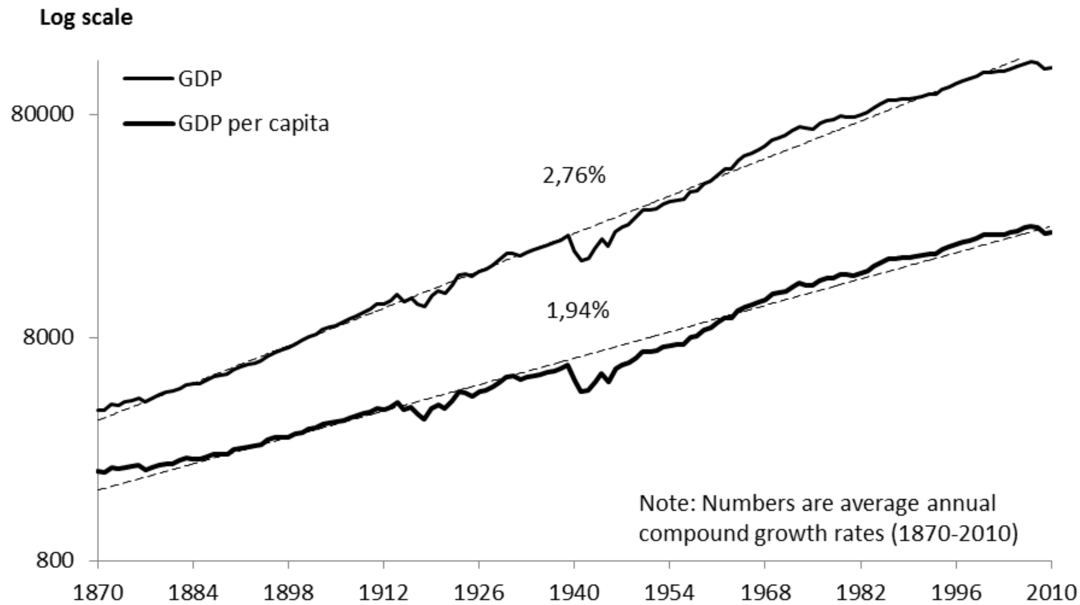


Figure 4.2: GDP and GDP per capita. Denmark 1870-2006. Sources: Bolt and van Zanden (2013); Maddison (2010); The Conference Board Total Economy Database (2013).

if *more* labor is required to reach a given output level for given capital. In fact, the opposite is the case, namely that T_t has risen so that *less* labor input is required. The idea is that the technological change − a certain percentage increase in T − affects the output level *as if* the labor input had been increased exactly by this percentage, and nothing else had happened.

The interpretation of Harrod neutrality is not that something miraculous happens to the labor input. The content of (4.2) is just that technological innovations are assumed to predominantly be such that not only do labor and capital *in combination* become more productive, but this happens to *manifest itself* in the form (4.2), that is, *as if* an improvement in the *quality* of the labor input had occurred.[1]

Kaldor's stylized facts

The reason that macroeconomists often assume that technological change at the aggregate level takes the Harrod-neutral form, as in (4.2), and not for example the

[1] As is usual in simple macroeconomic models, in both (4.1) and (4.2) it is simplifying assumed that technological progress is *disembodied.* This means that new technical and organizational knowledge increases the combined productivity of capital and workers independently of when the first were constructed and the latter educated, cf. Chapter 2.2.

form $Y_t = F(X_t K_t, T_t L_t)$ (where both X and T are changing over time, at least one of them growing), is the following. You want the long-run properties of the model to comply with Kaldor's list of "stylized facts" (Kaldor 1961) concerning the long-run evolution of certain "Great Ratios" of industrialized economies. Abstracting from short-run fluctuations, Kaldor's "stylized facts" are:

1. K/L and Y/L are growing over time and have roughly constant growth rates;

2. the output-capital ratio, Y/K, the income share of labor, wL/Y, and the economy-wide rate of return to capital, $(Y - wL - \delta K)/K$,[2] are roughly constant over time;

3. the growth rate of Y/L can vary substantially across countries for quite long time.

Ignoring the conceptual difference between the path of Y/L and that of Y *per capita* (a difference not so important in this context), the figures 4.1 and 4.2 illustrate Kaldor's "fact 1" about the long-run property of the Y/L path for the more developed countries. Japan had an extraordinarily high growth rate of GDP per capita for a couple of decades after World War II, usually explained by fast technology transfer from the most developed countries (the catching-up process which can only last until the technology gap is eliminated). Fig. 4.3 gives rough support for a part of Kaldor's "fact 2", namely the claim about long-run constancy of the labor income share of national income. "Fact 3" about large diversity across countries regarding the growth rate of Y/L over long time intervals is well documented empirically.[3]

It is fair to add, however, that the claimed regularities 1 and 2 do not fit all developed countries equally well. While Solow's famous growth model (Solow, 1956) can be seen as the first successful attempt at building a model consistent with Kaldor's "stylized facts", Solow himself once remarked about them: "There is no doubt that they are stylized, though it is possible to question whether they are facts" (Solow, 1970). Recently, several empiricists (see Literature notes) have questioned the methods which standard national income accounting applies to separate the income of entrepreneurs, sole proprietors, and unincorporated businesses into labor and capital income. It is claimed that these methods obscure a tendency of the labor income share to fall in recent decades.

[2] In this formula w is the real wage and δ is the capital depreciation rate. Land is ignored. For countries where land is a quantitatively important production factor, the denominator should be replaced by $K + p_J J$, where p_J is the real price of land, J.

[3] For a summary, see Pritchett (1997).

Figure 4.3: Labor's share of GDP in USA (1950-2011) and Denmark (1970-2011). Source: Feenstra, Inklaar and Timmer (2013), www.ggdc.net/pwt.

Notwithstanding these ambiguities, it is definitely a fact that many long-run models are constructed so a to comply with Kaldor's stylized facts. Let us briefly take a look at the Solow model (in discrete time) and check its consistency with Kaldor's "stylized facts". The point of departure of the Solow model and many other growth models is the *dynamic resource constraint for a closed economy*:

$$K_{t+1} - K_t = I_t - \delta K_t = S_t - \delta K_t \equiv Y_t - C_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \tag{4.3}$$

where I_t is gross investment, which in a closed economy equals gross saving, $S_t \equiv Y_t - C_t$; δ is a constant capital depreciation rate, $0 \leq \delta \leq 1$.

The Solow model and Kaldor's stylized facts

As is well-known, the Solow model postulates a constant aggregate saving-income *ratio*, $\hat{s}$, so that $S_t = \hat{s}Y_t$, $0 < \hat{s} < 1$.[4] Further, the model assumes that the aggregate production function is neoclassical and features Harrod-neutral technological progress. So, let F in (4.2) be Solow's production function. To this Solow adds assumptions of CRS and exogenous geometric growth in both the technology level T and the labour force L, i.e., $T_t = T_0(1+g)^t$, $g \geq 0$, and $L_t = L_0(1+n)^t$, $n > -1$. In view of CRS, we have $Y = F(K, AL) = TLF(\tilde{k}, 1) \equiv TLf(\tilde{k})$, where $\tilde{k} \equiv K/(TL)$ is the *effective capital-labor ratio* while $f' > 0$ and $f'' < 0$.

[4] Note that $\hat{s}$ is a *ratio* while the s in the Diamond model stands for the *saving* per young.

Substituting $S_t = \hat{s}Y_t$ into $K_{t+1} - K_t = S_t - \delta K_t$, dividing through by $T_t(1+g)L_t(1+n)$ and rearranging gives the "law of motion" of the Solow economy:

$$\tilde{k}_{t+1} = \frac{\hat{s}f(\tilde{k}_t) + (1-\delta)\tilde{k}_t}{(1+g)(1+n)} \equiv \varphi(\tilde{k}_t). \tag{4.4}$$

Defining $G \equiv (1+g)(1+n)$, we have $\varphi'(\tilde{k}) = (\hat{s}f'(\tilde{k}) + 1 - \delta)/G > 0$ and $\varphi''(\tilde{k}) = \hat{s}f''(\tilde{k})/G < 0$. If $G > 1 - \delta$ and f satisfies the Inada conditions $\lim_{\tilde{k}\to 0} f'(\tilde{k}) = \infty$ and $\lim_{\tilde{k}\to\infty} f'(\tilde{k}) = 0$, there is a unique and globally asymptotically stable steady state $\tilde{k}^* > 0$. The transition diagram looks entirely as in Fig. 3.4 of the previous chapter (ignoring the tildes).[5] The convergence of $\tilde{k}$ to $\tilde{k}^*$ implies that in the long run we have $K/L = \tilde{k}^*T$ and $Y/L = f(\tilde{k}^*)T$. Both K/L and Y/L are consequently growing at the same constant rate as T, the rate g. And constancy of $\tilde{k}$ implies that $Y/K = f(\tilde{k})/\tilde{k}$ is constant and so is the labor income share, $wL/Y = (f(\tilde{k}) - \tilde{k}f'(\tilde{k}))/f(\tilde{k})$, and hence also the net rate of return, $(1-wL/Y)Y/K-\delta$.

It follows that the Solow model complies with the stylized facts 1 and 2 above. Many different models aim at doing that. What these models must then have *in common* is a capability of generating *balanced growth*.

Balanced growth

With K_t, Y_t, and C_t denoting aggregate capital, output, and consumption as above, we define a balanced growth path the following way:

DEFINITION 1 A *balanced growth path*, BGP, is a path $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ along which the variables K_t, Y_t, and C_t are positive and grow at constant rates (not necessarily positive).

At least for a closed economy there is a general equivalence relationship between balanced growth and constancy of certain key ratios like Y/K and C/Y. This relationship is an implication of accounting based on the above aggregate dynamic resource constraint (4.3).

For an arbitrary variable $x_t \in \mathbb{R}_{++}$, we define $\Delta x_t \equiv x_t - x_{t-1}$. Whenever $x_{t-1} > 0$, the *growth rate* of x from $t-1$ to t, denoted $g_x(t)$, is defined by $g_x(t) \equiv \Delta x_t / x_{t-1}$. When there is no risk of confusion, we suppress the explicit dating and write $g_x \equiv \Delta x / x$.

PROPOSITION 1 *(the balanced growth equivalence theorem).* Let $P \equiv \{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ be a path along which K_t, Y_t, C_t, and $S_t \equiv Y_t - C_t$ are positive for all $t = 0, 1, 2, \ldots$.

[5] What makes the Solow model so easily tractable compared to the Diamond OLG model is the constant saving-income ratio which makes the transition function essentially dependent only on the production function in intensive form. Owing to dimishing marginal productivity of capital, this is a strictly concave function. Anyway, the Solow model emerges as a special case of the Diamond model, see Exercise IV.??.

Then, given the dynamic resource constraint for a closed economy, (4.3), the following holds:
(i) If P is a BGP, then $g_Y = g_K = g_C$ and the ratios Y/K and C/Y are constant.
(ii) If Y/K and C/Y are constant, then P is a BGP with $g_Y = g_K = g_C$, i.e., not only is balanced growth present but the constant growth rates of Y, K, and C are the same.

Proof Consider a path $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ along which K, Y, C, and $S_t \equiv Y - C_t$ are positive for all $t = 0, 1, 2, \ldots$.

(i) Suppose the path is a balanced growth path. Then, by definition, g_Y, g_K, and g_C are constant. Hence, by (4.3), $S/K = g_K + \delta$ must be constant, implying[6]

$$g_S = g_K. \tag{*}$$

By (4.3), $Y \equiv C + S$, and so

$$\begin{aligned} g_Y &= \frac{\Delta Y}{Y} = \frac{\Delta C}{Y} + \frac{\Delta S}{Y} = \frac{C}{Y}g_C + \frac{S}{Y}g_S = \frac{C}{Y}g_C + \frac{S}{Y}g_K \qquad \text{(by (*))} \\ &= \frac{C}{Y}g_C + \frac{Y-C}{Y}g_K = \frac{C}{Y}(g_C - g_K) + g_K. \qquad (**) \end{aligned}$$

Let us provisionally assume that $g_C \neq g_K$. Then (**) gives

$$\frac{C}{Y} = \frac{g_Y - g_K}{g_C - g_K}, \tag{***}$$

a constant since g_Y, g_K, and g_C are constant. Constancy of C/Y requires $g_C = g_Y$, hence, by (***), $C/Y = 1$, i.e., $C = Y$. In view of $Y \equiv C + S$, however, this implication contradicts the given condition that $S > 0$. Hence, our provisional assumption and its implication (***) are falsified. Instead we have $g_C = g_K$. By (**), this implies $g_Y = g_K = g_C$, but now without the condition $C/Y = 1$ being implied. It follows that Y/K and C/Y are constant.

(ii) Suppose Y/K and C/Y are positive constants. Applying that the ratio between two variables is constant if and only if the variables have the same (not necessarily constant or positive) growth rate, we can conclude that $g_Y = g_K = g_C$. By constancy of C/Y follows that $S/Y \equiv 1 - C/Y$ is constant. So $g_S = g_Y = g_K$, which in turn implies that S/K is constant. By (4.3),

$$\frac{S}{K} = \frac{\Delta K + \delta K}{K} = g_K + \delta,$$

[6] The ratio between two positive variables is constant if and only if the variables have the same growth rate (not necessarily constant or positive). For this and similar simple growth-arithmetic rules, see Appendix A.

so that also g_K is constant. This, together with constancy of Y/K and C/Y, implies that also g_Y and g_C are constant. □

Remark. It is part (i) of the proposition which requires the assumption $S > 0$ for all $t \geq 0$. If $S = 0$, we would have $g_K = -\delta$ and $C \equiv Y - S = Y$, hence $g_C = g_Y$ for all $t \geq 0$. Then there would be balanced growth if the common value of C and Y had a constant growth rate. This growth rate, however, could easily differ from that of K. Suppose $Y = AK^\alpha L^{1-\alpha}$, $0 < \alpha < 1$, $g_A = \gamma$ and $g_L = n$, where γ and n are constants. By the product and power function rule (see Appendix A), we would then have $1 + g_C = 1 + g_Y = (1 + \gamma)(1 - \delta)^\alpha(1 + n)^{1-\alpha}$, which could easily be larger than 1 and thereby different from $1 + g_K = 1 - \delta \leq 1$ so that (i) no longer holds. *Example:* If $\delta = n = 0 < \gamma$, then $1 + g_Y = 1 + \gamma > 1 = 1 + g_K$.

It is part (ii) of the proposition which requires the assumption of a closed economy. In an open economy we do not necessarily have $I = S$, hence constancy of S/K no longer implies constancy of $g_K = I/K - \delta$. □

For many long-run closed-economy models, including the Diamond OLG model, it holds that if and only if the dynamic system implied by the model is in a steady state, will the economy feature balanced growth, cf. Proposition 4 below. There *exist* cases, however, where this equivalence between steady state and balanced growth does not hold (some open economy models and some models with *embodied* technological change). Hence, we shall maintain a distinction between the two concepts.

Note that Proposition 1 pertains to *any* model for which (4.3) is valid. No assumption about market form and economic agents' behavior are involved. And except for the assumed constancy of the capital depreciation rate δ, no assumption about the technology is involved, not even that constant returns to scale is present.

Proposition 1 suggests that if one accepts Kaldor's stylized facts as a rough description of more than a century's growth experience and therefore wants the model to be consistent with them, one should construct the model so that it can generate balanced growth.

Balanced growth requires Harrod-neutrality

Our next proposition states that for a model to be capable of generating balanced growth, technological progress *must* take the Harrod-neutral form (i.e., be labor-augmenting). Also this proposition holds in a fairly general setting, but not as general as that of Proposition 1. Constant returns to scale and a constant growth rate in the labor force, two aspects about which Proposition 1 is silent, will now have a role to play.[7]

[7] On the other hand we do *not* imply that CRS is *always* necessary for a balanced growth path (see Exercise 4.??).

Consider an aggregate production function

$$Y_t = \tilde{F}(K_t, BL_t, t), \qquad B > 0,\ \tilde{F}_2' \geq 0, \tilde{F}_3' > 0, \tag{4.5}$$

where B is a constant that depends on measurement units, and the function $\tilde{F}$ is homogeneous of degree one with respect to the first two arguments (CRS) and is non-decreasing in its second argument and increasing in the third, time. The latter property represents technological progress: as time proceeds, unchanged inputs of capital and labor result in more and more output. Note that $\tilde{F}$ need not be neoclassical.

Let the labor force change at a constant rate:

$$L_t = L_0(1+n)^t, \qquad n > -1, \tag{4.6}$$

where $L_0 > 0$. The Japanese economist Hirofumi Uzawa (1928-) is famous for several contributions, not least his balanced growth theorem (Uzawa 1961).

PROPOSITION 2 *(Uzawa's balanced growth theorem).* Let $P \equiv \{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ be a path along which Y_t, K_t, C_t, and $S_t \equiv Y_t - C_t$ are positive for all $t = 0, 1, 2, \ldots$, and satisfy the dynamic resource constraint for a closed economy, (4.3), given the production function (4.5) and the labor force (4.6). Then:
(i) A *necessary* condition for the path P to be a BGP is that along P it holds that

$$Y_t = \tilde{F}(K_t, T_t L_t, 0), \tag{4.7}$$

where $T_t = T_0(1+g)^t$ with $T_0 = B$ and $1 + g \equiv (1 + g_Y)/(1+n) > 1$, g_Y being the constant growth rate of output along the BGP.
(ii) Assume $(1+g)(1+n) > 1 - \delta$. Then, for any $g \geq 0$ such that there is a $q > (1+g)(1+n) - (1-\delta)$ with the property that the production function $\tilde{F}$ in (4.5) allows an output-capital ratio equal to q at $t = 0$ (i.e., $\tilde{F}(1, \tilde{k}^{-1}, 0) = q$ for some real number $\tilde{k} > 0$), a *sufficient* condition for $\tilde{F}$ to be compatible with a BGP with output-capital ratio equal to q is that $\tilde{F}$ can be written as in (4.7) with $T_t = B(1+g)^t$.

Proof (i) Suppose the given path $\{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ is a BGP. By definition, g_K and g_Y are then constant so that $K_t = K_0(1+g_K)^t$ and $Y_t = Y_0(1+g_Y)^t$. With $t = 0$ in (4.5) we then have

$$Y_t(1+g_Y)^{-t} = Y_0 = \tilde{F}(K_0, BL_0, 0) = \tilde{F}(K_t(1+g_K)^{-t}, BL_t(1+n)^{-t}, 0). \tag{4.8}$$

In view of the assumption that $S_t \equiv Y_t - C_t > 0$, we know from (i) of Proposition 1, that Y/K is constant so that $g_Y = g_K$. By CRS, (4.8) then implies

$$\tilde{F}(K_t, B(1+g_Y)^t(1+n)^{-t}L_t, 0) = (1+g_Y)^t Y_0 = Y_t.$$

As $1+g \equiv (1+g_Y)/(1+n)$, this implies

$$Y_t = \tilde{F}(K_t, B(1+g)^t L_t, 0) = \tilde{F}(K_t, BL_t, t),$$

where the last equality comes from combining the first equality with (4.5). Now, the first equality shows that (4.7) holds for $T_t = B(1+g)^t = T_0(1+g)^t$. By $\tilde{F}_3'$ $(= \partial\tilde{F}/\partial t) > 0$ follows that for K_t and L_t fixed over time, Y_t is rising over time. For this to be consistent with the first equality, we must have $g > 0$.

(ii) See Appendix B. □

The form (4.7) indicates that along a BGP, technological progress must be Harrod-neutral. Moreover, by defining a new CRS production function F by $F(K_t, T_tL_t) \equiv \tilde{F}(K_t, T_tL_t, 0)$, we see that (i) of the proposition implies that at least along the BGP, we can rewrite the original production function this way:

$$Y_t = \tilde{F}(K_t, BL_t, t) = \tilde{F}(K_t, T_tL_t, 0) \equiv F(K_t, T_tL_t). \tag{4.9}$$

where F has CRS, and $T_t = T_0(1+g)^t$, with $T_0 = B$ and $1+g \equiv (1+g_Y)/(1+n)$.

What is the intuition behind the Uzawa result that for balanced growth to be possible, technological progress must at the aggregate level have the purely labor-augmenting form? We may first note that there is an asymmetry between capital and labor. Capital is an accumulated amount of non-consumed output and has thus at least a "tendency" to inherit the trend in output. In contrast, labor is a non-produced production factor. The labor force grows in an exogenous way and does *not* inherit the trend in output. Indeed, the ratio L_t/Y_t is free to adjust as t proceeds.

More specifically, consider the point of departure, the original production function (4.5). Because of CRS, it must satisfy

$$1 = \tilde{F}(\frac{K_t}{Y_t}, \frac{BL_t}{Y_t}, t). \tag{4.10}$$

We know from Proposition 1 that along a BGP, K_t/Y_t is constant. The assumption $\tilde{F}_3'$ $(= \partial\tilde{F}/\partial t) > 0$ implies that technological progress is present. Along a BGP, this progress must manifest itself in the form of a compensating change in L_t/Y_t in (4.10) as t proceeds, because otherwise the right-hand side of (4.10) would increase, which would contradict the constancy of the left-hand side. As we have in (4.5) assumed $\partial\tilde{F}/\partial L \geq 0$, the needed change in L_t/Y_t is a *fall*. The fall in L_t/Y_t must exactly offset the effect on $\tilde{F}$ of the rising t, when there is a fixed capital-output ratio and the left-hand side of (4.10) remains unchanged. It follows that along the considered BGP, L_t/Y_t is a decreasing function of t. The inverse, Y_t/L_t, is thus an *increasing* function of t. If we denote this function T_t, we end up with (4.9).

The generality of Uzawa's theorem is noteworthy. Like Proposition 1, Uzawa's theorem is about technically feasible paths, while economic institutions, market forms, and agents' behavior are not involved. The theorem presupposes CRS, but does not need that the technology has neoclassical properties not to speak of satisfying the Inada conditions. And the theorem holds for exogenous as well as endogenous technological progress.

A simple implication of the theorem is the following. Let y_t denote "labor productivity" in the sense of Y_t/L_t, k_t denote the capital-labor ratio, K_t/L_t, and c_t the consumption-labor ratio, C_t/L_t. We have:

COROLLARY Along a BGP with positive gross saving and the technology level T growing at a constant rate $g \geq 0$, output grows at the rate $(1+g)(1+n)-1$ ($\approx g+n$ for g and n "small") while labor productivity, y, capital-labor ratio, k, and consumption-labor ratio, c, all grow at the rate g.

Proof That $g_Y = (1+g)(1+n)-1$ follows from (i) of Proposition 2. As to g_y we have

$$y_t \equiv \frac{Y_t}{L_t} = \frac{Y_0(1+g_Y)^t}{L_0(1+n)^t} = y_0(1+g)^t,$$

since $1+g = (1+g_Y)/(1+n)$. This shows that y grows at the rate g. Moreover, $y/k = Y/K$, which is constant along a BGP, by (i) of Proposition 1. Hence k grows at the same rate as y. Finally, also $c/y \equiv C/Y$ is constant along a BGP, implying that also c grows at the same rate as y. □

Factor income shares

There is one facet of Kaldor's stylized facts which we have not yet related to Harrod-neutral technological progress, namely the claimed long-run "approximate" constancy of both the income share of labor and the rate of return on capital. It turns out that, if we assume (a) neoclassical technology, (b) profit maximizing firms, and (c) perfect competition in the output and factor markets, then these constancies are inherent in the combination of constant returns to scale and balanced growth.

To see this, let the aggregate production function be

$$Y_t = F(K_t, T_t L_t), \tag{4.11}$$

where F is neoclassical and has CRS. In view of perfect competition, the representative firm chooses inputs such that

$$\frac{\partial Y_t}{\partial K_t} = F_1(K_t, T_t L_t) = r_t + \delta, \qquad \text{and,} \tag{4.12}$$

$$\frac{\partial Y_t}{\partial L_t} = F_2(K_t, T_t L_t)T_t = w_t, \tag{4.13}$$

where the right-hand sides indicate the factor prices, r_t being the interest rate, δ the depreciation rate, and w_t the real wage.

In equilibrium the labor income share will be

$$\frac{w_t L_t}{Y_t} = \frac{\frac{\partial Y_t}{\partial L_t} L_t}{Y_t} = \frac{F_2(K_t, T_t L_t) T_t L_t}{Y_t}. \tag{4.14}$$

Since land as a production factor is ignored, gross capital income equals non-labor income, $Y_t - w_t L_t$. Denoting the gross capital income share by α_t, we thus have

$$\begin{aligned} \alpha_t &= \frac{Y_t - w_t L_t}{Y_t} = \frac{F(K_t, T_t L_t) - F_2(K_t, T_t L_t) T_t L_t}{Y_t} \\ &= \frac{F_1(K_t, T_t L_t) K_t}{Y_t} = \frac{\frac{\partial Y_t}{\partial K_t} K_t}{Y_t} = (r_t + \delta)\frac{K_t}{Y_t}, \end{aligned} \tag{4.15}$$

where we have used (4.13), Euler's theorem,[8] and then (4.12). Finally, when the capital good is nothing but a non-consumed output good, it has price equal to 1, and so the economy-wide rate of return on capital can be written

$$\frac{Y_t - w_t L_t - \delta K_t}{1 \cdot K_t} = \frac{Y_t - w_t L_t}{Y_t} \cdot \frac{Y_t}{K_t} - \delta = \alpha_t \cdot \frac{Y_t}{K_t} - \delta = r_t, \tag{4.16}$$

where the last equality comes from (4.15).

PROPOSITION 3 *(factor income shares under perfect competition)* Let the dynamic resource constraint for a closed economy be given as in (4.3). Assume F is neoclassical with CRS, and that the economy is competitive. Let the path $P = \{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ be BGP with positive gross saving. Then, along the path P:
(i) The gross capital income share equals some constant $\alpha \in (0, 1)$, and the labor income then equals $1 - \alpha$.
(ii) The rate of return on capital is $\alpha q - \delta$, where q is the constant output-capital ratio along the BGP.

Proof In view of CRS, $Y_t = F(K_t, T_t L_t) = T_t L_t F(\tilde{k}_t, 1) \equiv T_t L_t f(\tilde{k}_t)$, where $\tilde{k}_t \equiv K_t/(T_t L_t)$, and $f' > 0$, $f'' < 0$. From Proposition 1 follows that along the given path P, which is a BGP, Y_t/K_t is some constant, say q, equal to $f(\tilde{k}_t)/\tilde{k}_t$. Hence, $\tilde{k}_t$ is constant, say equal to $\tilde{k}^*$. Consequently, along P, $\partial Y_t/\partial K_t = f'(\tilde{k}^*) = r_t + \delta$. From this follows that r_t is a constant, r. (i) From (4.15) now follows that $\alpha_t = f'(\tilde{k}^*)/q \equiv \alpha$. Moreover, $0 < \alpha < 1$, since $0 < \alpha$ is implied by $f' > 0$, and $\alpha < 1$ is implied by the fact that $f'(\tilde{k}^*) < f(\tilde{k}^*)/\tilde{k}^* = Y/K = q$, where "$<$" is

[8] Indeed, from Euler's theorem follows that $F_1 K + F_2 TL = F(K, TL)$, when F is homogeneous of degree one.

due to $f'' < 0$ and $f(0) \geq 0$ (draw the graph of $f(\tilde{k})$). By the first equality in (4.15), the labor income share can be written $w_t L_t / Y_t = 1 - \alpha_t = 1 - \alpha$. (ii) Consequently, by (4.16), the rate of return on capital equals r_t $(= r) = \alpha q - \delta$. □

What this proposition amounts to is that a BGP in this economy exhibits both the first and the second "Kaldor fact" (point 1 and 2, respectively, in the list at the beginning of the chapter).

Although the proposition implies constancy of the factor income shares under balanced growth, it does not *determine* them. The proposition expresses the factor income shares in terms of the unknown constants α and q. These constants will generally depend on the effective capital-labor ratio in steady state, $\tilde{k}^*$, which will generally be an unknown as long as we have not formulated a theory of saving. This takes us back to Diamond's OLG model which provides such a theory.

4.2 The Diamond OLG model with Harrod-neutral technological progress

Recall from the previous chapter that in the Diamond OLG model people live in two periods, as young and as old. Only the young work and each young supplies one unit of labor inelastically. The period utility function, $u(c)$, satisfies the No Fast Assumption. The saving function of the young is $s_t = s(w_t, r_{t+1})$. We now include Harrod-neutral technological progress in the Diamond model.

Let (4.11) be the aggregate production function in the economy and assume, as before, that F is neoclassical with CRS. The technology level T_t grows at a constant exogenous rate:

$$T_t = T_0(1+g)^t, \qquad g \geq 0. \tag{4.17}$$

The initial level of technology, T_0, is historically given. The employment level L_t equals the number of young and thus grows at the constant exogenous rate $n > -1$.

Suppressing for a while the explicit dating of the variables, in view of CRS with respect to K and TL, we have

$$\tilde{y} \equiv \frac{Y}{TL} = F(\frac{K}{TL}, 1) = F(\tilde{k}, 1) \equiv f(\tilde{k}), \qquad f' > 0, f'' < 0,$$

where TL is *labor input in efficiency units;* $\tilde{k} \equiv K/(TL)$ is known as the *effective* or *technology-corrected capital-labor ratio* - also sometimes just called the "capital intensity". There is perfect competition in all markets. In each period the representative firm maximizes profit, $\Pi = F(K, TL) - \hat{r}K - wL$. Given the constant capital depreciation rate $\delta \in [0, 1]$, this leads to the first-order conditions

$$\frac{\partial Y}{\partial K} = \frac{\partial \left[TLf(\tilde{k})\right]}{\partial K} = f'(\tilde{k}) = r + \delta, \tag{4.18}$$

and

$$\frac{\partial Y}{\partial L} = \frac{\partial \left[TLf(\tilde{k})\right]}{\partial L} = \left[f(\tilde{k}) - f'(\tilde{k})\tilde{k}\right] T = w. \tag{4.19}$$

In view of $f'' < 0$, a $\tilde{k}$ satisfying (4.18) is unique. We let its value in period t be denoted $\tilde{k}_t^d$. Assuming equilibrium in the factor markets, this desired effective capital-labor ratio equals the effective capital-labor ratio from the supply side, $\tilde{k}_t \equiv K_t/(T_t L_t) \equiv k_t/T_t$, which is predetermined in every period. The equilibrium interest rate and real wage in period t are thus determined by

$$r_t = f'(\tilde{k}_t) - \delta \equiv r(\tilde{k}_t), \quad \text{where} \quad r'(\tilde{k}_t) = f''(\tilde{k}_t) < 0, \tag{4.20}$$

$$w_t = \left[f(\tilde{k}_t) - f'(\tilde{k}_t)\tilde{k}\right] T_t \equiv \tilde{w}(\tilde{k}_t)T_t, \quad \text{where } \tilde{w}'(\tilde{k}_t) = -\tilde{k}_t f''(\tilde{k}_t) > 0. \tag{4.21}$$

Here, $\tilde{w}(\tilde{k}_t) = w_t/T_t$ is known as the *technology-corrected real wage.*

The equilibrium path

The aggregate capital stock at the beginning of period $t + 1$ must still be owned by the old generation in that period and thus equal the aggregate saving these people had as young in the previous period. Hence, as before, $K_{t+1} = s_t L_t = s(w_t, r_{t+1})L_t$. In view of $K_{t+1} \equiv \tilde{k}_{t+1}T_{t+1}L_{t+1} = \tilde{k}_{t+1}T_t(1+g)L_t(1+n)$, together with (4.20) and (4.21), we get

$$\tilde{k}_{t+1} = \frac{s(\tilde{w}(\tilde{k}_t)T_t, r(\tilde{k}_{t+1}))}{T_t(1 + g)(1 + n)}. \tag{4.22}$$

This is the general version of the law of motion of the Diamond OLG model with Harrod-neutral technological progress.

For the model to comply with Kaldor's "stylized facts", the model should be capable of generating balanced growth. Essentially, this capability is equivalent to being able to generate a steady state. In the presence of technological progress this latter capability requires a restriction on the lifetime utility function, U. Indeed, we see from (4.22) that the model is consistent with existence of a steady state only if the time-dependent technology level, T_t, in the numerator and denominator cancels out. This requires that the saving function is homogeneous of degree one in its first argument such that $s(\tilde{w}(\tilde{k}_t)T_t, r(\tilde{k}_{t+1})) = s(\tilde{w}(\tilde{k}_t), r(\tilde{k}_{t+1}))T_t$. In turn this is so if and only if the lifetime utility function of the young is *homothetic.* So, in addition to the No Fast Assumption from Chapter 3, we impose the Homotheticity Assumption:

$$\text{the lifetime utility function } U \text{ is homothetic.} \tag{A4}$$

This property entails that if the value of the "endowment", here the human wealth w_t, is multiplied by a $\lambda > 0$, then the chosen c_{1t} and c_{2t+1} are also multiplied by this factor λ (see Appendix C). It then follows that s_t is multiplied by λ as well. Letting $\lambda = 1/(\tilde{w}(\tilde{k}_t)T_t)$, (A4) thus allows us to write

$$s_t = s(1, r(\tilde{k}_{t+1}))\tilde{w}(\tilde{k}_t)T_t \equiv \hat{s}(r(\tilde{k}_{t+1}))\tilde{w}(\tilde{k}_t)T_t, \tag{4.23}$$

where $\hat{s}(r(\tilde{k}_{t+1}))$ is the saving-wealth *ratio* of the young. The distinctive feature is that the homothetic lifetime utility function U allows a decomposition of the young's saving into two factors, where one is the saving-wealth ratio, which depends only on the interest rate, and the other is the human wealth. By (4.22), the law of motion of the economy reduces to

$$\tilde{k}_{t+1} = \frac{\hat{s}(r(\tilde{k}_{t+1}))}{(1+g)(1+n)}\tilde{w}(\tilde{k}_t). \tag{4.24}$$

The equilibrium path of the economy can be analyzed in a similar way as in the case of no technological progress. In the assumptions (A2) and (A3) from Chapter 3 we replace k by $\tilde{k}$ and $1+n$ by $(1+g)(1+n)$. As a generalization of Proposition 4 from Chapter 3, these generalized versions of (A2) and (A3), together with the No Fast Assumption (A1) and the Homotheticity Assumption (A4), guarantee that k_t over time converges to some steady state value $\tilde{k}^* > 0$.

Let an economy that can be described by the Diamond model be called a *Diamond economy.* Our conclusion is then that a Diamond economy will sooner or later settle down in a steady state. The convergence of $\tilde{k}$ implies convergence of many key variables, for instance the interest rate and the technology-corrected real wage. In view of (4.20) and (4.21), respectively, we get, for $t \to \infty$,

$$\begin{aligned} r_t &= f'(\tilde{k}_t) - \delta \to f'(\tilde{k}^*) - \delta \equiv r^*, \quad \text{and} \\ \frac{w_t}{T_t} &= f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \to f(k^*) - k^* f'(k^*) \equiv \tilde{w}^*. \end{aligned}$$

Moreover, for instance the labor income share converges to a constant:

$$\frac{w_t L_t}{Y_t} = \frac{w_t/T_t}{Y_t/(T_t L_t)} = \frac{f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)}{f(\tilde{k}_t)} \to 1 - \frac{k^* f'(k^*)}{f(k^*)} \equiv 1 - \alpha^* \text{ for } t \to \infty.$$

The prediction from the model is thus that a Diamond economy will in the long run behave in accordance with Kaldor's stylized facts. The background for this is that convergence to a steady state is, in this and many other models, equivalent to "convergence" to a BGP. This equivalence follows from:

PROPOSITION 4 Consider a Diamond economy with Harrod-neutral technological progress at the constant rate $g \geq 0$ and positive gross saving for all t.
(i) If the economy features balanced growth, then it is in a steady state.
(ii) If the economy is in a steady state, then it features balanced growth.

Proof (i) Suppose the economy features balanced growth. Then, by Proposition 1, Y/K is constant. As $Y/K = \tilde{y}/\tilde{k} = f(\tilde{k})/\tilde{k}$, also $\tilde{k}$ is constant. Thereby the economy is in a steady state. (ii) Suppose the economy is in a steady state, i.e., for some $\tilde{k}^* > 0$, (4.24) holds for $\tilde{k}_t = \tilde{k}_{t+1} = \tilde{k}^*$. The constancy of $\tilde{k} \equiv K/(TL)$ and $\tilde{y} \equiv Y/(TL) = f(\tilde{k})$ implies that both g_K and g_Y equal the constant g_{TL} $= (1+g)(1+n) - 1 > 0$. As $S \equiv Y - C$, constancy of g_K implies constancy of $S/K = (\Delta K + \delta K)/K = g_K + \delta$, so that also S grows at the rate g_K and thereby at the same rate as output. Hence S/Y, and thereby also $C/Y \equiv 1 - S/Y$, is constant. Hence, also C grows at the constant rate g_Y. All criteria for a BGP are thus satisfied. □

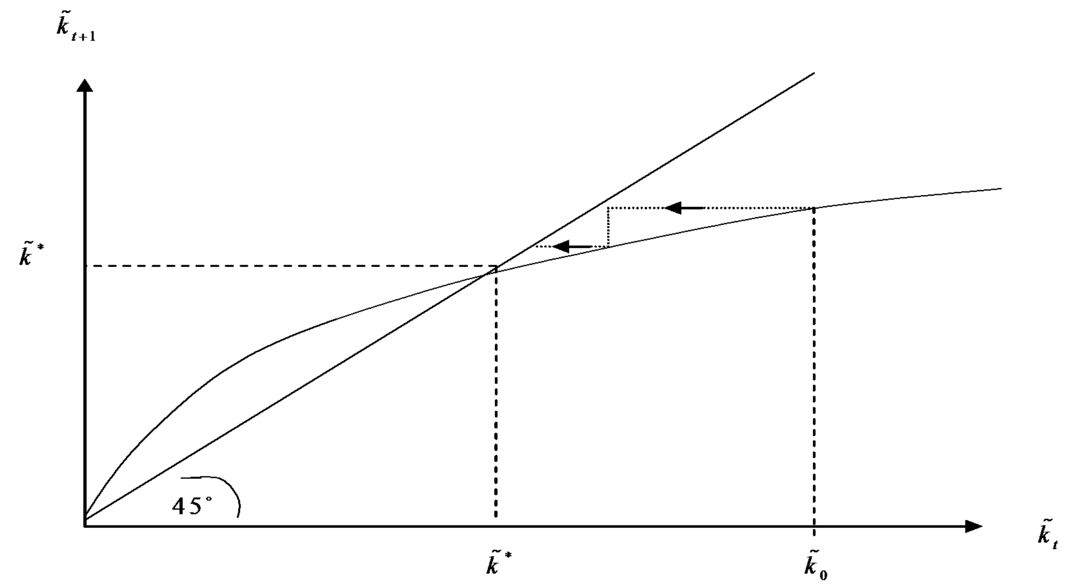


Figure 4.4: Transition curve for a well-behaved Diamond OLG model with Harrod-neutral technical progress.

Let us portray the dynamics by a transition diagram. Fig. 4.4 shows a "well-behaved" case in the sense that there is only one steady state. In the figure the initial effective capital-labor ratio, $\tilde{k}_0$, is assumed to be relatively large. This need not be interpreted as if the economy is highly developed and has a high initial capital-labor ratio, K_0/L_0. Indeed, the reason that $\tilde{k}_0 \equiv K_0/(T_0L_0)$ is large relative to its steady-steady value may be that the economy is "backward" in the sense of having a relatively low initial level of technology. Growing at a given rate g, the technology will in this situation grow faster than the capital-labor ratio, K/L, so that the effective capital-labor ratio declines over time. The process continues until the steady state is essentially reached with a real interest rate $r^* = f'(\tilde{k}^*) - \delta$. This is to remind ourselves that from an empirical point of view, the adjustment towards a steady state can be from above as well as from below.

The output growth rate in steady state, $(1+g)(1+n)-1$, is sometimes called the "natural rate of growth". Since $(1+g)(1+n)-1 = g+n+gn \approx g+n$ for g and n "small", the natural rate of growth approximately equals the sum of the rate of technological progress and the growth rate of the labor force.

Warning: When measured on an *annual* basis, the growth rates of technology and labor force, $\bar{g}$ and $\bar{n}$, do indeed tend to be "small", say $\bar{g} = 0.02$ and $\bar{n} = 0.005$, so that $\bar{g}+\bar{n}+\bar{g}\bar{n} = 0.0251 \approx 0.0250 = \bar{g}+\bar{n}$. But in the context of models like Diamond's, the period length is, say, 30 years. Then the corresponding g and n will satisfy the equations $1+g = (1+\bar{g})^{30} = 1.02^{30} = 1.8114$ and $1+n = (1+\bar{n})^{30} = 1.005^{30} = 1.1614$, respectively. We get $g+n = 0.973$, which is about 10 percent smaller than the true output growth rate over 30 years, which is $g+n+gn = 1.104$.

We end our account of Diamond's OLG model with some remarks on a popular special case of a homothetic utility function.

Example: CRRA utility

An example of a homothetic lifetime utility function is obtained by letting the period utility function take the CRRA form introduced in the previous chapter. Then

$$U(c_1, c_2) = \frac{c_1^{1-\theta}-1}{1-\theta} + (1+\rho)^{-1}\frac{c_2^{1-\theta}-1}{1-\theta}, \qquad \theta > 0. \tag{4.25}$$

Recall that the CRRA utility function with parameter θ has the property that the (absolute) elasticity of marginal utility of consumption equals the constant $\theta > 0$ for all $c > 0$. Up to a positive linear transformation it is, in fact, the only period utility function having this property. A proof that the utility function (4.25) is indeed homothetic is given in Appendix C.

One of the reasons that the CRRA function is popular in macroeconomics is that in *representative* agent models, the period utility function *must* have this form to obtain consistency with balanced growth and Kaldor's stylized facts (this is shown in Chapter 7). In contrast, a model with heterogeneous agents, like the

Diamond model, does not need CRRA utility to comply with the Kaldor facts. CRRA utility is just a convenient special case leading to homothetic lifetime utility. And *this* is what is needed for a BGP to exist and thereby for compatibility with Kaldor's stylized facts.

Given the CRRA assumption in (4.25), the saving-wealth ratio of the young becomes

$$\hat{s}(r) = \frac{1}{1 + (1+\rho)\left(\frac{1+r}{1+\rho}\right)^{(\theta-1)/\theta}}. \tag{4.26}$$

It follows that $\hat{s}'(r) \gtreqless 0$ for $\theta \lesseqgtr 1$.

When $\theta = 1$ (the case $u(c) = \ln c$), $\hat{s}(r) = 1/(2+\rho) \equiv \hat{s}$, a constant, and the law of motion (4.24) thus simplifies to

$$\tilde{k}_{t+1} = \frac{1}{(1+g)(1+n)(2+\rho)}\tilde{w}(\tilde{k}_t).$$

We see that in the $\theta = 1$ case, whatever the production function, $\tilde{k}_{t+1}$ enters only at the left-hand side of the fundamental difference equation, which thereby reduces to a simple transition function. Since $\tilde{w}'(\tilde{k}) > 0$, the transition curve is positively sloped everywhere. If the production function is Cobb-Douglas, $Y_t = K_t^\alpha (T_t L_t)^{1-\alpha}$, then $\tilde{w}(\tilde{k}_t) = (1-\alpha)\tilde{k}_t^\alpha$. Combining this with $\theta = 1$ yields a "well-behaved" Diamond model (thus having a unique and globally asymptotically stable steady state), cf. Fig. 4.4 above. In fact, as noted in Chapter 3, in combination with Cobb-Douglas technology, CRRA utility results in "well-behavedness" whatever the value of $\theta > 0$.

4.3 The golden rule under Harrod-neutral technological progress

Given that there is technological progress, consumption per unit of labor is likely to grow over time. Therefore the definition of the golden-rule capital-labor ratio from Chapter 3 has to be generalized to cover the case of growing consumption per unit of labor. To allow existence of steady states and BGPs, we maintain the assumption that technological progress is Harrod-neutral, that is, we maintain the production function (4.11) where the technology level, T, grows at a constant rate $g > 0$. We also maintain the assumption that the labor force, L_t, is fully employed and grows at a constant rate, $n \geq 0$.

Since we need not have a Diamond economy in mind, we can consider an arbitrary period length. It could be one year for instance. Consumption per unit of labor is

$$\begin{aligned} c_t &\equiv \frac{C_t}{L_t} = \frac{F(K_t, T_t L_t) - S_t}{L_t} = \frac{f(\tilde{k}_t) T_t L_t - (K_{t+1} - K_t + \delta K_t)}{L_t} \\ &= f(\tilde{k}_t) T_t - (1+g) T_t (1+n) \tilde{k}_{t+1} + (1-\delta) T_t \tilde{k}_t \\ &= \left[f(\tilde{k}_t) + (1-\delta)\tilde{k}_t - (1+g)(1+n)\tilde{k}_{t+1} \right] T_t. \end{aligned}$$

DEFINITION 2 The golden-rule capital intensity, $\tilde{k}_{GR}$, is that level of $\tilde{k} \equiv K/(TL)$ which gives the highest sustainable path for consumption per unit of labor in the economy.

To comply with the sustainability requirement, let us consider a steady state. So $\tilde{k}_{t+1} = \tilde{k}_t = \tilde{k}$ and therefore

$$c_t = \left[f(\tilde{k}) + (1-\delta)\tilde{k} - (1+g)(1+n)\tilde{k} \right] T_t \equiv \tilde{c}(\tilde{k}) T_t, \tag{4.27}$$

where $\tilde{c}(\tilde{k})$ is the "technology-corrected" level of consumption per unit of labor in the steady state. We see that in steady state, consumption per unit of labor will grow at the same rate as the technology. Thus, $\ln c_t = \ln \tilde{c}(\tilde{k}) + \ln T_0 + t \ln(1+g)$. Fig. 4.5 illustrates.

Since the evolution of the technology level T_t in (4.27) is exogenous, the highest possible path of c_t is found by maximizing $\tilde{c}(\tilde{k})$. This gives the first-order condition

$$\tilde{c}'(\tilde{k}) = f'(\tilde{k}) + (1-\delta) - (1+g)(1+n) = 0. \tag{4.28}$$

When $n \geq 0$, we have $(1+g)(1+n) - (1-\delta) > 0$ in view of $g > 0$. Then, by continuity, the equation (4.28) necessarily has a unique solution in $\tilde{k} > 0$, if the production function satisfies the condition

$$\lim_{\tilde{k} \to 0} f'(\tilde{k}) > (1+g)(1+n) - (1-\delta) > \lim_{\tilde{k} \to \infty} f'(\tilde{k}),$$

which we assume. This is a milder condition than the Inada conditions. Considering the second-order condition $\tilde{c}''(\tilde{k}) = f''(\tilde{k}) < 0$, the $\tilde{k}$ satisfying (4.28) does indeed maximize $\tilde{c}(\tilde{k})$. By definition, this $\tilde{k}$ is the golden-rule capital intensity, $\tilde{k}_{GR}$. Thus

$$f'(\tilde{k}_{GR}) - \delta = (1+g)(1+n) - 1 \approx g + n, \tag{4.29}$$

where the right-hand side is the "natural rate of growth". This says that the golden-rule capital intensity is that level of the capital intensity at which the net marginal productivity of capital equals the output growth rate in steady state.

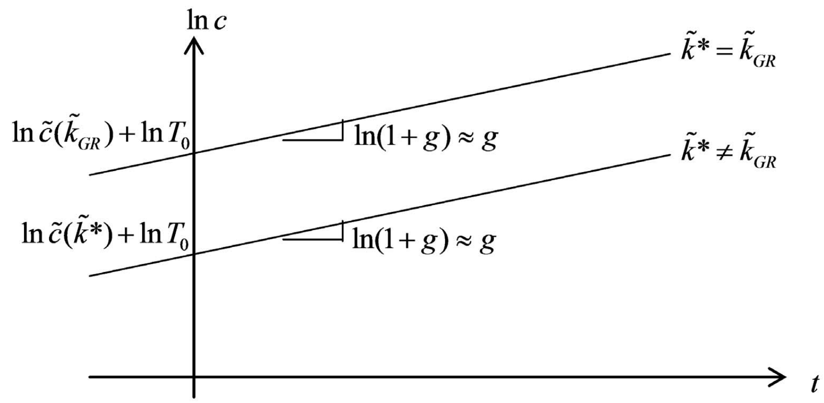


Figure 4.5: The highest sustainable path of consumption is where $\tilde{k}^* = \tilde{k}_{GR}$.

Has dynamic inefficiency been a problem in practice? As in the Diamond model without technological progress, it is theoretically possible that the economy ends up in a steady state with $\tilde{k}^* > \tilde{k}_{GR}$.[9] If this happens, the economy is dynamically inefficient and $r^* < (1 + g)(1 + n) - 1 \approx g + n$. To check whether dynamic inefficiency is a realistic outcome in an industrialized economy or not, we should compare the observed average GDP growth rate over a long stretch of time to the average real interest rate or rate of return in the economy. For the period after the Second World War the average GDP growth rate ($\approx g + n$) in Western countries is typically about 3 percent per year. But what interest rate should one choose? In simple macro models, like the Diamond model, there is no uncertainty and no need for money to carry out trades. In such models all assets earn the same rate of return, r, in equilibrium. In the real world there is a spectrum of interest rates, reflecting the different risk and liquidity properties of the different assets. The expected real rate of return on a short-term government bond is typically less than 3 percent per year (a relatively safe and liquid asset). This is much lower than the expected real rate of return on corporate stock, say 10 percent per year. Our model cannot tell which rate of return we should choose, but the conclusion hinges on that choice.

Abel et al. (1989) study the problem on the basis of a model with *uncertainty*. They show that a sufficient condition for dynamic efficiency is that gross investment, I, does not exceed the gross capital income in the long run, that is $I \leq Y - wL$. They find that for the U.S. and six other major OECD nations this seems to hold. Indeed, for the period 1929-85 the U.S. has, on average, $I/Y = 0.15$ and $(Y - wL)/Y = 0.29$. A similar difference is found for other industrialized countries, suggesting that they are dynamically efficient. At least in these countries,

[9] The proof is analogue to that in Chapter 3 for the case $g = 0$.

therefore, the potential coordination failure laid bare by OLG models does not seem to have been operative in practice.

4.4 The functional income distribution

By the *functional income distribution* is meant the distribution of national income on the different basic income categories: income to providers of labor, capital, and land (including other natural resources). Theory of the functional income distribution is thus theory about the determination and evolution of factor income shares.

The simplest theory about the functional income distribution is the *neoclassical theory* of the functional income distribution. It relies on competitive markets and an aggregate production function,

$$Y = F(K, L, J),$$

where K and L have the usual meaning, but the new symbol J measures the input of land. The production function is assumed to be neoclassical with CRS. When the representative firm maximizes profit and the factor markets clear, the equilibrium factor prices must satisfy:

$$\hat{r} = F_K(K, L, J), \quad w = F_L(K, L, J), \quad z = F_J(K, L, J),$$

where z denotes land rent (the charge for the use of land per unit of land), and, as usual, $\hat{r}$ is the rental rate per unit of capital, and w is the real wage per unit of labor. If in the given period, the supply of three production factors is predetermined, the three equations *determine* the three factor prices, and the factor income shares are determined as

$$\frac{\hat{r}K}{Y} = \frac{F_K(K, L, J)K}{F(K, L, J)}, \quad \frac{wL}{Y} = \frac{F_L(K, L, J)L}{F(K, L, J)}, \quad \frac{zJ}{Y} = \frac{F_J(K, L, J)J}{F(K, L, J)}.$$

The theory is also called the *marginal productivity theory* of the functional income distribution.

In advanced economies the role of land is relatively minor.[10] In fact many theoretical models completely ignore land. Below we follow that tradition, while considering the question: How is the direction of movement of the labor and capital income shares determined during the adjustment process from arbitrary initial conditions toward steady state?

First we consider the case of no technological progress.

[10] In 1750 land rent made up 20 percent of national income in England, in 1850 8 percent, and in 2010 less than 0.1 percent (Jones and Vollrath, 2013). The approximative numbers often used for the labor income share and capital income share in advanced economies are 2/3 and 1/3, respectively.

How the labor income share depends on the capital-labor ratio

Ignoring, to begin with, technological progress, we write aggregate output as $Y = F(K, L)$, where F is neoclassical with CRS. From Euler's theorem follows that $F(K, L) = F_1 K + F_2 L = f'(k)K + (f(k) - kf'(k))L$, where $k \equiv K/L$ and f is the production function in intensive form. In equilibrium under perfect competition we have

$$Y = \hat{r}K + wL,$$

where $\hat{r} = r + \delta = f'(k) \equiv \hat{r}(k)$ and $w = f(k) - kf'(k) \equiv w(k)$.

The *labor income share* is

$$\frac{wL}{Y} = \frac{f(k) - kf'(k)}{f(k)} \equiv \frac{w(k)}{f(k)} \equiv SL(k) = \frac{wL}{\hat{r}K + wL} = \frac{\frac{w/\hat{r}}{k}}{1 + \frac{w/\hat{r}}{k}}, \tag{4.30}$$

where the function $SL(\cdot)$ is the income share of labor function, $w/\hat{r}$ is the *factor price ratio*, and $(w/\hat{r})/k = w/(\hat{r}k)$ is the *factor income ratio*. As $\hat{r}'(k) = f''(k) < 0$ and $w'(k) = -kf''(k) > 0$, the relative factor price $w/\hat{r}$ is an increasing function of k.

Suppose that capital tends to grow faster than labor so that k rises over time. Unless the production function is Cobb-Douglas, this will under perfect competition affect the labor income share. But apriori it is not obvious in what direction. By (4.30) we see that the labor income share moves in the same direction as the factor *income* ratio, $(w/\hat{r})/k$. The latter goes up (down) depending on whether the percentage rise in the factor price ratio $w/\hat{r}$ is greater (smaller) than the percentage rise in k. So, if we let $\mathrm{E}\ell_x g(x)$ denote the elasticity of a function $g(x)$ w.r.t. x, that is, $xg'(x)/g(x))$, we can only say that

$$SL'(k) \gtreqless 0 \text{ for } \mathrm{E}\ell_k \frac{w}{\hat{r}} \gtreqless 1, \tag{4.31}$$

respectively. In words: if the production function is such that the ratio of the marginal productivities of the two production factors is strongly (weakly) sensitive to the capital-labor ratio, then the labor income share rises (falls) along with a rise in K/L.

Usually, however, the inverse elasticity is considered, namely $\mathrm{E}\ell_{w/\hat{r}} k \; (= 1/\mathrm{E}\ell_k \frac{w}{\hat{r}})$. This elasticity indicates how sensitive the cost minimizing capital-labor ratio, k, is to a given factor price ratio $w/\hat{r}$. Under perfect competition $\mathrm{E}\ell_{w/\hat{r}} k$ coincides with what is known as the *elasticity of factor substitution* (for a general definition, see below). The latter is often denoted σ. In the CRS case, σ will be a function of only k so that we can write $\mathrm{E}\ell_{w/\hat{r}} k = \sigma(k)$. By (4.31), we therefore have

$$SL'(k) \gtreqless 0 \text{ for } \sigma(k) \lesseqgtr 1,$$

respectively.

The size of the elasticity of factor substitution is a property of the production function, hence of the technology. In special cases the elasticity of factor substitution is a constant, i.e., independent of k. For instance, if F is Cobb-Douglas, i.e., $Y = K^{\alpha}L^{1-\alpha}, 0 < \alpha < 1$, we have $\sigma(k) \equiv 1$, as we will see in Section 4.6. In this case variation in k does not change the labor income share under perfect competition. Empirically there is not agreement about the "normal" size of the elasticity of factor substitution for industrialized economies, but the bulk of studies seems to conclude with $\sigma(k) < 1$ (see below).

Adding Harrod-neutral technical progress We now add Harrod-neutral technical progress. We write aggregate output as $Y = F(K, TL)$, where F is neoclassical with CRS, and $T = T_t = T_0(1+g)^t$, $g \geq 0$. Then the labor income share is

$$\frac{wL}{Y} = \frac{w/T}{Y/(TL)} \equiv \frac{\tilde{w}}{\tilde{y}}.$$

The above formulas still hold if we replace k by $\tilde{k} \equiv K/(TL)$ and w by $\tilde{w} \equiv w/T$. We get

$$SL'(\tilde{k}) \gtreqless 0 \text{ for } \sigma(\tilde{k}) \lesseqgtr 1,$$

respectively. We see that if $\sigma(\tilde{k}) < 1$ in the relevant range for $\tilde{k}$, then market forces tend to *increase* the income share of the factor that is becoming relatively more scarce. This factor is efficiency-adjusted labor, TL, if $\tilde{k}$ is increasing, which $\tilde{k}$ will be during the transitional dynamics in a well-behaved Diamond model if $\tilde{k}_0 < \tilde{k}^*$. And if instead $\sigma(\tilde{k}) > 1$ in the relevant range for $\tilde{k}$, then market forces tend to *decrease* the income share of the factor that is becoming relatively more scarce. This factor is K, if $\tilde{k}$ is decreasing, which $\tilde{k}$ will be during the transitional dynamics in a well-behaved Diamond model if $\tilde{k}_0 > \tilde{k}^*$, cf. Fig. 4 above. Note that, given the production function in intensive form, f, the elasticity of substitution between capital and labor does not depend on whether $g = 0$ or $g > 0$, but only on the function f itself and the level of $K/(TL)$. This follows from Section 4.6.

While k empirically is clearly growing, $\tilde{k} \equiv k/T$ is not necessarily so because also T is increasing. Indeed, according to Kaldor's "stylized facts", apart from short- and medium-term fluctuations, $\tilde{k}$ − and therefore also $\hat{r}$ and the labor income share − tend to be more or less constant over time. This can happen whatever the sign of $\sigma(\tilde{k}^*) - 1$, where $\tilde{k}^*$ is the long-run value of the effective capital-labor ratio $\tilde{k}$.

As alluded to earlier, there are empiricists who reject Kaldor's "facts" as a general tendency. For instance Piketty (2014) essentially claims that in the very long run the effective capital-labor ratio $\tilde{k}$ has an upward trend, temporarily braked by two world wars and the Great Depression in the 1930s. If so, the sign of $\sigma(\tilde{k}) - 1$ becomes decisive for in what direction wL/Y will move. Piketty interprets the econometric literature as favoring $\sigma(\tilde{k}) > 1$, which means there should be downward pressure on wL/Y. This particular source behind a falling wL/Y can be questioned, however. Indeed, $\sigma(\tilde{k}) > 1$ contradicts the more general empirical view.

Immigration*

The phenomenon of migration provides another example that illustrates how the size of $\sigma(\tilde{k})$ matters. Consider a competitive economy with perfect competition, a given aggregate capital stock K, and a given technology level T (entering the production function in the labor-augmenting way as above). Suppose that due to immigration an upward shift in aggregate labor supply, L, occurs. Full employment is maintained by the needed downward real wage adjustment. Given the present model, in what direction will aggregate labor income $wL = \tilde{w}(\tilde{k})TL$ then change? The effect of the larger L is to some extent offset by a lower w brought about by the lower effective capital-labor ratio. Indeed, in view of $d\tilde{w}/d\tilde{k} = -\tilde{k}f''(\tilde{k}) > 0$, we have $\tilde{k} \downarrow$ implies $w \downarrow$ for fixed T. So we cannot apriori sign the change in wL. The following relationship can be shown (Exercise ??), however:

$$\frac{\partial(wL)}{\partial L} = (1 - \frac{\alpha(\tilde{k})}{\sigma(\tilde{k})})w \gtreqless 0 \text{ for } \sigma(\tilde{k}) \gtreqless \alpha(\tilde{k}), \tag{4.32}$$

respectively, where $a(\tilde{k}) \equiv \tilde{k}f'(\tilde{k})/f(\tilde{k})$ is the output elasticity w.r.t. capital which under perfect competition equals the gross capital income share. It follows that the larger L will not be fully offset by the lower w as long as the elasticity of factor substitution, $\sigma(\tilde{k})$, exceeds the gross capital income share, $\alpha(\tilde{k})$. This condition seems confirmed by most of the empirical evidence, see next section.

The next section describes the concept of elasticity of factor substitution at a more general level. The subsequent section introduces the special case known as the CES production function.

4.5 The elasticity of factor substitution

We shall here discuss the concept of elasticity of factor substitution at a more general level. Fig. 4.6 depicts an isoquant, $F(K, L) = \bar{Y}$, for a given neoclassical production function, $F(K, L)$, which need not have CRS. Let MRS denote the marginal rate of substitution of K for L, i.e.,

$$MRS = -\frac{dK}{dL}_{|Y=\bar{Y}} = F_L(K,L)/F_K(K,L).$$

MRS thus measures how much extra of K is needed to compensate for a reduction in L by one unit?[11] At a given point (K, L) on the isoquant curve, MRS is given by the absolute value of the slope of the tangent to the isoquant at that point. This tangent coincides with that isocost line which, given the factor prices, has minimal intercept with the vertical axis while at the same time touching the isoquant. In view of $F(\cdot)$ being neoclassical, the isoquants are by definition strictly convex to the origin. Consequently, MRS is rising along the curve when L decreases and thereby K increases. Conversely, we can let MRS be the independent variable and consider the corresponding point on the indifference curve, and thereby the ratio K/L, as a function of MRS. If we let MRS rise along the given isoquant, the corresponding value of the ratio K/L will also rise.

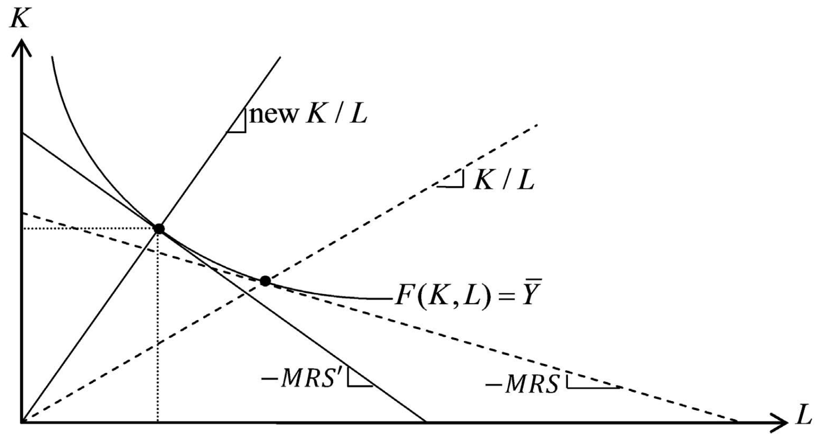


Figure 4.6: Substitution of capital for labor as the marginal rate of substitution increases from MRS to MRS'.

The *elasticity of substitution* between capital and labor, denoted $\hat{\sigma}(K, L)$, measures how sensitive K/L is vis-a-vis a rise in MRS. More precisely, $\hat{\sigma}(K, L)$ is defined as the elasticity of the ratio K/L with respect to MRS when moving along a given isoquant, evaluated at the point (K, L). Thus,

$$\hat{\sigma}(K,L) \equiv E\ell_{MRS} K/L = \frac{MRS}{K/L}\frac{d(K/L)}{dMRS}_{|Y=\bar{Y}} \approx \frac{\frac{\Delta(K/L)}{K/L}}{\frac{\Delta MRS}{MRS}}_{|Y=\bar{Y}}. \tag{4.33}$$

[11] When there is no risk of confusion as to what is up and what is down, we use MRS as a shorthand for the more precise notation MRS_{KL}.

Although the elasticity of factor substitution is a characteristic of the technology as such and is here defined without reference to markets and factor prices, it helps the intuition to refer to factor prices. At a cost-minimizing point, MRS equals the factor price ratio $w/\hat{r}$. Thus, the *elasticity of factor substitution* will under cost minimization coincide with *the percentage increase in the ratio of the cost-minimizing factor ratio induced by a one percentage increase in the inverse factor price ratio, holding the output level unchanged.*[12] The elasticity of factor substitution is thus a positive number and reflects how sensitive the capital-labor ratio K/L is under cost minimization to a one percentage increase in the factor price ratio $w/\hat{r}$ for a given output level. The less curvature the isoquant has, the greater is the elasticity of factor substitution. In an analogue way, in consumer theory one considers the elasticity of substitution between two consumption goods or between consumption today and consumption tomorrow, cf. Chapter 3. In that context the role of the given isoquant is taken over by an indifference curve. That is also the case when we consider the intertemporal elasticity of substitution in labor supply, cf. the next chapter.

Calculating the elasticity of substitution between K and L at the point (K, L), we get

$$\hat{\sigma}(K,L) = -\frac{F_K F_L (F_K K + F_L L)}{KL\left[(F_L)^2 F_{KK} - 2F_K F_L F_{KL} + (F_K)^2 F_{LL}\right]}, \tag{4.34}$$

where all the derivatives are evaluated at the point (K, L). When $F(K, L)$ has CRS, the formula (4.34) simplifies to

$$\hat{\sigma}(K,L) = \frac{F_K(K,L)F_L(K,L)}{F_{KL}(K,L)F(K,L)} = -\frac{f'(k)\left(f(k) - f'(k)k\right)}{f''(k)kf(k)} \equiv \sigma(k), \tag{4.35}$$

where $k \equiv K/L$.[13] We see that under CRS, the elasticity of substitution depends only on the capital-labor ratio k, not on the output level.

There is an alternative way of interpreting the substitution elasticity formula (4.33). This is based on the fact that any elasticity of a function $y = \varphi(x)$ can be written as a "double-log derivative": $\mathrm{E}\ell_x y \equiv (x/y)dy/dx = d\ln y/d\ln x$.[14] So, we can rewrite (4.33) as $\hat{\sigma}(K, L) = d\ln(K/L)/d\ln MRS$, which is a simple derivative when the data for K/L and MRS are given in logs.

[12] This characterization is equivalent to interpreting the elasticity of substitution as the percentage *decrease* in the factor ratio (when moving along a given isoquant) induced by a one-percentage *increase* in the *corresponding* factor price ratio.

[13] The formulas (4.34) and (4.35) are derived in Appendix D.

[14] To see this, let $X = \ln x$ and $Y = \ln y$. Then, by the chain rule,

$$\frac{d\ln y}{d\ln x} = \frac{dY}{dX} = \frac{dY}{dy}\frac{dy}{dx}\frac{dx}{dX} = \frac{1}{y}\frac{dy}{dx}e^X = \frac{x}{y}\frac{dy}{dx} = \mathrm{E}\ell_x y.$$

We will now consider the case where the elasticity of substitution is independent also of the capital-labor ratio.

4.6 The CES production function

It can be shown[15] that if a neoclassical production function with CRS has a constant elasticity of factor substitution different from one, it must be of the form

$$Y = A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}}, \tag{4.36}$$

where A, α, and β are parameters satisfying $A > 0$, $0 < \alpha < 1$, and $\beta < 1$, $\beta \neq 0$. This function has been used intensively in empirical studies and is called a *CES production function* (CES for Constant Elasticity of Substitution). For a given choice of measurement units, the parameter A reflects efficiency (or what is known as *total factor productivity*) and is thus called the *efficiency parameter.* The parameters α and β are called the *distribution parameter* and the *substitution parameter,* respectively. The restriction $\beta < 1$ ensures that the isoquants are strictly convex to the origin. Note that if $\beta < 0$, the right-hand side of (4.36) is not defined when either K or L (or both) equal 0. We can circumvent this problem by extending the domain of the CES function and assign the function value 0 to these points when $\beta < 0$. Continuity is maintained in the extended domain (see Appendix E).

By taking partial derivatives in (4.36) and substituting back we get

$$\frac{\partial Y}{\partial K} = \alpha A^{\beta}\left(\frac{Y}{K}\right)^{1-\beta} \quad \text{and} \quad \frac{\partial Y}{\partial L} = (1-\alpha)A^{\beta}\left(\frac{Y}{L}\right)^{1-\beta}, \tag{4.37}$$

where $Y/K = A\left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1}{\beta}}$ and $Y/L = A\left[\alpha k^{\beta} + 1 - \alpha\right]^{\frac{1}{\beta}}$.[16] The marginal rate of substitution of K for L therefore is

$$MRS = \frac{\partial Y/\partial L}{\partial Y/\partial K} = \frac{1-\alpha}{\alpha}k^{1-\beta} > 0.$$

Consequently,

$$\frac{dMRS}{dk} = \frac{1-\alpha}{\alpha}(1-\beta)k^{-\beta},$$

where the inverse of the right-hand side is the value of $dk/dMRS$. Substituting these expressions into (4.33) gives

$$\hat{\sigma}(K,L) \; (= \sigma(k)) = \frac{1}{1-\beta} \equiv \sigma, \tag{4.38}$$

[15] See, e.g., Arrow et al. (1961).

[16] The calculations are slightly simplified if we start from the transformation $Y^{\beta} = A^{\beta}\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]$.

confirming the constancy of the elasticity of substitution. Since $\beta < 1$, $\sigma > 0$ always. A higher substitution parameter, β, results in a higher elasticity of factor substitution, σ. And $\sigma \lesseqgtr 1$ for $\beta \lesseqgtr 0$, respectively.

Since $\beta = 0$ is not allowed in (4.36), at first sight we cannot get $\sigma = 1$ from this formula. Yet, $\sigma = 1$ can be introduced as the *limiting* case of (4.36) when $\beta \to 0$, which turns out to be the Cobb-Douglas function. Indeed, one can show[17] that, for fixed K and L,

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \to AK^{\alpha}L^{1-\alpha}, \text{ for } \beta \to 0 \text{ (so that } \sigma \to 1).$$

By a similar procedure as above we find that a Cobb-Douglas function always has elasticity of substitution equal to 1; this is exactly the value taken by σ in (4.38) when $\beta = 0$. In addition, the Cobb-Douglas function is the *only* production function that has unit elasticity of substitution whatever the capital-labor ratio.

Another interesting limiting case of the CES function appears when, for fixed K and L, we let $\beta \to -\infty$ so that $\sigma \to 0$. We get

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \to A\min(K, L), \text{ for } \beta \to -\infty \text{ (so that } \sigma \to 0). \quad (4.39)$$

So in this case the CES function approaches a Leontief production function, the isoquants of which form a right angle, cf. Fig. 4.7. In the limit there is *no* possibility of substitution between capital and labor. In accordance with this the elasticity of substitution calculated from (4.38) approaches zero when β goes to $-\infty$.

Finally, let us consider the "opposite" transition. For fixed K and L we let the substitution parameter rise towards 1 and get

$$A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} \to A\left[\alpha K + (1-\alpha)L\right], \text{ for } \beta \to 1 \text{ (so that } \sigma \to \infty).$$

Here the elasticity of substitution calculated from (4.38) tends to ∞ and the isoquants tend to straight lines with slope $-(1-\alpha)/\alpha$. In the limit, the production function thus becomes linear and capital and labor become *perfect substitutes.*

Fig. 4.7 depicts isoquants for alternative CES production functions and their limiting cases. In the Cobb-Douglas case, $\sigma = 1$, the horizontal and vertical asymptotes of the isoquant coincide with the coordinate axes. When $\sigma < 1$, the horizontal and vertical asymptotes of the isoquant belong to the interior of the positive quadrant. This implies that both capital and labor are essential inputs. When $\sigma > 1$, the isoquant terminates in points *on* the coordinate axes. Then neither capital, nor labor are essential inputs. Empirically there is not complete agreement about the "normal" size of the elasticity of factor substitution for

[17] Proofs of this and the further claims below are in Appendix E.

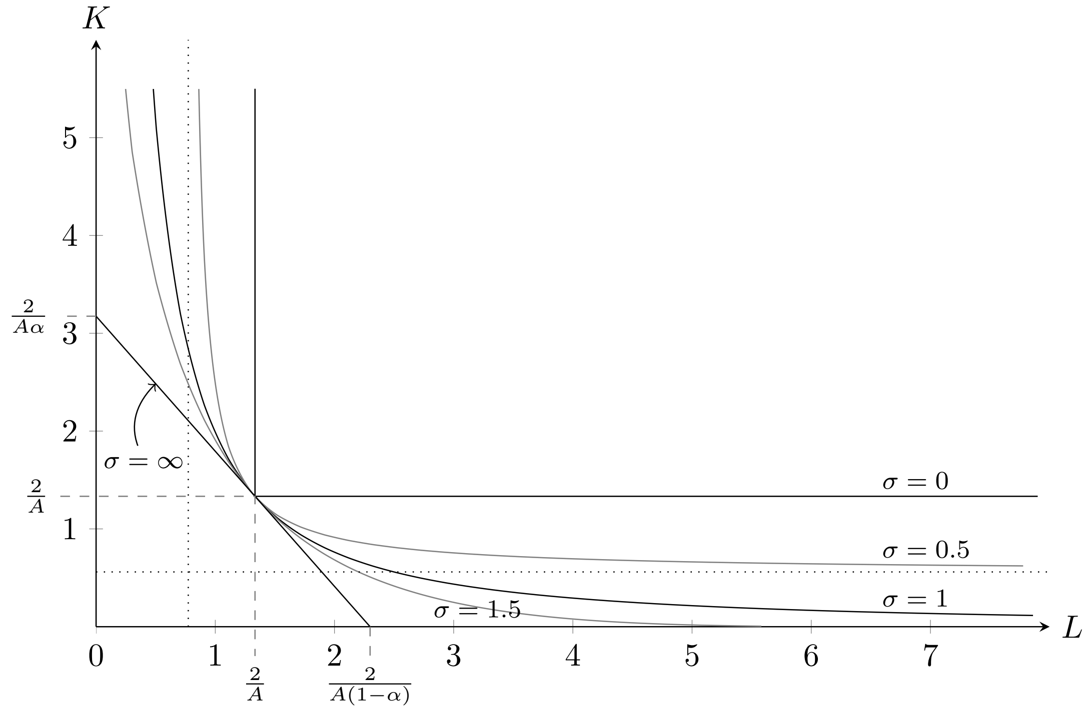


Figure 4.7: Isoquants for the CES function for alternative values of σ ($A = 1.5$, $\bar{Y} = 2$, and $\alpha = 0.42$).

industrialized economies. The elasticity also differs across the production sectors. A thorough econometric study (Antràs, 2004) of U.S. data indicate the aggregate elasticity of substitution to be in the interval $(0.5, 1.0)$. The survey by Chirinko (2008) concludes with the interval $(0.4, 0.6)$. Starting from micro data, a recent study by Oberfield and Raval (2014) finds that the elasticity of factor substitution for the US manufacturing sector as a whole has been stable since 1970 at about 0.7.

The CES production function in intensive form

Dividing through by L on both sides of (4.36), we obtain the CES production function in intensive form,

$$y \equiv \frac{Y}{L} = A(\alpha k^{\beta} + 1 - \alpha)^{\frac{1}{\beta}}, \tag{4.40}$$

where $k \equiv K/L$. The marginal productivity of capital can be written

$$MPK = \frac{dy}{dk} = \alpha A \left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1-\beta}{\beta}} = \alpha A^{\beta} \left(\frac{y}{k}\right)^{1-\beta},$$

which of course equals $\partial Y/\partial K$ in (4.37). We see that the CES function violates either the lower or the upper Inada condition for MPK, depending on the sign of β. Indeed, when $\beta < 0$ (i.e., $\sigma < 1$), then for $k \to 0$ both y/k and dy/dk approach an upper bound equal to $A\alpha^{1/\beta} < \infty$, thus violating the *lower* Inada condition for MPK (see the left-hand panel of Fig. 4.8). It is also noteworthy that in this case, for $k \to \infty$, y approaches an upper bound equal to $A(1-\alpha)^{1/\beta} < \infty$. These features reflect the low degree of substitutability when $\beta < 0$.

When instead $\beta > 0$, there is a high degree of substitutability ($\sigma > 1$). Then, for $k \to \infty$ both y/k and $dy/dk \to A\alpha^{1/\beta} > 0$, thus violating the *upper* Inada condition for MPK (see right-hand panel of Fig. 4.8). It is also noteworthy that for $k \to 0$, y approaches a positive lower bound equal to $A(1-\alpha)^{1/\beta} > 0$. Thus, when $\sigma > 1$, capital is not essential. At the same time $dy/dk \to \infty$ for $k \to 0$ (so the lower Inada condition for the marginal productivity of capital holds). Details are in Appendix E.

The marginal productivity of labor is

$$MPL = \frac{\partial Y}{\partial L} = (1-\alpha)A^{\beta}y^{1-\beta} = (1-\alpha)A(\alpha k^{\beta} + 1 - \alpha)^{(1-\beta)/\beta} \equiv w(k),$$

from (4.37). Under perfect competition, the equilibrium labor income share is thus

$$\frac{wL}{Y} = \frac{(1-\alpha)(\alpha k^{\beta} + 1 - \alpha)^{1/\beta - 1}}{(\alpha k^{\beta} + 1 - \alpha)^{\frac{1}{\beta}}} = \frac{1-\alpha}{\alpha k^{\beta} + 1 - \alpha}.$$

Since (4.36) is symmetric in K and L, we get a series of symmetric results by considering output per unit of capital as $x \equiv Y/K = A\left[\alpha + (1-\alpha)(L/K)^{\beta}\right]^{1/\beta}$. In total, therefore, when there is low substitutability ($\sigma < 1$), for fixed input of either of the production factors, there is an upper bound for how much an unlimited input of the other production factor can increase output. And when there is high substitutability ($\sigma > 1$), there is no such bound and an unlimited input of either production factor take output to infinity.

The Cobb-Douglas case, i.e., the limiting case for $\beta \to 0$, constitutes in several respects an intermediate case in that *all* four Inada conditions are satisfied and we have $y \to 0$ for $k \to 0$, and $y \to \infty$ for $k \to \infty$.

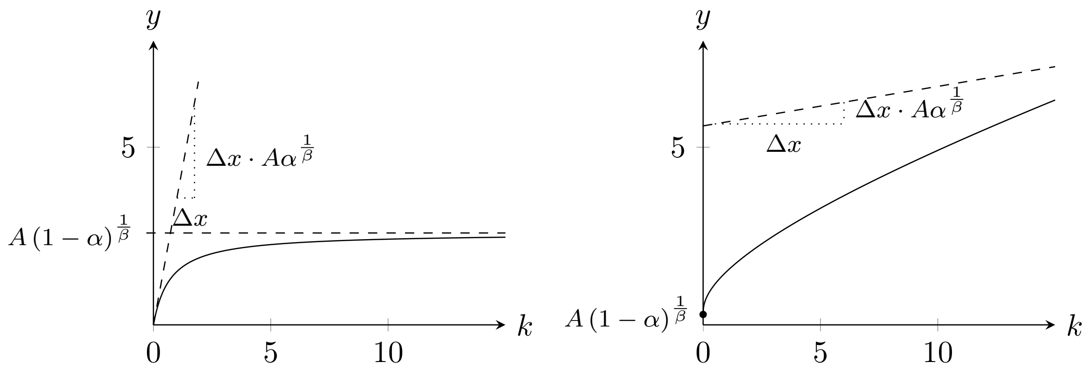
Returning to the general CES function, in case of Harrod-neutral technological progress, (4.36) and (4.40) are replaced by

$$Y = A\left[\alpha K^{\beta} + (1-\alpha)(TL)^{\beta}\right]^{\frac{1}{\beta}}$$

and

$$\tilde{y} \equiv \frac{Y}{TL} = A(\alpha \tilde{k}^{\beta} + 1 - \alpha)^{\frac{1}{\beta}},$$

respectively, where T is the technology level, and $\tilde{k} \equiv K/(TL)$.



a) The case of $\sigma < 1$. a) The case of $\sigma > 1$.

Figure 4.8: The CES production function in intensive form, $\sigma = 1/(1-\beta)$, $\beta < 1$.

Generalizations*

The CES production functions considered above have CRS. By adding an elasticity of scale parameter, γ, we get the generalized form (the case without technological progress):

$$Y = A\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{\gamma}{\beta}}, \qquad \gamma > 0. \tag{4.41}$$

In this form the CES function is homogeneous of degree γ. For $0 < \gamma < 1$, there are DRS, for $\gamma = 1$ CRS, and for $\gamma > 1$ IRS. If $\gamma \neq 1$, it may be convenient to consider $Q \equiv Y^{1/\gamma} = A^{1/\gamma}\left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{1/\beta}$ and $q \equiv Q/L = A^{1/\gamma}(\alpha k^{\beta} + 1 - \alpha)^{1/\beta}$.

The elasticity of substitution between K and L is $\sigma = 1/(1-\beta)$ whatever the value of γ. So including the limiting cases as well as non-constant returns to scale in the "family" of production functions with constant elasticity of substitution, we have the simple classification displayed in Table 4.1.

Table 4.1 The family of production functions with constant elasticity of substitution.

$\sigma = 0$	$0 < \sigma < 1$	$\sigma = 1$	$\sigma > 1$
Leontief	CES	Cobb-Douglas	CES

Note that only for $\gamma \leq 1$ is (4.41) a *neoclassical* production function. This is because, when $\gamma > 1$, the conditions $F_{KK} < 0$ and $F_{NN} < 0$ do not hold everywhere.

We may generalize further by assuming there are n inputs, in the amounts $X_1, X_2, ..., X_n$. Then the CES production function takes the form

$$Y = A\left[\alpha_1 X_1{}^{\beta} + \alpha_2 X_2{}^{\beta} + ...\alpha_n X_n{}^{\beta}\right]^{\frac{\gamma}{\beta}},\ \alpha_i > 0 \text{ for all } i, \sum_i \alpha_i = 1, \gamma > 0. \tag{4.42}$$

In analogy with (4.33), for an n-factor production function the *partial elasticity of substitution* between factor i and factor j is defined as

$$\sigma_{ij} = \frac{MRS_{ij}}{X_i/X_j}\frac{d(X_i/X_j)}{dMRS_{ij}}_{|Y=\bar{Y}},$$

where it is understood that not only the output level but also all X_k, $k \neq i$, j, are kept constant. Note that $\sigma_{ji} = \sigma_{ij}$. In the CES case considered in (4.42), all the partial elasticities of substitution take the same value, $1/(1-\beta)$.

4.7 Concluding remarks

(incomplete)

When speaking of "sustained growth" in variables like K, Y, and C, we do not mean growth in a narrow physical sense. Given limited natural resources (matter and energy), sustained exponential growth in a physical sense is not possible. But sustained exponential growth in terms of economic value is not ruled out. We should for instance understand K broadly as "produced means of production" of *rising quality* and *falling material intensity* (think of the rising efficiency of microprocessors). Similarly, C must be seen as a composite of consumer goods and services with declining material intensity over time. This accords with the empirical fact that as income rises, the share of consumption expenditures devoted to agricultural and industrial products declines and the share devoted to services, hobbies, and amusement increases. Although "economic development" is perhaps a more appropriate term (suggesting qualitative and structural change), we will in this book retain standard terminology and speak of "economic growth".

A further remark about terminology. In the branch of economics called economic growth theory, the term "economic growth" is used primarily for growth of *productivity* and *income per capita* rather than just growth of GDP.

4.8 Literature notes

1. We introduced the assumption that at the macroeconomic level the "direction" of technological progress tends to be Harrod-neutral. Otherwise the model

will not be consistent with Kaldor's stylized facts. The Harrod-neutrality of the "direction" of technological progress is in the present model just an exogenous feature. This raises the question whether there are *mechanisms* tending to generate Harrod-neutrality. Fortunately new growth theory provides clues as to the sources of the speed as well as the direction of technological change. A facet of this theory is that the direction of technological change is linked to the same economic forces as the speed, namely profit incentives. See Acemoglu (2003) and Jones (2006).

2. Recent literature discussing Kaldor's "stylized facts" includes Rognlie (2015), Gollin (2002), Elsby et al. (2013), and Karabarbounis and Neiman (2014). The latter three references conclude with serious scepticism. Attfield and Temple (2010) and others, however, find support for the Kaldor "facts" considering the US and UK based on time-series econometrics. This means an observed evolution roughly obeying balanced growth in terms of *aggregate* variables. *Structural change* is not ruled out by this. *A* changing sectorial composition of the economy is under certain conditions compatible with balanced growth (in a generalized sense) at the aggregate level, cf. the "Kuznets facts" (see Kongsamut et al., 2001, and Acemoglu, 2009).

3. In Section 4.2 we claimed that from an empirical point of view, the adjustment towards a steady state can be from above as well as from below. Indeed, Cho and Graham (1996) find that "on average, countries with a lower income per adult are above their steady-state positions, while countries with a higher income are below their steady-state positions".

4. As to the assessment of whether dynamic inefficiency is - or at least has been - part of reality, in addition to Abel et al. (1989) other useful sources include Ball et al. (1998), Blanchard and Weil (2001), and Barbie, Hagedorn, and Kaul (2004). A survey is given in Weil (2008).

5. In the Diamond OLG model as well as in many other models, a steady state and a balanced growth path imply each other. Indeed, they are two sides of the same process. There *exist* cases, however, where this equivalence does not hold (some open economy models and some models with *embodied* technological change, see Groth et al., 2010). Therefore, it is recommendable always to maintain a terminological distinction between the two concepts.

6. On the declining material intensity of consumer goods and services as technology develops, see Fagnart and Germain (2011).

From here incomplete:

The term "Great Ratios" of the economy was coined by Klein and Kosubud (1961).

La Grandville (1989): normalization of the CES function. La Grandville (2009) contains a lot about analytical aspects linked to the CES production function and the concept of elasticity of factor substitution.

Piketty (2014), Zucman ( ).

According to Summers (2014), Piketty's interpretation of data relevant for estimation of the elasticity of factor substitution relies on conflating gross and net returns to capital. Krusell and Smith (2015) and Ronglie (2015).

Demange and Laroque (1999, 2000) extend Diamond's OLG model to uncertain environments.

For expositions in depth of OLG modeling and dynamics in discrete time, see Azariadis (1993), de la Croix and Michel (2002), and Bewley (2007).

Dynamic inefficiency, see also Burmeister (1980).

Uzawa's theorem: Uzawa (1961), Schlicht (2006).

The way the intuition behind the Uzawa theorem was presented in Section 4.1 draws upon Jones and Scrimgeour (2008).

For more general and flexible production functions applied in econometric work, see, e.g., Nadiri (1982).

Other aspects of life cycle behavior: education. OLG where people live three periods. Also Eggertsson and Mehrotra (2015).

4.9 Appendix

A. Growth and interest arithmetic in discrete time

Let $t = 0, \pm 1, \pm 2, \ldots$, and consider the variables z_t, x_t, and y_t, assumed positive for all t. Define $\Delta z_t = z_t - z_{t-1}$ and Δx_t and Δy_t similarly. These Δ's need not be positive. The *growth rate* of x_t from period $t-1$ to period t is defined as the relative rate of increase in x, i.e., $\Delta x_t / x_{t-1} \equiv x_t / x_{t-1}$. And the *growth factor* for x_t from period $t-1$ to period t is defined as $1 + x_t / x_{t-1}$.

As we are here interested not in the time evolution of growth rates, we simplify notation by suppressing the t's. So we write the growth rate of x as $g_x \equiv \Delta x / x_{-1}$ and similarly for y and z.

PRODUCT RULE If $z = xy$, then $1 + g_z = (1 + g_x)(1 + g_y)$ and $g_z \approx g_x + g_y$, when g_x and g_y are "small".

Proof. By definition, $z = xy$, which implies $z_{-1} + \Delta z = (x_{-1} + \Delta x)(y_{-1} + \Delta y)$. Dividing by $z_{-1} = x_{-1} y_{-1}$ gives $1 + \Delta z / z_{-1} = (1 + \Delta x / x_{-1})(1 + \Delta y / y_{-1})$ as claimed. By carrying out the multiplication on the right-hand side of this equation, we get $1 + \Delta z / z_{-1} = 1 + \Delta x / x_{-1} + \Delta y / y_{-1} + (\Delta x / x_{-1})(\Delta y / y_{-1}) \approx 1 + \Delta x / x_{-1} + \Delta y / y_1$ when $\Delta x / x_{-1}$ and $\Delta y / y_{-1}$ are "small". Subtracting 1 on both sides gives the stated approximation. □

So the product of two positive variables will grow at a rate approximately equal to the sum of the growth rates of the two variables.

QUOTIENT RULE If $z = \frac{x}{y}$, then $1 + g_z = \frac{1+g_x}{1+g_y}$ and $g_z \approx g_x - g_y$, when g_x and g_y are "small".

Proof. By interchanging z and x in Product Rule and rearranging, we get $1 + \Delta z/z_{-1} = \frac{1+\Delta x/x_{-1}}{1+\Delta y/y_{-1}}$, as stated in the first part of the claim. Subtracting 1 on both sides gives $\Delta z/z_{-1} = \frac{\Delta x/x_{-1} - \Delta y/y_{-1}}{1+\Delta y/y_{-1}} \approx \Delta x/x_{-1} - \Delta y/y_{-1}$, when $\Delta x/x_{-1}$ and $\Delta y/y_{-1}$ are "small". This proves the stated approximation. □

So the ratio between two positive variables will grow at a rate approximately equal to the excess of the growth rate of the numerator over that of the denominator. An implication of the first part of Claim 2 is: the ratio between two positive variables is constant if and only if the variables have the same growth rate (not necessarily constant or positive).

POWER FUNCTION RULE If $z = x^\alpha$, then $1 + g_z = (1 + g_x)^\alpha$.

Proof. $1 + g_z \equiv z/z_{-1} = (x/x_{-1})^\alpha \equiv (1 + g_x)^\alpha$. □

Given a time series $x_0, x_1, ..., x_n$, by the *average growth rate* per period, or more precisely, the *average compound growth rate,* is meant a g which satisfies $x_n = x_0(1 + g)^n$. The solution for g is $g = (x_n/x_0)^{1/n} - 1$.

Finally, the following approximation may be useful (for intuition) if used with caution:

THE GROWTH FACTOR With n denoting a positive integer above 1 but "not too large", the growth factor $(1 + g)^n$ can be approximated by $1 + ng$ when g is "small". For $g \neq 0$, the approximation error is larger the larger is n.

Proof. (i) We prove the claim by induction. Suppose the claim holds for a fixed $n \geq 2$, i.e., $(1 + g)^n \approx 1 + ng$ for g "small". Then $(1 + g)^{n+1} = (1 + g)^n(1 + g) \approx (1 + ng)(1 + g) = 1 + ng + g + ng^2 \approx 1 + (n + 1)g$ since g "small" implies g^2 "very small" and therefore ng^2 "small" if n is not "too" large. So the claim holds also for $n + 1$. Since $(1 + g)^2 = 1 + 2g + g^2 \approx 1 + 2g$, for g "small", the claim does indeed hold for $n = 2$. □

THE EFFECTIVE ANNUAL RATE OF INTEREST Suppose interest on a loan is charged n times a year at the rate r/n per year. Then the *effective annual interest rate* is $(1 + r/n)^n - 1$.

B. Proof of the sufficiency part of Uzawa's theorem

For convenience we restate the full theorem here:

PROPOSITION 2 *(Uzawa's balanced growth theorem).* Let $P \equiv \{(K_t, Y_t, C_t)\}_{t=0}^{\infty}$ be a path along which Y_t, K_t, C_t, and $S_t \equiv Y_t - C_t$ are positive for all $t = 0, 1, 2, \ldots$, and satisfy the dynamic resource constraint for a closed economy, (4.3), given the production function (4.5) and the labor force (4.6). Then:
(i) A *necessary* condition for the path P to be a BGP is that along P it holds that

$$Y_t = \tilde{F}(K_t, T_t L_t, 0), \qquad (*)$$

where $T_t = T_0(1+g)^t$ with $T_0 = B$ and $1 + g \equiv (1 + g_Y)/(1 + n) > 1$, g_Y being the constant growth rate of output along the BGP.
(ii) Assume $(1 + g)(1 + n) > 1 - \delta$. Then, for any $g \geq 0$ such that there is a $q > (1 + g)(1 + n) - (1 - \delta)$ with the property that the production function $\tilde{F}$ in (4.5) allows an output-capital ratio equal to q at $t = 0$ (i.e., $\tilde{F}(1, \tilde{k}^{-1}, 0) = q$ for some real number $\tilde{k} > 0$), a *sufficient* condition for $\tilde{F}$ to be compatible with a BGP with output-capital ratio equal to q is that $\tilde{F}$ can be written as in (4.7) with $T_t = B(1 + g)^t$.

Proof (i) See Section 4.1. (ii) Suppose (*) holds with $T_t = B(1 + g)^t$. Let $g \geq 0$ be given such that there is a $q > (1 + g)(1 + n) - (1 - \delta) > 0$ with the property that

$$\tilde{F}(1, \tilde{k}^{-1}, 0) = q \qquad (**)$$

for some constant $\tilde{k} > 0$. Our strategy is to prove the claim by construction of a path $P = (Y_t, K_t, C_t)_{t=0}^{\infty}$ which satisfies it. We let P be such that the saving-income ratio is a constant $\hat{s} \equiv [(1 + g)(1 + n) - (1 - \delta)]/q \in (0, 1)$, i.e., $Y_t - C_t \equiv S_t = \hat{s}Y_t$ for all $t = 0, 1, 2, \ldots$. Inserting this, together with $Y_t = f(\tilde{k}_t)T_t L_t$, where $f(\tilde{k}_t) \equiv \tilde{F}(\tilde{k}_t, 1, 0)$ and $\tilde{k}_t \equiv K_t/(T_t L_t)$, into (4.3), rearranging gives the Solow equation (4.4), which we may rewrite as

$$\tilde{k}_{t+1} - \tilde{k}_t = \frac{\hat{s}f(\tilde{k}_t) - [(1 + g)(1 + n) - (1 - \delta)]\,\tilde{k}_t}{(1 + g)(1 + n)}.$$

We see that $\tilde{k}_t$ is constant if and only if $\tilde{k}_t$ satisfies the equation $f(\tilde{k}_t)/\tilde{k}_t = [(1 + g)(1 + n) - (1 - \delta)]/\hat{s} \equiv q$. By (**) and the definition of f, the required value of $\tilde{k}_t$ is $\tilde{k}$, which is thus the steady state for the constructed Solow model. Letting K_0 satisfy $K_0 = \tilde{k}BL_0$, where $B = T_0$, we thus have $\tilde{k}_0 = K_0/(T_0 L_0) = \tilde{k}$. So that the initial value of $\tilde{k}_t$ equals the steady-state value. It follows that $\tilde{k}_t = \tilde{k}$ for all $t = 0, 1, 2, \ldots$, and so $Y_t/K_t = f(\tilde{k}_t)/\tilde{k}_t = f(\tilde{k})/\tilde{k} = q$ for all $t \geq 0$. In addition, $C_t = (1 - \hat{s})Y_t$, so that C_t/Y_t is constant along the path P. As both Y/K and C/Y are thus constant along the path P, by (ii) of Proposition 1 follows that P is a BGP. □

It is noteworthy that the proof of the sufficiency part of the theorem is *constructive.* It provides a method for constructing a BGP with a given technology growth rate and a given output-capital ratio.

C. Homothetic utility functions

Generalities A set C in $\mathbb{R}^n$ is called a *cone* if $x \in C$ and $\lambda > 0$ implies $\lambda x \in C$. A function $f(\mathbf{x}) = f(x_1,\ldots,x_n)$ is *homothetic* in the cone C if for all $\mathbf{x}, \mathbf{y} \in C$ and all $\lambda > 0$, $f(\mathbf{x}) = f(\mathbf{y})$ implies $f(\lambda\mathbf{x}) = f(\lambda\mathbf{y})$.

Consider the continuous utility function $U(x_1, x_2)$, defined in $\mathbb{R}^2_+$. Suppose U is *homothetic* and that the consumption bundles (x_1, x_2) and (y_1, y_2) are on the same indifference curve, i.e., $U(x_1, x_2) = U(y_1, y_2)$. Then for any $\lambda > 0$ we have $U(\lambda x_1, \lambda x_2) = U(\lambda y_1, \lambda y_2)$ so that the bundles $(\lambda x_1, \lambda x_2)$ and $(\lambda y_1, \lambda y_2)$ are also on the same indifference curve.

For a continuous utility function $U(x_1, x_2)$, defined in $\mathbb{R}^2_+$ and increasing in each of its arguments (as is our life time utility function in the Diamond model), one can show that U is homothetic if and only if U can be written $U(x_1, x_2) \equiv F(f(x_1, x_2))$ where the function f is homogeneous of degree one and F is an increasing function. The "if" part is easily shown. Indeed, if $U(x_1, x_2) = U(y_1, y_2)$, then $F(f(x_1, x_2)) = F(f(y_1, y_2))$. Since F is increasing, this implies $f(x_1, x_2) = f(y_1, y_2)$. Because f is homogeneous of degree one, if $\lambda > 0$, then $f(\lambda x_1, \lambda x_2) = \lambda f(x_1, x_2)$ and $f(\lambda y_1, \lambda y_2) = \lambda f(y_1, y_2)$ so that $U(\lambda x_1, \lambda x_2) = F(f(\lambda x_1, \lambda x_2)) = F(f(\lambda y_1, \lambda y_2)) = U(\lambda y_1, \lambda y_2)$, which shows that U is homothetic. As to the "only if" part, see Sydsaeter et al. (2002).

Using differentiability of our homothetic time utility function $U(x_1, x_2) \equiv F(f(x_1, x_2))$, we find the marginal rate of substitution of good 2 for good 1 to be

$$MRS = \frac{dx_2}{dx_1}_{|U=\bar{U}} = \frac{\partial U/\partial x_1}{\partial U/\partial x_2} = \frac{F' f_1(x_1, x_2)}{F' f_2(x_1, x_2)} = \frac{f_1(1, \frac{x_2}{x_1})}{f_2(1, \frac{x_2}{x_1})}. \tag{4.43}$$

The last equality is due to Euler's theorem saying that when f is homogeneous of degree 1, then the first-order partial derivatives of f are homogeneous of degree 0. Now, (4.43) implies that for a given MRS, in optimum reflecting a given relative price of the two goods, the same consumption ratio, x_2/x_1, will be chosen whatever the budget. For a given relative price, a rising budget (rising wealth) will change the position of the budget line, but not its slope. So MRS will not change, which implies that the chosen pair (x_1, x_2) will move outward along a given ray in $\mathbb{R}^2_+$. Indeed, as the intercepts with the axes rise proportionately with the budget (the wealth), so will x_1 and x_2.

Proof that the utility function in (4.25) is homothetic In Section 4.2 we claimed that (4.25) is a homothetic utility function. This can be proved in the following way. There are two cases to consider. *Case 1:* $\theta > 0$, $\theta \neq 1$. We rewrite (4.25) as

$$U(c_1, c_2) = \frac{1}{1-\theta}\left[(c_1^{1-\theta} + \beta c_2^{1-\theta})^{1/(1-\theta)}\right]^{1-\theta} - \frac{1+\beta}{1-\theta},$$

where $\beta \equiv (1+\rho)^{-1}$. The function $x = g(c_1, c_2) \equiv (c_1^{1-\theta} + \beta c_2^{1-\theta})^{1/(1-\theta)}$ is homogeneous of degree one and the function $G(x) \equiv (1/(1-\theta))x^{1-\theta} - (1+\beta)/(1-\theta)$ is an increasing function, given $\theta > 0$, $\theta \neq 1$. *Case 2:* $\theta = 1$. Here we start from $U(c_1, c_2) = \ln c_1 + \beta \ln c_2$. This can be written

$$U(c_1, c_2) = (1+\beta)\ln\left[(c_1 c_2^{\beta})^{1/(1+\beta)}\right],$$

where $x = g(c_1, c_2) = (c_1 c_2^{\beta})^{1/(1+\beta)}$ is homogeneous of degree one and $G(x) \equiv (1+\beta)\ln x$ is an increasing function. □

D. General formulas for the elasticity of factor substitution

We here prove (4.34) and (4.35). Given the neoclassical production function $F(K, L)$, the slope of the isoquant $F(K, L) = \bar{Y}$ at the point $(\bar{K}, \bar{L})$ is

$$\frac{dK}{dL}_{|Y=\bar{Y}} = -MRS = -\frac{F_L(\bar{K}, \bar{L})}{F_K(\bar{K}, \bar{L})}. \tag{4.44}$$

We consider this slope as a function of the value of $k \equiv K/L$ as we move along the isoquant. The derivative of this function is

$$\begin{aligned} -\frac{dMRS}{dk}_{|Y=\bar{Y}} &= -\frac{dMRS}{dL}_{|Y=\bar{Y}}\frac{dL}{dk}_{|Y=\bar{Y}} \\ &= -\frac{(F_L)^2F_{KK} - 2F_KF_LF_{KL} + (F_K)^2F_{LL}}{F_K^3}\frac{dL}{dk}_{|Y=\bar{Y}} \end{aligned} \tag{4.45}$$

by (2.53) of Chapter 2. In view of $L \equiv K/k$ we have

$$\frac{dL}{dk}_{|Y=\bar{Y}} = \frac{k\frac{dK}{dk}_{|Y=\bar{Y}} - K}{k^2} = \frac{k\frac{dK}{dL}_{|Y=\bar{Y}}\frac{dL}{dk}_{|Y=\bar{Y}} - K}{k^2} = \frac{-kMRS\frac{dL}{dk}_{|Y=\bar{Y}} - K}{k^2}.$$

From this we find

$$\frac{dL}{dk}_{|Y=\bar{Y}} = -\frac{K}{(k+MRS)k},$$

to be substituted into (4.45). Finally, we substitute the inverse of (4.45) together with (4.44) into the definition of the elasticity of factor substitution:

$$\begin{aligned}
\sigma(K,L) &\equiv \frac{MRS}{k}\frac{dk}{dMRS}_{|Y=\bar{Y}} \\
&= -\frac{F_L/F_K}{k}\frac{(k+F_L/F_K)k}{K}\frac{F_K^3}{[(F_L)^2F_{KK}-2F_KF_LF_{KL}+(F_K)^2F_{LL}]} \\
&= -\frac{F_KF_L(F_KK+F_LL)}{KL\left[(F_L)^2F_{KK}-2F_KF_LF_{KL}+(F_K)^2F_{LL}\right]},
\end{aligned}$$

which is the same as (4.34).

Under CRS, this reduces to

$$\begin{aligned}
\sigma(K,L) &= -\frac{F_KF_LF(K,L)}{KL\left[(F_L)^2F_{KK}-2F_KF_LF_{KL}+(F_K)^2F_{LL}\right]} \quad \text{(from (2.54) with } h=1) \\
&= -\frac{F_KF_LF(K,L)}{KLF_{KL}\left[-(F_L)^2L/K-2F_KF_L-(F_K)^2K/L\right]} \quad \text{(from (2.60))} \\
&= \frac{F_KF_LF(K,L)}{F_{KL}(F_LL+F_KK)^2} = \frac{F_KF_L}{F_{KL}F(K,L)}, \quad \text{(using (2.54) with } h=1)
\end{aligned}$$

which proves the first part of (4.35). The second part is an implication of rewriting the formula in terms of the production in intensive form.

E. Properties of the CES production function

The generalized CES production function is

$$Y = A\left[\alpha K^{\beta}+(1-\alpha)L^{\beta}\right]^{\frac{\gamma}{\beta}}, \tag{4.46}$$

where A, α, and β are parameters satisfying $A > 0$, $0 < \alpha < 1$, and $\beta < 1$, $\beta \neq 0, \gamma > 0$. If $\gamma < 1$, there is DRS, if $\gamma = 1$, CRS, and if $\gamma > 1$, IRS. The elasticity of substitution is always $\sigma = 1/(1-\beta)$. Throughout below, k means K/L.

The limiting functional forms We claimed in the text that, for fixed $K > 0$ and $L > 0$, (4.46) implies:

$$\begin{aligned}
\lim_{\beta\to 0} Y &= A(K^{\alpha}L^{1-\alpha})^{\gamma} = AL^{\gamma}k^{\alpha\gamma}, && (*) \\
\lim_{\beta\to -\infty} Y &= A\min(K^{\gamma},L^{\gamma}) = AL^{\gamma}\min(k^{\gamma},1). && (**)
\end{aligned}$$

Proof. Let $q \equiv Y/(AL^\gamma)$. Then $q = (\alpha k^\beta + 1 - \alpha)^{\gamma/\beta}$ so that

$$\ln q = \frac{\gamma \ln(\alpha k^\beta + 1 - \alpha)}{\beta} \equiv \frac{m(\beta)}{\beta}, \tag{4.47}$$

where

$$m'(\beta) = \frac{\gamma \alpha k^\beta \ln k}{\alpha k^\beta + 1 - \alpha} = \frac{\gamma \alpha \ln k}{\alpha + (1-\alpha)k^{-\beta}}. \tag{4.48}$$

Hence, by L'Hôpital's rule for "0/0",

$$\lim_{\beta \to 0} \ln q = \lim_{\beta \to 0} \frac{m'(\beta)}{1} = \gamma \alpha \ln k = \ln k^{\gamma\alpha},$$

so that $\lim_{\beta \to 0} q = k^{\gamma\alpha}$, which proves (*). As to (**), note that

$$\lim_{\beta \to -\infty} k^\beta = \lim_{\beta \to -\infty} \frac{1}{k^{-\beta}} \to \begin{cases} 0 \text{ for } k > 1, \\ 1 \text{ for } k = 1, \\ \infty \text{ for } k < 1. \end{cases}$$

Hence, by (4.47),

$$\lim_{\beta \to -\infty} \ln q = \begin{cases} 0 \text{ for } k \geq 1, \\ \lim_{\beta \to -\infty} \frac{m'(\beta)}{1} = \gamma \ln k = \ln k^\gamma \text{ for } k < 1, \end{cases}$$

where the result for $k < 1$ is based on L'Hôpital's rule for "∞/-∞". Consequently,

$$\lim_{\beta \to -\infty} q = \begin{cases} 1 \text{ for } k \geq 1, \\ k^\gamma \text{ for } k < 1, \end{cases}$$

which proves (**). □

Properties of the isoquants of the CES function The absolute value of the slope of an isoquant for (4.46) in the (L, K) plane is

$$MRS_{KL} = \frac{\partial Y/\partial L}{\partial Y/\partial K} = \frac{1-\alpha}{\alpha} k^{1-\beta} \to \begin{cases} 0 \text{ for } k \to 0, \\ \infty \text{ for } k \to \infty. \end{cases} \tag{*}$$

This holds whether $\beta < 0$ or $0 < \beta < 1$.

Concerning the asymptotes and terminal points, if any, of the isoquant $Y = \bar{Y}$ we have from (4.46) $\bar{Y}^{\beta/\gamma} = A\left[\alpha K^\beta + (1-\alpha)L^\beta\right]$. Hence,

$$\begin{aligned} K &= \left(\frac{\bar{Y}^{\frac{\beta}{\gamma}}}{A\alpha} - \frac{1-\alpha}{\alpha} L^\beta \right)^{\frac{1}{\beta}}, \\ L &= \left(\frac{\bar{Y}^{\frac{\beta}{\gamma}}}{A(1-\alpha)} - \frac{\alpha}{1-\alpha} K^\beta \right)^{\frac{1}{\beta}}. \end{aligned}$$

From these two equations follows, when $\beta < 0$ (i.e., $0 < \sigma < 1$), that

$$\begin{aligned} K &\rightarrow (A\alpha)^{-\frac{1}{\beta}}\bar{Y}^{\frac{1}{\gamma}} \text{ for } L \rightarrow \infty, \\ L &\rightarrow [A(1-\alpha)]^{-\frac{1}{\beta}}\bar{Y}^{\frac{1}{\gamma}} \text{ for } K \rightarrow \infty. \end{aligned}$$

When instead $\beta > 0$ (i.e., $\sigma > 1$), the same limiting formulas obtain for $L \rightarrow 0$ and $K \rightarrow 0$, respectively.

Properties of the CES function on intensive form Given $\gamma = 1$, i.e., CRS, we have $y \equiv Y/L = A(\alpha k^\beta + 1 - \alpha)^{1/\beta}$ from (4.46). Then

$$\frac{dy}{dk} = A\frac{1}{\beta}(\alpha k^\beta + 1 - \alpha)^{\frac{1}{\beta}-1}\alpha\beta k^{\beta-1} = A\alpha\left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1-\beta}{\beta}}.$$

Hence, when $\beta < 0$ (i.e., $0 < \sigma < 1$),

$$\begin{aligned} y &= \frac{A}{(ak^\beta + 1 - \alpha)^{-1/\beta}} \rightarrow \begin{cases} 0 \text{ for } k \rightarrow 0, \\ A(1-\alpha)^{1/\beta} \text{ for } k \rightarrow \infty. \end{cases} \\ \frac{dy}{dk} &= \frac{A\alpha}{[\alpha + (1-\alpha)k^{-\beta}]^{(\beta-1)/\beta}} \rightarrow \begin{cases} A\alpha^{1/\beta} \text{ for } k \rightarrow 0, \\ 0 \text{ for } k \rightarrow \infty. \end{cases} \end{aligned}$$

If instead $\beta > 0$ (i.e., $\sigma > 1$),

$$\begin{aligned} y &\rightarrow \begin{cases} A(1-\alpha)^{1/\beta} \text{ for } k \rightarrow 0, \\ \infty \text{ for } k \rightarrow \infty. \end{cases} \\ \frac{dy}{dk} &\rightarrow \begin{cases} \infty \text{ for } k \rightarrow 0, \\ A\alpha^{1/\beta} \text{ for } k \rightarrow \infty. \end{cases} \end{aligned}$$

The output-capital ratio is $y/k = A\left[\alpha + (1-\alpha)k^{-\beta}\right]^{\frac{1}{\beta}}$ and has the same limiting values as dy/dk, when $\beta > 0$.

Continuity at the boundary of $\mathbb{R}^2_+$ When $0 < \beta < 1$, the right-hand side of (4.46) is defined and continuous also on the boundary of $\mathbb{R}^2_+$. Indeed, we get

$$Y = F(K, L) = A\left[\alpha K^\beta + (1-\alpha)L^\beta\right]^{\frac{\gamma}{\beta}} \rightarrow \begin{cases} A\alpha^{\frac{\gamma}{\beta}}K^\gamma \text{ for } L \rightarrow 0, \\ A(1-\alpha)^{\frac{\gamma}{\beta}}L^\gamma \text{ for } K \rightarrow 0. \end{cases}$$

When $\beta < 0$, however, the right-hand side is not defined on the boundary. We circumvent this problem by redefining the CES function in the following way when $\beta < 0$:

$$Y = F(K, L) = \begin{cases} A\left[\alpha K^\beta + (1-\alpha)L^\beta\right]^{\frac{\gamma}{\beta}} \text{ when } K > 0 \text{ and } L > 0, \\ 0 \text{ when either } K \text{ or } L \text{ equals } 0. \end{cases} \tag{4.49}$$

We now show that continuity holds in the extended domain. When $K > 0$ and $L > 0$, we have

$$Y^{\frac{\beta}{\gamma}} = A^{\frac{\beta}{\gamma}} \left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right] \equiv A^{\frac{\beta}{\gamma}} G(K,L). \tag{4.50}$$

Let $\beta < 0$ and $(K,L) \to (0,0)$. Then, $G(K,L) \to \infty$, and so $Y^{\beta/\gamma} \to \infty$. Since $\beta/\gamma < 0$, this implies $Y \to 0 = F(0,0)$, where the equality follows from the definition in (4.49). Next, consider a fixed $L > 0$ and rewrite (4.50) as

$$\begin{aligned} Y^{\frac{1}{\gamma}} &= A^{\frac{1}{\gamma}} \left[\alpha K^{\beta} + (1-\alpha)L^{\beta}\right]^{\frac{1}{\beta}} = A^{\frac{1}{\gamma}} L(\alpha k^{\beta} + 1 - \alpha)^{\frac{1}{\beta}} \\ &= \frac{A^{\frac{1}{\gamma}} L}{(ak^{\beta} + 1 - \alpha)^{-1/\beta}} \to 0 \text{ for } k \to 0, \end{aligned}$$

when $\beta < 0$. Since $1/\gamma > 0$, this implies $Y \to 0 = F(0,L)$, from (4.49). Finally, consider a fixed $K > 0$ and let $L/K \to 0$. Then, by an analogue argument we get $Y \to 0 = F(K,0)$, (4.49). So continuity is maintained in the extended domain.

4.10 Exercises

4.1 *(the aggregate saving rate in steady state)*

a) In a well-behaved Diamond OLG model let n be the rate of population growth and k^* the steady state capital-labor ratio (until further notice, we ignore technological progress). Derive a formula for the long-run aggregate net saving rate, S^N/Y, in terms of n and k^*. *Hint:* use that for a closed economy $S^N = K_{t+1} - K_t$.

b) In the Solow growth model without technological change a similar relation holds, but with a different interpretation of the causality. Explain.

c) Compare your result in a) with the formula for S^N/Y in steady state one gets in *any* model with the same CRS-production function and no technological change. Comment.

d) Assume that $n = 0$. What does the formula from a) tell you about the level of net aggregate savings in this case? Give the intuition behind the result in terms of the aggregate saving by any generation in two consecutive periods. One might think that people's rate of impatience (in Diamond's model the rate of time preference ρ) affect S^N/Y in steady state. Does it in this case? Why or why not?

e) Suppose there is Harrod-neutral technological progress at the constant rate $g > 0$. Derive a formula for the aggregate net saving rate in the long run in a well-behaved Diamond model in this case.

f) Answer d) with "from a)" replaced by "from e)". Comment.

g) Consider the statement: "In Diamond's OLG model any generation saves as much when young as it dissaves when old." True or false? Why?

4.2 (*increasing returns to scale and balanced growth)*

Chapter 6

Long-run aspects of fiscal policy and public debt

We consider an economy with a government providing public goods and services. It finances its spending by taxation and borrowing. The term *fiscal policy* refers to the government's decisions about spending and the financing of this spending, be it by taxes or debt issue. The government's choice concerning the level and composition of its spending and how to finance it, may aim at:

1 affecting resource allocation (provide public goods that would otherwise not be supplied in a sufficient amount, correct externalities and other markets failures, prevent monopoly inefficiencies, provide social insurance);

2 affecting income distribution, be it a) within generations or b) between generations;

3 contribute to macroeconomic stabilization (dampening of business cycle fluctuations through aggregate demand policies).

The design of fiscal policy with regard to the aims 1 and 2 at a disaggregate level is a major theme within the field of public economics. Macroeconomics studies ways of dealing with aim 3 as well as big-picture aspects of 1 and 2, like overall policies to maintain and promote sustainable prosperity.

In this chapter we address fiscal sustainability and long-run implications of debt finance. This relates to one of the conditions that constrain public financing instruments. To see the issue of fiscal sustainability in a broader context, Section 6.1 provides an overview of conditions and factors that constrain public financing instruments. Section 6.2 introduces the basics of government budgeting and Section 6.3 defines the concepts of *government solvency* and *fiscal sustainability.* In Section 6.4 the analytics of debt dynamics is presented. As an example, the

Stability and Growth Pact of the EMU (the Economic and Monetary Union of the European Union) is discussed. Section 6.5 looks more closely at the link between government solvency and the government's *No-Ponzi-Game condition* and *intertemporal budget constraint.* In Section 6.6 we widen public sector accounting by introducing separate operating and capital budgets so as to allow for proper accounting of public investment. A theoretical claim, known as the *Ricardian equivalence* proposition, is studied in Section 6.7. The question whether Ricardian equivalence is likely to be a good approximation to reality, is addressed, applying the Diamond OLG framework extended with a public sector.

6.1 An overview of government spending and financing issues

Before entering the more specialized sections, it is useful to have a general idea about circumstances that condition public spending and financing. These circumstances include:

(i) financing by debt issue is constrained by the need to remain solvent and avoid catastrophic debt dynamics;

(ii) financing by taxes is limited by problems arising from:

  (a) distortionary supply-side effects of many kinds of taxes;

  (b) tax evasion (cf. the rise of the shadow economy, tax havens used by multinationals, etc.).

(iii) time lags in spending as well as taxing may interfere with attempts to stabilize the economy (recognition lag, decision lag, implementation lag, and effect lag);

(iv) credibility problems due to time-inconsistency;

(v) conditions imposed by political processes, bureaucratic self-interest, lobbying, and rent seeking.

Point (i) is the main focus of sections 6.2-6.6. Point (ii) is briefly considered in Section 6.4.1 in connection with the *Laffer curve.* In Section 6.6 point (iii) is briefly commented on. The remaining points, (iv) - (v), are not addressed specifically in this chapter. They should always be kept in mind, however, when discussing fiscal policy in practice. Hence some remarks at the end of the chapter.

Now to the specifics of government budget accounting and debt financing.

6.2 The government budget

We generally perceive the *public sector* (or the *nation state*) as consisting of the *national government* and a *central bank.* In economics the term "government" does not generally refer to the particular administration in office at a point in time. The term is rather used in a broad sense, encompassing both legislation and central and local administration. The aspects of legislation and administration in focus in macroeconomics are the rules and decisions concerning spending on public consumption, public investment, transfers, and subsidies on the expenditure side and on levying taxes and incurring debts on the financing side. Within certain limits the national government has usually delegated the management of the nation's currency to the central bank, a separate governmental institution, often called the monetary authority. Yet, from an overall macroeconomic point of view it is useful to treat "government budgeting" as covering the public sector as a whole: the consolidated government and central bank. Government bonds held by the central bank are thus excluded from what we call "government debt".

The basics of government budget accounting cannot be described without including money, nominal prices, and inflation. Elementary aspects of money and inflation will therefore be included in this section. We shall not, however, consider money and inflation in any systematic way until later chapters. Whether the economy considered is a closed or open economy will generally not be important in this chapter. We use the terms *government debt* and *public debt* synonymously.

Table 6.1 lists key variables of government budgeting.

Table 6.1. List of main variable symbols

Symbol	*Meaning*
Y_t	real GDP (= real GNP if the economy is closed)
C_t^g	public consumption
I_t^g	public fixed capital investment
G_t	$\equiv C_t^g + I_t^g$ real public purchases (spending on goods and services)
X_t	real transfer payments
$\tilde{T}_t$	real gross tax revenue
T_t	$\equiv \tilde{T}_t - X_t$ real net tax revenue
M_t	the monetary base (currency and bank reserves in the central bank)
P_t	price level (in money) for goods and services (the GDP deflator)
D_t	nominal net public debt (including possible debt of local government)
B_t	$\equiv \frac{D_t}{P_{t-1}}$ real net public debt
b_t	$\equiv \frac{B_t}{Y_t}$ government debt-to-income ratio
i_t	nominal short-term interest rate
Δx_t	$\equiv x_t - x_{t-1}$ (where x is some arbitrary variable)
π_t	$\equiv \frac{\Delta P_t}{P_{t-1}} \equiv \frac{P_t - P_{t-1}}{P_{t-1}}$ inflation rate
$1 + r_t$	$\equiv \frac{P_{t-1}(1+i_t)}{P_t} \equiv \frac{1+i_t}{1+\pi_t}$ real short-term interest rate

Note that Y_t, G_t, and T_t are quantities defined *per period*, or more generally, *per time unit,* and are thus flow variables. On the other hand, M_t, D_t, and B_t are stock variables, that is, quantities defined at a given point in time, here at the *beginning* of period t. We measure D_t and B_t *net* of financial claims held by the government. Almost all countries have positive government net debt, but in principle $D_t < 0$ is possible.[1] The monetary base, M_t, is currency plus fully liquid deposits in the central bank held by the private sector at the beginning of period t; M_t is by definition nonnegative.

Until further notice, we shall in this chapter ignore uncertainty and default risk. We shall also ignore the fact that government bonds are usually more liquid (easier to quickly convert into cash) than other financial assets. Under these circumstances the market interest rate on government bonds must be the same as that on other interest-bearing assets. There is thus only one interest rate, i_t, in the economy. For ease of exposition we imagine that all government bonds are *one-period bonds.* That is, each government bond promises a payout equal to one unit of account at the end of the period and then the bond expires. Given the interest rate, i_t, the market value of a bond at the start of period t is $v_t = 1/(1 + i_t)$. If the number of outstanding bonds (the quantity of bonds) in

[1] If $D_t < 0$, the government has positive net financial claims on the private sector and earns interest on these claims − which is then an additional source of government revenue besides taxation.

period t is q_t, the government debt has face value (value at maturity) equal to q_t. The market value at the start of period t of this quantity of bonds will be $D_t \equiv q_t v_t = q_t/(1+i_t)$. The nominal expenditure to be made at the end of the period to redeem the outstanding debt can then be written

$$q_t = D_t(1+i_t). \tag{6.1}$$

This is the usual way of writing the expenditure to be made, namely as *if* the government debt were like a given bank loan of size D_t with a variable rate of interest. We should not forget, however, that given the quantity, q_t, of the bonds, the value, D_t, of the government debt at the issue date depends negatively on i_t.

Anyway, the total nominal government expenditure in period t can be written

$$P_t(G_t + X_t) + D_t(1+i_t).$$

It is common to refer to this expression as expenditure "in period t". Yet, in a discrete time model (with a period length of a year or a quarter corresponding to typical macroeconomic data) one has to imagine that the *payment* for goods and services delivered in the period occurs either at the beginning or the end of the period. We follow the latter interpretation and so the nominal price level P_t for period-t goods and services refers to payment occurring at the *end* of period t. As an implication, the real value, B_t, of government debt at the beginning of period t (= end of period $t-1$) is D_t/P_{t-1}. This may look a little awkward but is nevertheless meaningful. Indeed, D_t is a *stock* of liabilities at the beginning of period t while P_{t-1} is a price referring to a flow *paid for* at the *end* of period $t-1$ which is essentially the same point in time as the beginning of period t. Anyway, whatever timing convention is chosen, some kind of awkwardness will always arise in discrete time analysis. This is because the discrete time approach artificially treats the continuous flow of time as a sequence of discrete points in time.[2]

The government's expenditure is financed, effectively, by a combination of taxes, bonds issue, and increase in the monetary base:

$$P_t\tilde{T}_t + D_{t+1} + \Delta M_{t+1} = P_t(G_t + X_t) + D_t(1+i_t). \tag{6.2}$$

By rearranging we have

$$\Delta D_{t+1} + \Delta M_{t+1} = P_t(G_t + X_t - \tilde{T}_t) + i_t D_t. \tag{6.3}$$

Although in many developed countries the central bank is prohibited from buying government bonds directly from the government, it may buy them from private

[2]In a theoretical model this kind of problems is avoided when government budgeting is formulated in continuous time, cf. Chapter 13.

entities shortly after these have bought them from the government. Over the year the newly issued government debt may thus be more or less "monetarized".

In customary government budget accounting the nominal *government budget deficit, GBD*, is defined as the excess of total government spending over government revenue, $P\tilde{T}$. That is, according to this definition the right-hand side of (6.3) is the nominal budget deficit in period t, GBD_t. The first term on the right-hand side, $P_t(G_t + X_t - \tilde{T}_t)$, is named the nominal *primary budget deficit* (non-interest spending less taxes). The second term, $i_t D_t$, is called the nominal *debt service*. Similarly, $P_t(\tilde{T}_t - X_t - G_t)$ is called the nominal *primary budget surplus*. A negative value of a "deficit" thus amounts to a positive value of a corresponding "surplus", and a negative value of a "surplus" amounts to a positive value of a corresponding "deficit".

We immediately see that this accounting deviates from "normal" principles. Business companies typically have sharply separated capital and operating budgets. In contrast, the budget deficit defined above treats that part of G which represents government *net investment* as parallel to government consumption. Government net investment is attributed as an *expense* in a single year's account. According to "normal" principles it is only the *depreciation* on the public capital that should figure as an expense. Likewise, the above accounting does not consider that a part of D, or perhaps more than D, may be backed by the value of public physical capital. And if the government sells a physical asset to the private sector, the sale will appear as a reduction of the government budget deficit while in reality it is merely a conversion of an asset from a physical form to a financial form. The expense and asset aspects of government net investment are thus not properly dealt with in the standard public accounting.[3]

With the exception of Section 6.6 we will nevertheless stick to the traditional vocabulary. Where this might create logical difficulties, it helps to imagine that:

(a) all of G *is* public consumption, i.e., $G_t = C_t^g$ for all t;

(b) there is *no* public physical capital.

Now, from (6.2) and the definition $T_t \equiv \tilde{T}_t - X_t$ (net tax revenue) follows that

[3] Another anomaly is related to the fact that some countries, for instance Denmark, have large implicit government assets due to deferred taxes on the part of personal income invested in pension funds. If the government then decides to reverse the deferred taxation (as the Danish government did 2012 and 2014 to comply better with the 3%-deficit rule of the Stability and Growth Pact of the EMU), the official budget deficit is reduced. But essentially, all that has happened is that one government asset has been replaced by another.

real government debt at the beginning of period $t+1$ is:

$$\begin{aligned}
B_{t+1} &\equiv \frac{D_{t+1}}{P_t} = G_t + X_t - \tilde{T}_t + (1+i_t)\frac{D_t}{P_t} - \frac{\Delta M_{t+1}}{P_t} \\
&= G_t - T_t + (1+i_t)\frac{D_t/P_{t-1}}{P_t/P_{t-1}} - \frac{\Delta M_{t+1}}{P_t} = G_t - T_t + \frac{1+i_t}{1+\pi_t}B_t - \frac{\Delta M_{t+1}}{P_t} \\
&\equiv (1+r_t)B_t + G_t - T_t - \frac{\Delta M_{t+1}}{P_t}. \qquad (6.4)
\end{aligned}$$

This is the law of motion of real government debt.

The last term, $\Delta M_{t+1}/P_t$, in (6.4) is *seigniorage,* i.e., public sector revenue obtained by issuing base money (ignoring the diminutive cost of printing money). To get a sense of this variable, suppose real output grows at the constant rate g_Y so that $Y_{t+1} = (1+g_Y)Y_t$. Then the public debt-to-income ratio can be written

$$b_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}} = \frac{1+r_t}{1+g_Y}b_t + \frac{G_t - T_t}{(1+g_Y)Y_t} - \frac{\Delta M_{t+1}}{P_t(1+g_Y)Y_t}. \qquad (6.5)$$

Apart from the growth-correcting factor, $(1+g_Y)^{-1}$, the last term is the seigniorage-income ratio,

$$\frac{\Delta M_{t+1}}{P_t Y_t} = \frac{\Delta M_{t+1}}{M_t}\frac{M_t}{P_t Y_t}.$$

If in the long run the base money growth rate, $\Delta M_{t+1}/M_t$, as well as the nominal interest rate (i.e., the opportunity cost of holding money) are constant, then the velocity of money and its inverse, the money-nominal income ratio, $M_t/(P_t Y_t)$, are also likely to be roughly constant. So is, therefore, the seigniorage-income ratio.[4] For the more developed countries this ratio tends to be a fairly small number although not immaterial. For emerging economies with poor institutions for collecting taxes seigniorage matters more.[5]

The U.S. has a single monetary authority, the central bank, and a single fiscal authority, the treasury. The seigniorage created is immediately transferred from the first to the latter. The Eurozone has a single monetary authority but

[4] A reasonable money demand function is $M_t^d = P_t Y_t e^{-\alpha i}$, $\alpha > 0$, where i is the nominal interest rate. With clearing in the money market, we thus have $M_t/(P_t Y_t) = e^{-\alpha i}$. In view of $1 + i \equiv (1+r)(1+\pi)$, when r and π are constant, so is i and, thereby, $M_t/(P_t Y_t)$.

[5] In the U.S. over the period 1909-1950s seigniorage fluctuated a lot and peaked 4 % of GDP in the 1930s and 3 % of GDP at the end of WW II. But over the period from the late 1960s to 1986 seigniorage fluctuated less around an average close to 0.5 %.of GDP (Walsh, 2003, p. 177). In Denmark seigniorage was around 0.2 % of GDP during the 1990s (*Kvartalsoversigt 4. kvartal 2000*, Danmarks Nationalbank). In Bolivia, up to the event of hyperinflation 1984-85, seigniorage reached 5 % of GDP and more than 50 % of government revenue (Sachs and Larrain, 1993).

multiple fiscal authorities, namely the treasuries of the member countries. The seigniorage created by the ECB is every year shared by the national central banks of the Eurozone countries in proportion to their equity share in the ECB. And the national central banks then transfer their share to the national treasuries. This makes up a ΔM_{t+1} term for the consolidated public sector of the individual Eurozone countries.

In monetary unions and countries with their own currency, government budget deficits are thus, from a macroeconomic point of view, generally financed both by debt creation and money creation, as envisioned by the above equations. Nonetheless, from now on, for simplicity, in this chapter we will predominantly ignore the seigniorage term in (6.5) and only occasionally refer to the modifications implied by taking it into account.

We thus proceed with the simple government accounting equation:

$$B_{t+1} - B_t = r_t B_t + G_t - T_t, \tag{DGBC}$$

where the right-hand side is the *real budget deficit.* This equation is often called the *dynamic government budget constraint* (or DGBC for short). It is in fact just an accounting identity conditional on $\Delta M = 0$. It says that if the real budget deficit is positive and there is essentially no financing by money creation, then the real public debt grows. We come closer to a *constraint* when combining (DGBC) with the requirement that the government stays *solvent.*

A terminological remark before proceeding: One is tempted to call the right-hand side of (DGBC) the *real* budget deficit. And there is nothing wrong with that as long as one keeps in mind that right-hand side of (DGBC) is *not* the same as the nominal budget deficit deflated by P_t. Indeed,

$$r_t B_t + G_t - T_t = \left(\frac{1+i_t}{1+\pi_t} - 1\right)\frac{D_t}{P_{t-1}} + G_t - T_t = \frac{i_t - \pi_t}{1+\pi_t}\frac{D_t}{P_{t-1}} + G_t - T_t = \frac{GBD_t - \pi_t D_t}{P_t},$$

by definition of the nominal budget deficit GBD_t. The reason that the term $\pi_t D_t$ is subtracted is that *inflation* curtails the increase in *real* debt, given the nominal interest rate

6.3 Government solvency and fiscal sustainability

To be *solvent* means being able to meet the financial commitments as they fall due. In practice this concept is closely related to the government's No-Ponzi-Game condition and intertemporal budget constraint (to which we return in Section 6.5), but at the theoretical level it is more fundamental.

We may view the public sector as an infinitely-lived agent in the sense that there is no last date where all public debt has to be repaid. Nevertheless, as we shall see, there tends to be stringent constraints on government debt creation in the long run.

6.3.1 The critical role of the growth-corrected interest factor

Very much depends on whether the real interest rate in the long-run is higher than the growth rate of GDP or not.

To see this, suppose the country considered has positive government debt at time 0 and that the government levies taxes equal to its non-interest spending:

$$\tilde{T}_t = G_t + X_t \quad \text{or} \quad T_t \equiv \tilde{T}_t - X_t = G_t \text{ for all } t \geq 0. \tag{6.6}$$

So taxes cover only the primary expenses while interest payments (and debt repayments when necessary) are financed by issuing new debt. That is, the government attempts a permanent *roll-over* of the debt including the interest due for payment. In view of (DGBC), this implies that $B_{t+1} = (1 + r_t)B_t$, saying that the debt grows at the rate r_t. Assuming, for simplicity, that $r_t = r$ (a given constant), the law of motion for the public debt-to-income ratio is

$$b_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}} = \frac{1+r}{1+g_Y}\frac{B_t}{Y_t} \equiv \frac{1+r}{1+g_Y}b_t, \qquad b_0 > 0,$$

where we have maintained the assumption of a constant output growth rate, g_Y. The solution to this linear difference equation then becomes

$$b_t = b_0\left(\frac{1+r}{1+g_Y}\right)^t,$$

where we consider both r and g_Y as exogenous. We see that the growth-corrected interest rate, $\frac{1+r}{1+g_Y} - 1 \approx r - g_Y$ (for g_Y and r "small") plays a key role. There are contrasting cases to discuss.

Case 1: $r > g_Y$. In this case, $b_t \to \infty$ for $t \to \infty$. Owing to compound interest, the debt grows so large in the long run that the government will be unable to find buyers for the newly issued debt. Permanent debt roll-over is thus not feasible. Imagine for example an economy described by the Diamond OLG model. Here the buyers of the debt are the young who place part of their saving in government bonds. But if the stock of these bonds grows at a higher rate than income, the saving of the young cannot in the long run keep track with the fast-growing government debt. In this situation the private sector will understand that bankruptcy is threatening and nobody will buy government bonds except at a low price, which means a high interest rate. The high interest rate only aggravates the problem. That is, the fiscal policy (6.6) breaks down. Either the government defaults on the debt or T must be increased or G decreased (or both) until the growth rate of the debt is no longer higher than g_Y.

If the debt is denominated in the country's own currency, an alternative way out is of course a shift to money financing of the budget deficit, that is, seigniorage. When capacity utilization is high, this leads to rising inflation and thus the real value of the debt is eroded. Bond holders will then demand a higher nominal interest rate, thus aggravating the fiscal difficulties. The economic and social chaos of *hyperinflation* threatens.[6] The hyperinflation in Germany 1922-23 peaked in Nov. 1923 at 29,525% per month; it eroded the real value of the huge government debt of Germany after WW I by 95 percent.

Case 2: $r = g_Y$. If $r = g_Y$, we get $b_t = b_0$ for all $t \geq 0$. Since the debt, increasing at the rate r, does not increase faster than national income, the government has no problem finding buyers of its newly issued bonds − the government stays solvent. Thereby the government is able to finance its interest payments simply by issuing new debt. The growing debt is passed on to ever new generations with higher income and saving and the debt roll-over implied by (6.6) can continue forever.

Case 3: $r < g_Y$. Here we get $b_t \to 0$ for $t \to \infty$, and the same conclusion holds *a fortiori*.

In Case 2 as well as Case 3, where the interest rate is not higher than the growth rate of the economy, the government can thus pursue a permanent debt roll-over policy as implied by (6.6) and still remain solvent. But in Case 1, permanent debt roll-over is impossible and sooner or later the interest payments must be tax financed.

Which of the cases is relevant in real life? Fig. 6.1 shows for Denmark (upper panel) and the US (lower panel) the time paths of the real short-term interest rate and the GDP growth rate, both on an annual basis. Overall, the levels of the two are more or less the same, although on average the interest rate is in Denmark slightly higher but in the US somewhat lower than the growth rate. (Note that the interest rates referred to are not the average rate of return in the economy but a proxy for the lower interest rate on government bonds.)

Nevertheless, many macroeconomists believe there is good reason for paying attention to the case $r > g_Y$, also for a country like the US. This is because we live

[6] In economists' standard terminology "hyperinflation" is present when the inflation rate exceeds 50 percent *per month*. As we shall see in Chapter 18, the monetary financing route comes to a dead end if the needed seigniorage reaches the backward-bending part of the "seigniorage Laffer curve".

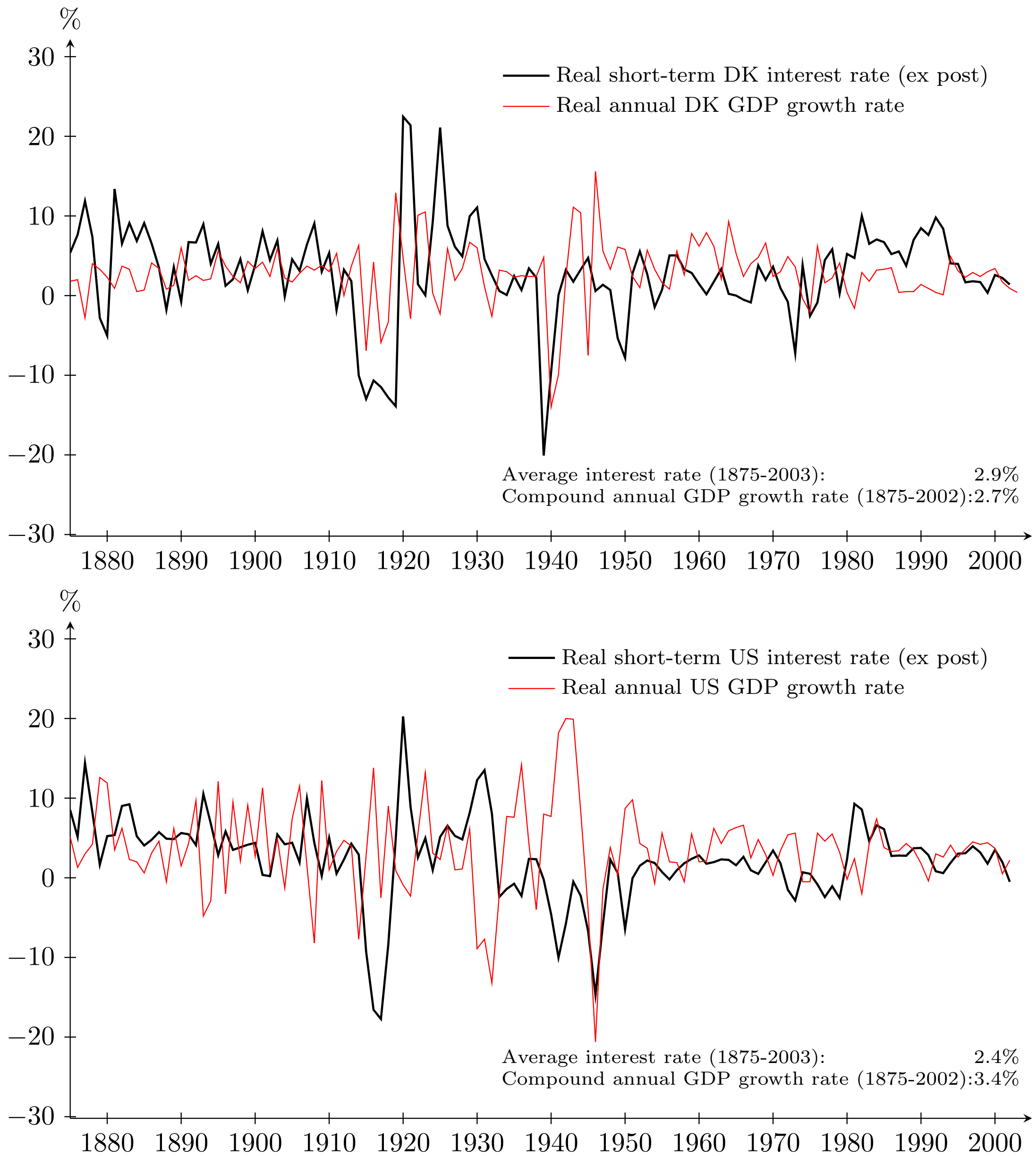


Figure 6.1: Real short-term interest rate and annual growth rate of real GDP in Denmark and the US since 1875. The real short-term interest rate is calculated as the money market rate minus the contemporaneous rate of consumer price inflation. Source: Abildgren (2005) and Maddison (2003).

in a world of *uncertainty,* with many different interest rates, and imperfect credit markets, aspects the above line of reasoning has not incorporated. The prudent debt policy needed whenever, under certainty, $r > g_Y$ can be shown to apply to a larger range of circumstances when uncertainty is present (see Literature notes). To give a flavor we may say that a prudent debt policy is needed when the average interest rate on the public debt exceeds $g_Y - \varepsilon$ for some "small" but positive ε.[7] On the other hand there is a different feature which draws the matter in the opposite direction. This is the possibility that a tax, $\tau \in (0, 1)$, on interest income is in force so that the net interest rate on the government debt is $(1 - \tau)r$ rather than r.

6.3.2 Sustainable fiscal policy

The concept of sustainable fiscal policy is closely related to the concept of government solvency. As already noted, to be *solvent* means being able to meet the financial commitments as they fall due. A given fiscal policy is called *sustainable* if by applying its spending and tax rules forever, the government stays solvent. "Sustainable" conveys the intuitive meaning. The issue is: can the current tax and spending rules continue forever?

To be more specific, suppose G_t and T_t are determined by fiscal policy rules represented by the functions

$$G_t = \mathcal{G}(x_{1t}, ..., x_{nt}, t), \quad \text{and} \quad T_t = \mathcal{T}(x_{1t}, ..., x_{nt}, t),$$

where $t = 0, 1, 2, \ldots,$ and x_{1t},..., x_{nt} are key macroeconomic and demographic variables (like national income, old-age dependency ratio, rate of unemployment, extraction of natural resources, say oil from the North Sea, etc.). In this way a given fiscal policy is characterized by the rules $\mathcal{G}(\cdot)$ and $\mathcal{T}(\cdot)$. Suppose further that we have an economic model, $\mathcal{M}$, of how the economy functions.

DEFINITION Let the current period be period 0 and let the public debt at the beginning of period 0 be given. Then, given a forecast of the evolution of the demographic and foreign economic environment in the future and given the economic model $\mathcal{M}$, the fiscal policy $(\mathcal{G}(\cdot), \mathcal{T}(\cdot))$ is said to be *sustainable* relative to this model if the forecast calculated on the basis of $\mathcal{M}$ is that the government stays solvent under this policy. The fiscal policy $(\mathcal{G}(\cdot), \mathcal{T}(\cdot))$ is called *unsustainable*, if it is not sustainable.

This definition of fiscal sustainability is silent about the presence of uncertainty. Without going into detail about this difficult issue, suppose the model $\mathcal{M}$ is stochastic and let ε be a "small" positive number. Then we may say that

[7] This is only a "rough" characterization, see, e.g., Blanchard and Weil (2001).

the fiscal policy $(\mathcal{G}(\cdot), \mathcal{T}(\cdot))$ with 100-ε percent probability is *sustainable* relative to the model $\mathcal{M}$ if the forecast calculated on the basis of $\mathcal{M}$ is that with 100-ε percent probability the government stays solvent under this policy.

Governments, rating agencies, and other institutions evaluate sustainability of fiscal policy on the basis of simulations of giant macroeconometric models. Essentially, the operational criterion for sustainability is whether the fiscal policy can be deemed compatible with upward boundedness of the public debt-to-income ratio. Normally, the income measure applied here is GDP. Other measures are conceivable such as GNP, taxable income, or after-tax income. Moreover, even if a debt spiral is not (yet) underway in a given country, a high *level* of the debt-income ratio may in itself be worrisome. This is because a high level of debt under certain conditions may trigger a spiral of self-fulfilling expectations of default. We come back to this in the section to follow.

Owing to the increasing pressure on public finances caused by factors such as reduced birth rates, increased life expectancy, and a fast-growing demand for medical care, many industrialized countries have for a long time been assessed to be in a situation where their fiscal policy is not sustainable (Elmendorf and Mankiw 1999). The implication is that sooner or later one or more expenditure rules and/or tax rules (in a broad sense) will probably have to be changed.

Two major kinds of strategies have been suggested. One kind of strategy is the *pre-funding strategy*. The idea is to prevent sharp future tax increases by ensuring a fiscal consolidation prior to the expected future demographic changes. Another strategy (alternative or complementary to the former) is to attempt a gradual increase in the labor force by letting the age limits for retirement and pension increase along with expected lifetime − this is the *indexed retirement strategy*. The first strategy implies that current generations bear a large part of the adjustment cost. In the second strategy the costs are shared by current and future generations in a way more similar to the way the benefits in the form of increasing life expectancy are shared. We shall not go into detail about these matters here, but refer the reader to a large literature about securing fiscal sustainability in the ageing society, see Literature notes.

6.4 Debt arithmetic

A key tool for evaluating fiscal sustainability is *debt arithmetic*, i.e., the analytics of debt dynamics. The previous section described the important role of the growth-corrected interest rate. The next subsection considers the minimum primary budget surplus required for fiscal sustainability in different situations.

6.4.1 The required primary budget surplus

Ignoring the seigniorage term $\Delta M_{t+1}/P_t$ in the dynamic government budget identity (6.4) and assuming a constant interest rate r, we have:

$$B_{t+1} = (1+r)B_t - (T_t - G_t), \qquad \text{(DGBC)}$$

where $T_t - G_t \equiv \tilde{T}_t - X_t - G_t$ is the primary budget surplus in real terms. Suppose aggregate income, Y_t, grows at a given constant rate g_Y. Let the spending-to-income ratio, G_t/Y_t, and the (net) tax revenue-to-income ratio, T_t/Y_t, be constants, γ and τ, respectively. We assume that interest income on government bonds is not taxed. It follows that the public debt-to-income ratio $b_t \equiv B_t/Y_t$ (from now just denoted debt-income ratio) changes over time according to

$$b_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}} = \frac{1+r}{1+g_Y}b_t - \frac{\tau-\gamma}{1+g_Y}, \qquad (6.7)$$

where we have assumed a constant interest rate, r. There are (again) three cases to consider.

Case 1: $r > g_Y$. As emphasized above this case is generally considered the one of most practical relevance. And it is in this case that *latent debt instability* is present and the government has to pay attention to the danger of runaway debt dynamics. To see this, note that the solution of the linear difference equation (6.7) is

$$b_t = (b_0 - b^*)\left(\frac{1+r}{1+g_Y}\right)^t + b^*, \quad \text{where} \qquad (6.8)$$

$$b^* = -\frac{\tau-\gamma}{1+g_Y}\left(1-\frac{1+r}{1+g_Y}\right)^{-1} = \frac{\tau-\gamma}{r-g_Y} \equiv \frac{s}{r-g_Y}, \qquad (6.9)$$

where s is the *primary surplus as a share of GDP*. Here b_0 is historically given. But the steady-state debt-income ratio, b^*, depends on fiscal policy. The important feature is that the growth-corrected interest factor is in this case higher than 1 and has the exponent t. Therefore, if fiscal policy is such that $b^* < b_0$, the debt-income ratio exhibits geometric growth. The solid curve in the topmost panel in Fig. 6.2 shows a case where fiscal policy is such that $\tau - \gamma < (r - g_Y)b_0$ whereby we get $b^* < b_0$ when $r > g_Y$, so that the debt-income ratio, b_t, grows without bound. This reflects that with $r > g_Y$, compound interest is stronger than compound growth. The sequence of discrete points implied by our discrete-time model is in the figure smoothed out as a continuous curve.

The American economist and Nobel Prize laureate George Akerlof (2004, p. 6) came up with this analogy:

> "It takes some time after running off the cliff before you begin to fall. But the law of gravity works, and that fall is a certainty".

Somewhat surprisingly, perhaps, when $r > g_Y$, there can be debt explosion in the long run even if $\tau > \gamma$, namely if $0 < \tau - \gamma < (r - g_Y)b_0$. Debt explosion can also arise if $b_0 < 0$, namely if $\tau - \gamma < (r - g_Y)b_0 < 0$.

The only way to avoid the snowball effects of compound interest when the growth-corrected interest rate is positive is to ensure a primary budget surplus as a share of GDP, $\tau - \gamma$, high enough such that $b^* \geq b_0$. So the *minimum* primary surplus as a share of GDP, $\hat{s}$, required for fiscal sustainability is the one implying $b^* = b_0$, i.e., by (6.9),

$$\hat{s} = (r - g_Y)b_0. \tag{6.10}$$

If by adjusting τ and/or γ, the government obtains $\tau - \gamma = \hat{s}$, then $b^* = b_0$ whereby $b_t = b_0$ for all $t \geq 0$ according to (6.8), cf. the second from the top panel in Fig. 6.2. The difference between $\hat{s}$ and the actual primary surplus as a share of GDP is named the *primary surplus gap* or the *sustainability gap*.

Note that $\hat{s}$ will be larger:

- the higher is the initial level of debt, b_0; and,
- when $b_0 > 0$, the higher is the growth-corrected interest rate, $r - g_Y$.

Delaying the adjustment increases the size of the needed policy action, since the debt-income ratio, and thereby $\hat{s}$, will become higher in the meantime.

For fixed spending-income ratio γ, the minimum tax-to-income ratio needed for fiscal sustainability is

$$\hat{\tau} = \gamma + (r - g_Y)b_0. \tag{6.11}$$

Given b_0 and γ, this tax-to-income ratio is sometimes called the *sustainable tax rate*. The difference between this rate and the actual tax rate, τ, indicates the size of the needed tax adjustment, were it to take place at time 0, assuming a given γ.

Suppose that the debt build-up can be − and is − prevented already at time 0 by ensuring that the primary surplus as a share of income, $\tau - \gamma$, at least equals $\hat{s}$ so that $b^* \geq b_0$. The solid curve in the midmost panel in Fig. 6.2 illustrates the resulting evolution of the debt-income ratio if b^* is at the level corresponding to the hatched horizontal line while b_0 is unchanged compared with the top panel. Presumably, the government would in such a state of affairs relax its fiscal policy after a while in order not to accumulate large government financial net wealth. Yet, the pre-funding strategy vis-a-vis the fiscal challenge of population ageing (referred to above) is in fact based on accumulating some positive public financial net wealth as a buffer before the substantial effects of population ageing set in. In

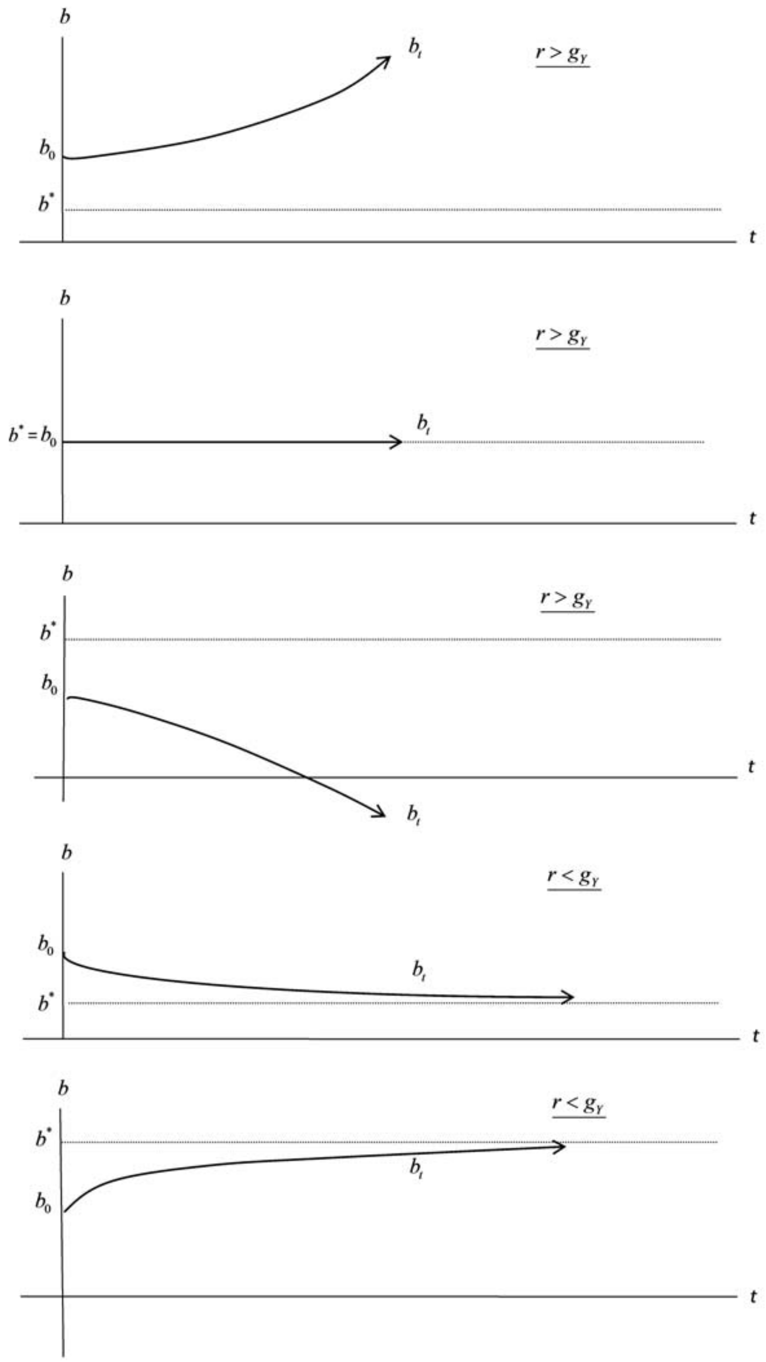


Figure 6.2: Evolution of the debt-income ratio, depending on the sign of $b_0 - b^*$, in the cases $r > g_Y$ (the three upper panels) and $r < g_Y$ (the two lower panels), respectively.

this context, the higher the growth-corrected interest rate, the shorter the time needed to reach a given positive net wealth position.

Case 2: $r = g_Y$. In this knife-edge case there is still a danger of runaway dynamics, but of a less explosive form. The formula (6.8) is no longer valid. Instead the solution of (6.7) is $b_t = b_0 + [(\gamma - \tau)/(1 + g_Y)]\, t = b_0 - [(\tau - \gamma)/(1 + g_Y)]\, t$. Here, a non-negative primary surplus is both necessary and sufficient to avoid $b_t \to \infty$ for $t \to \infty$.

Case 3: $r < g_Y$. This is the case of stable debt dynamics. The formula (6.8) is again valid, but now implying that the debt-income ratio is non-explosive. Indeed, $b_t \to b^*$ for $t \to \infty$, whatever the level of the initial debt-income ratio and whatever the sign of the budget surplus. Moreover, when $r < g_Y$,

$$b^* = \frac{\tau - \gamma}{r - g_Y} \lesseqqgtr 0 \text{ for } \tau - \gamma \gtreqqless 0. \tag{*}$$

So, if there is a forever positive primary surplus, the result is a negative long-run debt, i.e., a positive government financial net wealth in the long run. And if there is a forever negative primary surplus, the result is not debt explosion but just convergence toward some positive long-run debt-income ratio. The second from bottom panel in Fig. 6.2 illustrates this case for a situation where $b_0 > b^*$ and $b^* > 0$, i.e., $\tau - \gamma < 0$, by (*). When the GDP growth rate continues to exceed the interest rate on government debt, a large debt-income ratio can be brought down quite fast, as witnessed by the evolution of both UK and US government debt in the first three decades after the second world war. Indeed, if the growth-corrected interest rate remains negative, permanent debt roll-over can handle the financing, and taxes need never be levied.[8]

Finally, the bottom panel in Fig. 6.2 shows the case where, with a *large* primary deficit ($\tau - \gamma < 0$ but large in absolute value), excess of output growth over the interest rate still implies convergence towards a constant debt-income ratio, albeit a high one.

In this discussion we have treated r as exogenous. But r may to some extent be dependent on prolonged budget deficits. Indeed, in Chapter 13 we shall see that with prolonged budget deficits, r tends to become higher than otherwise. Everything else equal, this reduces the likelihood of Case 2 and Case 3.

Laffer curve*

We return to Case 1 because we have ignored supply-side effects of taxation, and such effects could be important in Case 1.

[8] On the other hand, we should not forget that this analysis presupposes absence of uncertainty. As touched on in Section 6.3.1, in the presence of uncertainty and therefore existence of many interest rates, the issue becomes more complicated.

A *Laffer curve* (so named after the American economist Arthur Laffer, 1940-) refers to a hump-shaped relationship between the income tax rate and the tax revenue. For simplicity, suppose the (gross) tax revenue equals taxable income times a given average tax rate. A 0% tax rate and likely also a 100% tax rate generate no tax revenue. As the tax rate increases from a low initial level, a rising tax revenue is obtained. But after a certain point some people may begin to work less (in the legal economy), stop reporting all their income, and stop investing. So it is reasonable to think of a tax rate above which the tax revenue begins to decline.

While Laffer was wrong about where USA was "on the curve" (see, e.g., Fullerton 2008), and while, strictly speaking, there is no such thing as *the* Laffer curve and *the* tax rate,[9] Laffer's intuition is hardly controversial. Ignoring, for simplicity, transfers, we therefore now assume that for a *given* tax *system* there is a gross tax-income ratio, τ_L, above which the tax revenue declines. Then, if the presumed sustainable tax-income ratio, $\hat{\tau}$, in (6.11) exceeds τ_L, the tax revenue aimed at can not be realized.

To see what the value of τ_L could be, suppose aggregate taxable income before tax is a function, φ, of the net-of-tax share $1-\tau$. Then tax revenue is

$$\tilde{T} = \tau \cdot \varphi(1-\tau) \equiv R(\tau) \ ,$$

which we assume is a hump-shaped function of τ in the interval $[0,1]$. Taking logs and differentiating w.r.t. τ gives the first-order condition $R'(\tau)/R(\tau) = 1/\tau - \varphi'(1-\tau)/\varphi(1-\tau) = 0$, which holds for $\tau = \tau_L$, the tax-income ratio that maximizes R. It follows that $1/\tau_L = \varphi'(1-\tau_L)/\varphi(1-\tau_L)$, hence

$$\frac{1-\tau_L}{\tau_L} = \frac{1-\tau_L}{\varphi(1-\tau_L)}\varphi'(1-\tau_L) \equiv \mathrm{E}\ell_{1-\tau}\varphi(1-\tau_L).$$

Rearranging gives

$$\tau_L = \frac{1}{1+\mathrm{E}\ell_{1-\tau}\varphi(1-\tau_L)}.$$

If the elasticity of income w.r.t. $1-\tau$ is given as 0.4,[10] we get $\tau_L \simeq 0.7$. Thus, if the required tax-income ratio, $\hat{\tau}$, calculated on the basis of (6.11) (under the simplifying assumption of no transfers), exceeds 0.7, fiscal sustainability can not be obtained by just raising taxation.

[9] A lot of contingencies are involved: income taxes are typically progressive (i.e., average tax rates rise with income); it matters whether a part of tax revenue is spent to reduce tax evasion, etc.

[10] As suggested for the U.S. by Gruber and Saez (2002).

The level of the debt-income ratio and self-fulfilling expectations of default

We again consider Case 1: $r > g_Y$. As incumbent chief economist at the IMF, Olivier Blanchard remarked in the midst of the 2010-2012 debt crisis in the Eurozone:

> "The higher the level of debt, the smaller is the distance between solvency and default".[11]

The background for this remark is the following. There is likely to be an upper bound for the tax-income ratio deemed politically or economically feasible by the government as well as the market participants. Similarly, a lower bound for the spending-income ratio is likely to exist, be it for economic or political reasons. In the present framework we therefore let the government face the constraints $\tau \leq \bar{\tau}$ and $\gamma \geq \bar{\gamma}$, where $\bar{\tau}$ is the least upper bound for the tax-income ratio and $\bar{\gamma}$ is the greatest lower bound for the spending-income ratio. We assume that $\bar{\tau} > \bar{\gamma}$. Then the actual primary surplus, s, can at most equal $\bar{s} \equiv \bar{\tau} - \bar{\gamma}$.

Suppose that at first the situation in the considered country is as in the second from the top panel in Fig. 6.2. That is, initially, $b_0 > 0$ and

$$s = \tau - \gamma = \hat{s} = (r - g_Y)b_0 \leq \bar{s} \equiv \bar{\tau} - \bar{\gamma}, \tag{6.12}$$

with $b_0 > 0$. Define $\bar{r}$ to be the value of r satisfying

$$(\bar{r} - g_Y)b_0 = \bar{s}, \text{ i.e., } \bar{r} = \frac{\bar{s}}{b_0} + g_Y. \tag{6.13}$$

Thereby $\bar{r}$ is the maximum level of the interest rate consistent with absence of an explosive debt-income ratio.

According to (6.12), fundamentals (tax- and spending-income ratios, growth-corrected interest rate, and initial debt) are consistent with absence of an explosive debt-income ratio as long as r is unchanged. Nevertheless, financial investors may be worried about default if b_0 is high. Investors are aware that a rise in the actual interest rate, r, can always happen and that if it does, a situation with $r > \bar{r}$ is looming, in particular if the country has high debt. The larger is b_0, the lower is the critical interest rate, $\bar{r}$, as witnessed by (6.13).

The worrying scenario is that the fear of default triggers a risk premium, and if the resulting level of the interest rate on the debt, say r', exceeds $\bar{r}$, unpleasant debt dynamics like that in the top panel of Fig. 6.2 set in. To r' corresponds a new value of the primary surplus, say $\hat{s}'$, defined by $\hat{s}' = (r' - g_Y)b_0$. So $\hat{s}'$ is the

[11] Blanchard (2011).

minimum primary surplus (as a share of GDP) required for a non-accelerating debt-income ratio in the new situation. With $b_0 > 0$ and $r' > \bar{r}$, we get

$$\hat{s}' = (r' - g_Y)b_0 > (\bar{r} - g_Y)b_0 = \bar{s},$$

where $\bar{s}$ is given in (6.12). The government could possibly increase its primary surplus, s, but at most up to $\bar{s}$, and this will not be enough since the required primary surplus, $\hat{s}'$, exceeds $\bar{s}$. The situation would be as illustrated in the top panel of Fig. 6.2 with b^* given as $\bar{s}/(r' - g_Y) < b_0$.

That is, *if* the actual interest rate should rise above the critical interest rate, $\bar{r}$, runaway debt dynamics would take off and debt default follow. A fear that it *may* happen may be enough to trigger a fall in the market price of government bonds which means a rise in the actual interest rate, r. So financial investors' fear can be a self-fulfilling prophesy. Moreover, as we saw in connection with (6.13), the risk that r becomes greater than $\bar{r}$ is larger the larger is b_0.

It is not so that across countries there is a common threshold value for a "too large" public debt-to-income ratio. This is because variables like $\bar{\tau}$, $\bar{\gamma}$, r, and g_Y, as well as the net foreign debt position and the current account deficit (not in focus in this chapter), *differ* across countries. Late 2010 Greece had (gross) government debt of 148 percent of GDP and the interest rate on 10-year government bonds skyrocketed. Conversely Japan had (gross) government debt of more than 200 percent of GDP while the interest rate on 10-year government bonds remained very low.

Finer shades

1. As we have just seen, even when in a longer-run perspective a solvency problem is unlikely, self-fulfilling expectations can here and now lead to default. Such a situation is known as a *liquidity crisis* rather than a true *solvency crisis.* In a liquidity crisis there is an acute problem of insufficient cash to pay the next bill on time ("cash-flow insolvency") because borrowing is difficult due to actual and potential creditors' *fear* of default. A liquidity crisis can be braked by the central bank stepping in and acting as a "lender of last resort" by printing money. In a country with its own currency, the central bank can do so and thereby prevent a bad self-fulfilling expectations equilibrium to unfold.[12]

[12] In a monetary union which is not also a fiscal union (think of the eurozone), the situation is more complicated. A single member country with large government debt (or large debt in commercial banks for that matter) may find itself in an acute liquidity crisis without its own means to solve it. Indeed, the elevation of interest rates on government bonds in the Southern part of the eurozone in 2010-2012 can be seen as a manifestation of investors' fear of the governments running into difficulties of paying their way. The elevation was not reversed until the European Central Bank in September 2012 declared its willingness to effectively act as a "lender of last resort" (on a conditional basis), see Box 6.2 in Section 6.4.2.

2. In the above analysis we simplified by assuming that several variables, including γ, τ, and r, are constants. The upward trend in the old-age dependency ratio, due to a decreased birth rate and rising life expectancy, together with a rising request for medical care is likely to generate upward pressure on γ. Thereby a high initial debt-income ratio becomes *more* challenging.

3. On the other hand, rB_t is income to the private sector and can be taxed at the same average tax rate τ as factor income, Y_t. Then the benign inequality is no longer $r \leq g_Y$ but $(1-\tau)r \leq g_Y$, which is more likely to hold. Taxing interest income is thus supportive of fiscal sustainability (cf. Exercise B.28).

4. Having ignored seigniorage, there is an upward bias in our measure (6.10) of the *minimum* primary surplus as a share of GDP, $\hat{s}$, required for fiscal sustainability when $r > g_Y$. Imposing stationarity of the debt-income ratio at the level $\bar{b}$ into the general debt-accumulation formula (6.5), multiplying through by $1 + g_Y$, and cancelling out, we find

$$\hat{s} = (r - g_Y)\bar{b} - \frac{\Delta M_{t+1}}{P_t Y_t} = (r - g_Y)\bar{b} - \frac{\Delta M_{t+1}}{M_t} \cdot \frac{M_t}{P_t Y_t}.$$

With $r = 0.04$, $g_Y = 0.03$, and $\bar{b} = 0.60$, we get $(r - g_Y)\bar{b} = 0.006$. With a seigniorage-income ratio even as small as 0.003, the "true" required primary surplus is 0.003 rather than 0.006. As long as the seigniorage-income ratio is approximately constant, our original formula, given in (6.10), for the required primary surplus as a share of GDP is in fact valid if we interpret τ as the (tax+seigniorage)-income ratio.

5. Having assumed a constant g_Y, we have ignored business cycle fluctuations. Allowing for booms and recessions, the *timing* of fiscal consolidation in a country with a structural primary surplus gap ($\hat{s} - s > 0$) becomes a crucial issue. The case study in the next section will be an opportunity to touch upon this issue.

6.4.2 Case study: The Stability and Growth Pact of the EMU

The European Union (EU) is approaching its aim of establishing a "single market" (unrestricted movement of goods and services, workers, and financial capital) across the territory of its member countries, 28 sovereign nations. Nineteen of these have joined the common currency, the euro. They constitute what is known as the Eurozone with the European Central Bank (ECB) as supranational institution responsible for conducting monetary policy in the Eurozone. The Eurozone countries as well as the nine EU countries outside the Eurozone (including UK, Denmark, Sweden, and Poland) are, with minor exceptions, required to abide with a set of *fiscal rules*, first formulated already in the Treaty of Maastrict from

1992. In that year a group of European countries decided a road map leading to the establishment of the euro in 1999 and a set of criteria for countries to join. These fiscal rules included a deficit rule as well as a debt rule. The *deficit rule* says that the annual nominal government budget deficit must not be above 3 percent of nominal GDP. The *debt rule* says that the government debt should not be above 60 percent of GDP. The fiscal rules were upheld and in minor respects tightened in the *Stability and Growth Pact* (SGP) which was implemented in 1997 as the key fiscal constituent of the Economic and Monetary Union (EMU). The latter name is a popular umbrella term for the fiscal and monetary legislation of the EU. The EU member countries that have adopted the euro are often referred to as "the full members of the EMU".

Some of the EU member states (Belgium, Italy, and Greece) had debt-income ratios above 100 percent since the early 1990s − and still have. Committing to the requirement of a gradual reduction of their debt-income ratios, they became full members of the EMU essentially from the beginning (that is, 1999 except Greece, 2001). The 60 percent debt rule of the SGP is to be understood as a long-run ceiling that, by the stock nature of debt, can not be accomplished here and now if the country is highly indebted.

The deficit and debt rules (with associated detailed contingencies and arrangements including ultimate pecuniary fines for defiance) are meant as discipline devices aiming at "sound budgetary policy", alternatively called "fiscal prudence". The motivation is protection of the ECB against political demands to loosen monetary policy in situations of fiscal distress. A fiscal crisis in one or more of the Eurozone countries, perhaps "too big to fail", could set in and entail a state of affairs approaching default on government debt and chaos in the banking sector with rising interest rates spreading to neighboring member countries (a negative externality). This could lead to open or concealed political pressure on the ECB to inflate away the real value of the debt, thus challenging the ECB's one and only concern with "price stability".[13] Or a fiscal crisis might at least result in demands on the ECB to curb soaring interest rates by purchasing government bonds from the country in trouble. In fact, such a scenario is close to what we have seen in southern Europe in the wake of the Great Recession triggered by the financial crisis starting 2007. Such "bailing out" could give governments incentives to be relaxed about deficits and debts (a "moral hazard" problem). And the lid on deficit spending imposed by the SGP should help to prevent needs for "bailing out" to arise.

[13] In recent years the ECB has interpreted "price stability" as a consumer price inflation rate "below, but close to, 2 percent per year over the medium term".

The link between the deficit and the debt rule

Whatever the virtues or vices of the design of the deficit and debt rules, one may ask the plain question: what is the arithmetical relationship, if any, between the 3 percent and 60 percent tenets?

First a remark about measurement. The measure of government debt, called the EMU debt, used in the SGP criterion is based on the book value of the financial liabilities rather than the market value. In addition, the EMU debt is more of a *gross* nature than the theoretical net debt measure represented by our D. The EMU debt measure allows fewer of the government financial assets to be subtracted from the government financial liabilities.[14] In our calculation and subsequent discussion we ignore these complications.

Consider a deficit rule saying that the (total) nominal budget deficit must never be above $\alpha \cdot 100$ percent of nominal GDP. By (6.3) with ΔM_{t+1} "small" enough to be ignored, this deficit rule is equivalent to the requirement

$$D_{t+1} - D_t = GBD_t = i_t D_t + P_t(G_t - T_t) \leq \alpha P_t Y_t. \qquad (6.14)$$

In the SGP, $\alpha = 0.03$. Here we consider the general case: $\alpha > 0$. To see the implication for the (public) debt-to-income ratio in the long run, let us first imagine a situation where the deficit ceiling, α, is always *binding* for the economy we look at. Then $D_{t+1} = D_t + \alpha P_t Y_t$ and so

$$b_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}} \equiv \frac{D_{t+1}}{P_t Y_{t+1}} = \frac{D_t}{(1+\pi)P_{t-1}(1+g_Y)Y_t} + \frac{\alpha}{1+g_Y},$$

assuming constant output growth rate, g_Y, and inflation rate π. This reduces to

$$b_{t+1} = \frac{1}{(1+\pi)(1+g_Y)} b_t + \frac{\alpha}{1+g_Y}. \qquad (6.15)$$

Assuming that $(1+\pi)(1+g_Y) > 1$ (as is normal over the medium run), this linear difference equation has the stable solution

$$b_t = (b_0 - b^*) \left(\frac{1}{(1+\pi)(1+g_Y)} \right)^t + b^* \to b^* \text{ for } t \to \infty, \qquad (6.16)$$

where

$$b^* = \frac{(1+\pi)}{(1+\pi)(1+g_Y) - 1} \alpha. \qquad (6.17)$$

[14] For instance for Denmark the difference between the EMU and the net debt is substantial. In 2013 the Danish EMU debt was 44.6% of GDP while the government net debt was 5.5% of GDP (Danish Ministry of Finance, 2014).

Consequently, if the deficit rule (6.14) is always binding, the debt-income ratio tends in the long run to be proportional to the deficit bound α. The factor of proportionality is a decreasing function of the long-run growth rate of real GDP and the inflation rate. This result confirms the general tenet that if there is economic growth, perpetual budget deficits need not lead to fiscal problems.

If on the other hand the deficit rule is *not* always binding, then the budget deficit is on average smaller than above so that the debt-income ratio will in the long run be *smaller* than b^*.

The conclusion is the following. With one year as the time unit, suppose the deficit rule has $\alpha = 0.03$ and that $g_Y = 0.03$ (which by the architects of the Maastrict Treaty was considered the "natural" GDP growth rate) and $\pi = 0.02$ (which is the upper end of the inflation interval aimed at by the ECB). Suppose further the deficit rule is never violated. Then in the long run the debt-income ratio will be *at most* $b^* = 1.02 \times 0.03/(1.02 \times 1.03 - 1) \approx 0.60$. This is in agreement with the debt rule of the SGP according to which the maximum value allowed for the debt-income ratio is 60%.

Although there is nothing sacred about either of the numbers 0.60 or 0.03, they are mutually consistent, given $\pi = 0.02$ and $g_Y = 0.03$.

We observe that the deficit rule (6.14) implies that:

- The upper bound, b^*, on the long-run debt income ratio is lower the higher is inflation. The reason is that the growth factor $\beta \equiv [(1+\pi)\ (1+g_Y)]^{-1}$ for b_t in (6.15) depends negatively on the inflation rate, π. So does therefore b^* since, by (6.16), $b^* \equiv \alpha(1+g_Y)^{-1}(1-\beta)^{-1}$.

- For a *given* π, the upper bound on the long-run debt income ratio is *independent* of both the nominal and real interest rate (this follows from the indicated formula for the growth factor for b_t and the fact that $(1+i)(1+r)^{-1} = 1+\pi$).

The debate about the design of the SGP

In addition to the aimed long-run implications, by its design the SGP has short-run implications for the economy. Hence an evaluation of the SGP cannot ignore the way the economy functions in the short run. How changes in government spending and taxation affects the economy depends on the "state of the business cycle": is the economy in a boom with full capacity utilization or in a slump with slack aggregate demand?

Much of the debate about the SGP has centered around the consequences of the deficit rule in an economic recession triggered by a collapse of aggregate demand (for instance due to private deleveraging in the wake of a banking crisis).

Although the Eurozone countries are economically quite different, they are subject to the same one-size-fits-all monetary policy. Facing dissimilar shocks, the single member countries in need of aggregate demand stimulation in a recession have by joining the euro renounced on both interest rate policy and currency depreciation.[15] The only policy tool left for demand stimulation is therefore fiscal policy. Instead of a supranational fiscal authority responsible for handling the problem, it is up to the individual member countries to act − and to do so within the constraints of the SGP.

On this background, the critiques of the deficit rule of the SGP include the following points. (It may here be useful to have at the back of one's mind the simple Keynesian income-expenditure model, where output is below capacity and demand-determined whereas the general price level is sticky.)

Critiques 1. When considering the need for fiscal stimuli in a recession, a ceiling at 0.03 is too low unless the country has almost no government debt in advance. Such a deficit rule gives too little scope for counter-cyclical fiscal policy, including the free working of the *automatic fiscal stabilizers* (i.e., the provisions, through tax and transfer codes, in the government budget that automatically cause tax revenues to fall and spending to rise when GDP falls).[16] As an economy moves towards recession, the deficit rule may, bizarrely, force the government to tighten fiscal policy although the situation calls for stimulation of aggregate demand. The pact has therefore sometimes been called the "Instability and Depression Pact" − it imposes a *wrong timing* of fiscal consolidation.[17]

2. Since what really matters is long-run fiscal sustainability, a deficit rule should be designed in a more flexible way than the 3% rule of the SGP. A meaningful deficit rule would relate the deficit to the *trend* nominal GDP, which we may denote $(PY)^*$. Such a criterion would imply

$$GBD \leq \alpha(PY)^*. \tag{6.18}$$

[15]Denmark is in a similar situation. In spite of not joining the euro after the referendum in 2000, the Danish krone has been linked to the euro through a fixed exchange rate since 1999.

[16]Over the first 13 years of existence of the euro even Germany violated the 3 percent rule five of the years.

[17]The SGP has an exemption clause referring to "exceptional" circumstances. These circumstances were originally defined as "severe economic recession", interpreted as an annual fall in real GDP of at least 1-2%. By the reform of the SGP in March 2005, the interpretation was changed into simply "negative growth". Owing to the international economic crisis that broke out in 2008, the deficit rule was thus suspended in 2009 and 2010 for most of the EMU countries. But the European Commission brought the rule into effect again from 2011, which according to many critics was much too early, given the circumstances.

Then

$$\frac{GBD}{PY} \leq \alpha \frac{(PY)^*}{PY}.$$

In recessions the ratio $(PY)^*/(PY)$ is high, in booms it is low. This has the advantage of allowing more room for budget deficits when they are needed − without interfering with the long-run aim of stabilizing government debt below some specified ceiling.

3. A further step in this direction is a rule directly in terms of the *structural* or *cyclically adjusted* budget deficit rather than the actual year-by-year deficit. The cyclically adjusted budget deficit in a given year is defined as the value the deficit would take in case actual output were equal to trend output in that year. Denoting the cyclically adjusted budget deficit GBD^*, the rule would be

$$\frac{GBD^*}{(PY)^*} \leq \alpha.$$

In fact, in its original version as of 1997 the SGP contained an *additional* rule like that, but in the very strict form of $\alpha \approx 0$. This requirement was implicit in the directive that the cyclically adjusted budget "should be close to balance or in surplus". By this requirement it is imposed that the debt-income ratio should be close to zero in the long run. Many EMU countries certainly had − and have − larger cyclically adjusted deficits. Taking steps to comply with such a low structural deficit ceiling may be hard and endanger national welfare by getting in the way of key tasks of the public sector. The minor reform of the SGP endorsed in March 2005 allowed more contingencies, also concerning this structural bound. By the more recent reform in 2012, the Fiscal Pact, the lid on the cyclically adjusted deficit-income ratio was raised to 0.5% and to 1.0% for members with a debt-income ratio "significantly below 60%". These are still quite small numbers. Abiding by the 0.5% or 1.0% rule implies a long-run debt-income ratio of at most 10% or 20%, respectively, given structural inflation and structural GDP growth at 2% and 3% per year, respectively.[18]

4. Regarding the *composition* of government expenditure, critics have argued that the SGP pact entails a problematic disincentive for public investment. The view is that a fiscal rule should be based on a proper accounting of public investment instead of simply ignoring the composition of government expenditure. We consider this issue in Section 6.6 below.

5. At a more general level critics have contended that policy rules and surveillance procedures imposed on sovereign nations will hardly be able to do their job unless they encompass stronger incentive-compatible elements. Enforcement mechanisms are bound to be week. The SGP's threat of pecuniary fines to a

[18] Again apply (6.17).

country which during a recession has difficulties to reduce its budget deficit lacks credibility and has, at the time of writing (June 2015), not been made use of so far. Moreover, abiding by the fiscal rules of the SGP prior to the Great Recession was certainly no guarantee of not ending up in a fiscal crisis in the wake of a crisis in the banking sector, as witnessed by Ireland and Spain. A seemingly strong fiscal position can vaporize fast, particularly if banks, "too big to fail", need be bailed out.

Counter-arguments Among the counter-arguments raised against the criticisms of the SGP has been that the potential benefits of the proposed alternative rules are more than offset by the costs in terms of reduced simplicity, measurability, and transparency. The lack of flexibility may even be a good thing because it helps "tying the hands of elected policy makers". Tight rules are needed because of a *deficit bias* arising from short-sighted policy makers' temptation to promise spending without ensuring the needed financing, especially before an upcoming election. These points are sometimes linked to the view that market economies are generally self-regulating: Keynesian stabilization policy is not needed and may do more harm than good.

Box 6.1. The 2010-2012 debt crisis in the Eurozone

What began as a banking crisis became a deep economic recession combined with a government debt crisis.

At the end of 2009, in the aftermath of the global economic downturn, it became evident that Greece faced an acute debt crisis driven by three factors: high government debt, low ability to collect taxes, and lack of competitiveness due to cost inflation. Anxiety broke out about the debt crisis spilling over to Spain, Portugal, Italy, and Ireland, thus widening bond yield spreads in these countries vis-a-vis Germany in the midst of a serious economic recession. Moreover, the solvency of big German and French banks that were among the prime creditors of Greece was endangered. The major Eurozone governments and the International Monetary Fund (IMF) reached an agreement to help Greece (and thereby its creditors) with loans and guarantees for loans, conditional on the government of Greece imposing yet another round of harsh fiscal austerity measures. The elevated bond interest rates of Greece, Italy, and Spain were not convincingly curbed, however, until in August-September 2012 the president of the ECB, Mario Draghi, launched the "Outright Monetary Transactions" (OMT) program according to which, under certain conditions, the ECB will buy government bonds in the secondary market with the aim of "safeguarding an appropriate monetary policy transmission and the singleness of the monetary policy" and with "no ex ante quantitative limits". Considerably reduced government bond spreads followed and so the sheer announcement of the program seemed effective in its own right. Doubts raised by the German Constitutional Court about its legality vis-à-vis Treaties of the European Union were finally repudiated by the European Court of Justice mid-June 2015. At the time of writing (late June 2015) the OMT program has not been used in practice. Early 2015, a different massive program for purchases of government bonds, including long-term bonds, in the secondary market as well as private asset-backed bonds was decided and implemented by the ECB. The declared aim was to brake threatening deflation and return to "price stability", by which is meant inflation close to 2 percent per year.

So much about the monetary policy response. What about fiscal policy? On the basis of the SGP, the EU Commission imposed "fiscal consolidation" initiatives to be carried out in most EU countries in the period 2011-2013 (some of the countries were required to start already in 2010). With what consequences? By many observers, partly including the research department of the IMF, the initiatives were judged self-defeating. When at the same time comprehensive deleveraging in the private sector is going on, "austerity" policy deteriorates aggregate demand further and raises unemployment. Thereby, instead of budget deficits being decreased, it is the denominator of the debt-income ratio, $D/(PY)$, that is decreased. Fiscal multipliers are judged to be large ("in the 0.9 to 1.7 range since the Great Recession", according to IMF's *World Economic Outlook*, Oct. 2012) in a situation of idle resources where monetary policy aims at low interest rates; and negative spillover effects through trade linkages when "fiscal consolidation" is synchronized across countries. The unemployment rate in the Eurozone countries was elevated from 7.5 percent in 2008 to 12 percent in 2013. The British economists, Holland and Portes (2012), concluded: "It is ironic that, given that the EU was set up in part to avoid coordination failures in economic policy, it should deliver the exact opposite".

The whole crisis has pointed to a basic difficulty faced by the Eurozone. In spite of the member countries being economically very different sovereign nations, they are subordinate to the same one-size-fits-all monetary policy without sharing a federal government ready to use fiscal instruments to mitigate regional consequences of country-specific shocks. Adverse demand shocks may lead to sharply rising budget deficits in some countries, and financial investors may loose confidence and so elevate government bond interest rates. A liquidity crisis may arise, thereby amplifying adverse shocks. Even when a common negative demand shock hits all the member countries in a similar way, and a general relaxation of both monetary and fiscal policy is called for, there is the problem that the individual countries, in fear of boosting their budget deficit and facing the risk of exceeding the deficit or debt limit, may wait for the others to initiate fiscal expansion. The possible consequence of this "free rider" problem is general under-stimulation of the economies.

The dismal experience regarding the ability of the Eurozone to handle the Great Recession has incited proposals along at least two dimensions. One dimension is about allowing the ECB greater scope for acting as a "lender of last resort". The other dimension is about centralizing a larger part of the national budgets into a common union budget (see, e.g., De Grauwe, 2014). (END OF BOX)

6.5 Solvency, the NPG condition, and the intertemporal government budget constraint

Up to now we have considered the issue of government solvency from the perspective of dynamics of the government debt-to-income ratio. It is sometimes useful to view government solvency from another angle – the intertemporal budget constraint (GIBC). Under a certain condition stated below, the intertemporal budget constraint is, essentially, as relevant for a government as for private agents.

A simple condition closely linked to whether the government's intertemporal budget constraint is satisfied or not is what is known as the government's No-Ponzi-Game (NPG) condition. It is convenient to first focus on this condition. We concentrate on government *net* debt, measured in *real* terms, and ignore seigniorage.

6.5.1 When is the NPG condition necessary for solvency?

Consider a situation with a given constant interest rate, r. Suppose taxes are lump sum or at least that there is no tax on interest income from owning government bonds. Then the government's *NPG condition* is that the present discounted value of the public debt in the far future is not positive, i.e.,

$$\lim_{t\to\infty} B_t(1+r)^{-t} \leq 0. \tag{NPG}$$

This condition says that government debt is not allowed to grow in the long run at a rate as high as (or even higher than) the interest rate.[19] That is, a fiscal policy satisfying the NPG condition rules out a permanent debt rollover. Indeed, as we saw in Section 6.3.1, with $B_0 > 0$, a permanent debt rollover policy (financing all interest payments and perhaps even also part of the primary government spending) by debt issue leads to $B_t \geq B_0(1+r)^t$ for $t = 0, 1, 2, \ldots$. Substituting into (NPG) gives $\lim_{t\to\infty} B_t \geq B_0(1+r)^t(1+r)^{-t} = B_0 > 0$, thus violating (NPG).

The designation No-Ponzi-Game condition refers to a guy from Boston, Charles Ponzi, who in the 1920s made a fortune out of an investment scam based on the

[19] If there is effective taxation of interest income at the rate $\tau_r \in (0,1)$, then the after-tax interest rate, $(1-\tau_r)r$, is the relevant discount rate, and the NPG condition would read $\lim_{t\to\infty} B_t\left[1+(1-\tau_r)r\right]^{-t} \leq 0$.

chain-letter principle. The principle is to pay off old investors with money from new investors, keeping the remainder of that money to oneself. Ponzi was sentenced to many years in prison for his transactions; he died poor − and without friends!

To our knowledge, this kind of financing behavior is nowhere forbidden for the government as it generally is for private agents. But under "normal" circumstances a government *has* to plan its expenditures and taxation so as to comply with its NPG condition since otherwise not enough lenders will be forthcoming.

As the state is in principle infinitely-lived, however, there is no final date where all government debt should be over and done with. Indeed, the NPG condition does not even require that the debt has ultimately to be non-increasing. The NPG condition "only" says that the debtor, here the government, can not let the debt grow forever at a rate as high as (or higher than) the interest rate. For instance the U.K. as well as the U.S. governments have had positive debt for centuries − and high debt after both WW I and WW II.

Suppose Y (GDP) grows at the given constant rate g_Y (actually, for most of the following results it is enough that $\lim_{t\rightarrow\infty} Y_{t+1}/Y_t = 1 + g_Y$). We have:

PROPOSITION 1 Interpret "solvency" as absence of an for ever accelerating debt-income ratio, $b_t \equiv B_t/Y_t$. Then:

(i) if $r > g_Y$, solvency requires (NPG) satisfied;

(ii) if $r \leq g_Y$, the government can remain solvent without (NPG) being satisfied.

Proof. When $b_t \neq 0$,

$$\lim_{t\rightarrow\infty} \frac{b_{t+1}}{b_t} \equiv \lim_{t\rightarrow\infty} \frac{B_{t+1}/Y_{t+1}}{B_t/Y_t} = \lim_{t\rightarrow\infty} \frac{B_{t+1}/B_t}{Y_{t+1}/Y_t} = \lim_{t\rightarrow\infty} \frac{B_{t+1}/B_t}{1+g_Y}. \qquad (6.19)$$

Case (i): $r > g_Y$. If $\lim_{t\rightarrow\infty} B_t \leq 0$, then (NPG) is trivially satisfied. Assume $\lim_{t\rightarrow\infty} B_t > 0$. For this situation we prove the statement by contradiction. Suppose (NPG) is not satisfied. Then, $\lim_{t\rightarrow\infty} B_t(1+r)^{-t} > 0$, implying that $\lim_{t\rightarrow\infty} B_{t+1}/B_t \geq 1 + r$. In view of (6.19) this implies that $\lim_{t\rightarrow\infty} b_{t+1}/b_t \geq (1+r)/(1+g_Y) > 1$. Thus, $b_t \rightarrow \infty$, which violates solvency. By contradiction, this proves that solvency implies (NPG) when $r > g_Y$.

Case (ii): $r \leq g_Y$. Consider the permanent debt roll-over policy $T_t = G_t$ for all $t \geq 0$, and assume $B_0 > 0$. By (DGBC) of Section 6.2 this policy yields $B_{t+1}/B_t = 1 + r$; hence, in view of (6.19), $\lim_{t\rightarrow\infty} b_{t+1}/b_t = (1 + r)/(1 + g_Y) \leq 1$. The policy consequently implies solvency. On the other hand the solution of the difference equation $B_{t+1} = (1 + r)B_t$ is $B_t = B_0(1 + r)^t$. Thus $B_t(1 + r)^{-t} = B_0 > 0$ for all t, thus violating (NPG). □

Hence imposition of the NPG condition on the government relies on the interest rate being in the long run higher than the growth rate of GDP. If instead $r \leq g_Y$, the government can cut taxes, run a budget deficit, and postpone the tax burden indefinitely. In that case the government can thus run a Ponzi Game and still stay solvent. Nevertheless, as alluded to earlier, if uncertainty is added to the picture, there will be many different interest rates, and matters become more complicated. Then qualifications to Proposition 1 are needed (Blanchard and Weil, 2001). The prevalent view among macroeconomists is that imposition of the NPG condition on the government is generally warranted.

While in the case $r > g_Y$, the NPG condition is *necessary* for solvency, it is *not sufficient*. Indeed, we could have

$$1 + g_Y < \lim_{t\to\infty} B_{t+1}/B_t < 1 + r. \tag{6.20}$$

Here, by the upper inequality, (NPG) is satisfied, yet, by the lower inequality together with (6.19), we have $\lim_{t\to\infty} b_{t+1}/b_t > 1$ so that the debt-income ratio explodes.

EXAMPLE 1 Let GDP $= Y$, a constant, and $r > 0$; so $r > g_Y = 0$. Let the budget deficit in real terms equal $\varepsilon B_t + \alpha$, where $0 \leq \varepsilon < r$ and $\alpha > 0$. Assuming no money-financing of the deficit, government debt evolves according to $B_{t+1} - B_t = \varepsilon B_t + \alpha$ which implies a simple linear difference equation:

$$B_{t+1} = (1 + \varepsilon)B_t + \alpha. \tag{*}$$

Case 1: $\varepsilon = 0$. Then the solution of (*) is

$$B_t = B_0 + \alpha t, \tag{**}$$

B_0 being historically given. Then $B_t(1+r)^{-t} = B_0(1+r)^{-t} + \alpha t(1+r)^{-t} \to 0$ for $t \to \infty$. So, (NPG) is satisfied. Yet the debt-GDP ratio, B_t/Y, goes to infinity for $t \to \infty$. That is, in spite of (NPG) being satisfied, solvency is not present. For $\varepsilon = 0$ we thus get the insolvency result even though the lower *strict* inequality in (6.20) is *not* satisfied. Indeed, (**) implies $B_{t+1}/B_t = 1 + \alpha/B_t \to 1$ for $t \to \infty$ and $1 + g_Y = 1$.

Case 2: $0 < \varepsilon < r$. Then the solution of (*) is

$$B_t = (B_0 + \frac{\alpha}{\varepsilon})(1 + \varepsilon)^t - \frac{\alpha}{\varepsilon} \to \infty \text{ for } t \to \infty,$$

if $B_0 > -\alpha/\varepsilon$. So $B_t/Y \to \infty$ for $t \to \infty$ and solvency is violated. Nevertheless $B_t(1 + r)^{-t} \to 0$ for $t \to \infty$ so that (NPG) holds.

The example of this case fully complies with both strict inequalities in (6.20) because $B_{t+1}/B_t = 1 + \varepsilon + \alpha/B_t \to 1 + \varepsilon$ for $t \to \infty$. □

An approach to fiscal budgeting that *ensures* debt stabilization and thereby solvency is the following. First impose that the cyclically adjusted primary budget surplus as a share of GDP equals a constant, s. Next adjust taxes and/or spending such that $s \geq \hat{s} = (r - g_Y)b_0$, ignoring short-run differences between Y_{t+1}/Y_t and $1 + g_Y$ and between r_t and its long-run value, r. As in (6.10), $\hat{s}$ is the minimum primary surplus as a share of GDP required to obtain $b_{t+1}/b_t \leq 1$ for all $t \geq 0$ (Example 2 below spells this out in detail). This $\hat{s}$ is a measure of the burden that the government debt imposes on tax payers. If the policy steps needed to realize at least $\hat{s}$ are not taken, the debt-income ratio will grow, thus worsening the fiscal position in the future by increasing $\hat{s}$.

6.5.2 Equivalence of NPG and GIBC

The condition under which the NPG condition is necessary for solvency is also the condition under which the government's intertemporal budget constraint is necessary. To show this we let t denote the current period and $t + i$ denote a period in the future. As above, we ignore seigniorage. Debt accumulation is then described by

$$B_{t+1} = (1 + r)B_t + G_t + X_t - \tilde{T}_t, \qquad \text{where } B_t \text{ is given.} \tag{6.21}$$

The *government intertemporal budget constraint* (GIBC), as seen from the beginning of period t, is the requirement

$$\sum_{i=0}^{\infty}(G_{t+i} + X_{t+i})(1 + r)^{-(i+1)} \leq \sum_{i=0}^{\infty}\tilde{T}_{t+i}(1 + r)^{-(i+1)} - B_t. \tag{GIBC}$$

This condition requires that the present value (PV) of current and expected future government spending does not exceed the government's net wealth. The latter equals the PV of current and expected future tax revenue minus existing government debt. By the symbol $\sum_{i=0}^{\infty} x_i$ we mean $\lim_{I\to\infty}\sum_{i=0}^{I} x_i$. Until further notice we assume this limit exists.

What connection is there between the dynamic accounting relationship (6.21) and the intertemporal budget constraint, (GIBC)? To find out, we rearrange

(6.21) and use forward substitution to get

$$
\begin{aligned}
B_t &= (1+r)^{-1}(\tilde{T}_t - X_t - G_t) + (1+r)^{-1}B_{t+1} \\
&= \sum_{i=0}^{j}(1+r)^{-(i+1)}(\tilde{T}_{t+i} - X_{t+i} - G_{t+i}) + (1+r)^{-(j+1)}B_{t+j+1} \\
&= \sum_{i=0}^{\infty}(1+r)^{-(i+1)}(\tilde{T}_{t+i} - X_{t+i} - G_{t+i}) + \lim_{j\to\infty}(1+r)^{-(j+1)}B_{t+j+1} \\
&\leq \sum_{i=0}^{\infty}(1+r)^{-(i+1)}(\tilde{T}_{t+i} - X_{t+i} - G_{t+i}), \qquad (6.22)
\end{aligned}
$$

if and only if the government debt ultimately grows at a rate less than r so that

$$\lim_{j\to\infty}(1+r)^{-(j+1)}B_{t+j+1} \leq 0. \qquad (6.23)$$

This latter condition is exactly the NPG condition above (replace t in (6.23) by 0 and j by $t-1$). And the condition (6.22) is just a rewriting of (GIBC). We conclude:

PROPOSITION 2 Given the book-keeping relation (6.21), then:

(i) (NPG) is satisfied if and only if (GIBC) is satisfied;

(ii) there is strict equality in (NPG) if and only if there is strict equality in (GIBC).

We know from Proposition 1 that in the "normal case" where $r > g_Y$, (NPG) is needed for government solvency. The message of (i) of Proposition 2 is then that also (GIBC) need be satisfied. Given $r > g_Y$, to appear solvent a government has to realistically plan taxation and spending profiles such that the PV of current and expected future primary budget surpluses matches the current debt, cf. (6.22). Otherwise debt default is looming and forward-looking investors will refuse to buy government bonds or only buy them at a reduced price, thereby aggravating the fiscal conditions.[20]

In view of the remarks around the inequalities in (6.20), however, satisfying the condition (6.22) is only a necessary condition (if $r > g_Y$), not in itself a sufficient condition for solvency. A simple condition under which satisfying the

[20] Government debt defaults have their own economic as well as political costs, including loss of credibility. Yet, they occur now and then. Recent examples include Russia in 1998 and Argentina in 2001-2002. During 2010-12, Greece was on the brink of debt default. At the time of writing (June 2015) such a situation has turned up again for Greece.

condition (6.22) *is* sufficient for solvency is that both G_t and T_t are proportional to Y_t, cf. Example 2.

EXAMPLE 2 Consider a small open economy facing an exogenous constant real interest rate r. Suppose that at time t government debt is $B_t > 0$, GDP is growing at the constant rate g_Y, and $r > g_Y$. Assume $G_t = \gamma Y_t$ and $T_t \equiv \tilde{T}_t - X_t = \tau Y_t$, where γ and τ are positive constants. What is the minimum size of the primary budget surplus as a share of GDP required for satisfying the government's intertemporal budget constraint as seen from time t? Inserting into the formula (6.22), with strict equality, yields $\sum_{i=0}^{\infty}(1+r)^{-(i+1)}(\tau-\gamma)Y_{t+i} = B_t$. This gives $\frac{\tau-\gamma}{1+g_Y}Y_t\sum_{i=0}^{\infty}\left(\frac{1+g_Y}{1+r}\right)^{(i+1)} = \frac{\tau-\gamma}{r-g_Y}Y_t = B_t$, where we have used the rule for the sum of an infinite geometric series. Rearranging, we conclude that the required primary surplus as a share of GDP is

$$\tau - \gamma = (r - g_Y)\frac{B_t}{Y_t}.$$

This is the same result as in (6.10) above if we substitute $\hat{s} = \tau - \gamma$ and $t = 0$. Thus, maintaining G_t/Y_t and T_t/Y_t constant while satisfying the government's intertemporal budget constraint ensures a constant debt-income ratio and thereby government solvency. □

On the other hand, if $r \leq g_Y$, it follows from propositions 1 and 2 together that the government can remain solvent *without* satisfying its intertemporal budget constraint (at least as long as we ignore uncertainty).[21] The background for this fact may become more apparent when we recognize how the condition $r \leq g_Y$ affects the constraint (GIBC). Indeed, to the extent that the tax revenue tends to grow at the same rate as national income, we have $\tilde{T}_{t+i} = \tilde{T}_t(1+g_Y)^i$. Then

$$\sum_{i=0}^{\infty}\tilde{T}_{t+i}(1+r)^{-(i+1)} = \frac{\tilde{T}_t}{1+g_Y}\sum_{i=0}^{\infty}\left(\frac{1+g_Y}{1+r}\right)^{(i+1)},$$

which is clearly infinite if $r \leq g_Y$. The PV of expected future tax revenues is thus unbounded in this case. Suppose that also government spending, $G_{t+i} + X_{t+i}$, grows at the rate g_Y. Then the evolution of the primary surplus is described by $\tilde{T}_{t+i} - X_{t+i} - G_{t+i} = (\tilde{T}_t - (G_t + X_t))(1+g_Y)^i$, $i = 1, 2, \ldots$. Although in this case also the PV of future government spending is infinite, (6.22) shows that any

[21] Of course, this statement is a contradiction in terms if one thinks of "solvency" in the standard *financial* sense where solvency requires that the debt does not exceed the present value of future surplusses, i.e., that (6.22) holds. As noted in Section 6.3 we use the term *solvency* in the broader meaning of "being able to meet the financial commitments as they fall due".

positive initial primary budget surplus, $\tilde{T}_t - (G_t + X_t)$, ever so small can repay any level of initial debt in finite time.

In (GIBC) and (6.23) we allow strict inequalities to obtain. What is the interpretation of a strict inequality here? The answer is:

COROLLARY OF PROPOSITION 2 Given the book-keeping relation (6.21), then strict inequality in (GIBC) is equivalent to the government in the long run accumulating positive net financial wealth.

Proof. Strict inequality in (GIBC) is equivalent to strict inequality in (6.22), which in turn, by (ii) of Proposition 2, is equivalent to strict inequality in (6.23), which is equivalent to $\lim_{j\to\infty}(1+r)^{-(j+1)}B_{t+j+1} < 0$. This latter inequality is equivalent to $\lim_{j\to\infty} B_{t+j+1} < 0$, that is, positive net financial wealth in the long run. Indeed, by definition, $r > -1$, hence $\lim_{j\to\infty}(1+r)^{-(j+1)} \geq 0$. □

It is common to consider as the *regular case* the case where the government does not attempt to accumulate positive net financial wealth in the long run and thereby become a net creditor vis-à-vis the private sector. Returning to the assumption $r > g_Y$, in the regular case fiscal solvency thus amounts to the requirement

$$\sum_{i=0}^{\infty} \tilde{T}_{t+i}(1+r)^{-(i+1)} = \sum_{i=0}^{\infty}(G_{t+i} + X_{t+i})(1+r)^{-(i+1)} + B_t, \qquad \text{(GIBC')}$$

which is obtained by rearranging (GIBC) and replacing weak inequality with strict equality. It is certainly *not* required that the budget is balanced all the time. The point is "only" that for a given planned expenditure path, a government should plan realistically a stream of future tax revenues the PV of which matches the PV of planned expenditure *plus* the current debt.

We may rewrite (GIBC') as

$$\sum_{i=0}^{\infty} \left(\tilde{T}_{t+i} - (G_{t+i} + X_{t+i})\right)(1+r)^{-(i+1)} = B_t. \qquad \text{(GIBC")}$$

This expresses the basic principle that when $r > g_Y$, solvency requires that *the present value of planned future primary surpluses equals the initial debt.* We have thus shown:

PROPOSITION 3 Consider the regular case. Assume $r > g_Y$. Then:

(i) if debt is positive today, the government has to run a positive primary budget surplus for a sufficiently long time in the future;

(ii) if an unplanned deficit arises so as to create an unexpected rise in public debt, then higher taxes than otherwise must be levied in the future.

Finer shades

1. If the real interest rate varies over time, all the above formulas remain valid if $(1+r)^{-(i+1)}$ is replaced by $\Pi_{j=0}^{i}(1+r_{t+j})^{-1}$.

2. We have essentially ignored seigniorage. Under "normal" circumstances seigniorage is present and this relaxes (GIBC") somewhat. Indeed, as noted in Section 6.2, the money-nominal income ratio, M/PY, tend to be roughly constant over time, reflecting that money and nominal income tend to grow at the same rate. So a rough indicator of g_M is the sum $\pi + g_Y$. Seigniorage is $S \equiv \Delta M/P = g_M M/P = sY$, where s is the seigniorage-income ratio. Taking seigniorage into account amounts to subtracting the present value of expected future seigniorage, PV(S), from the right-hand side of (GIBC"). With s constant and Y growing at the constant rate $g_Y < r$, PV(S) can be written

$$\begin{aligned} \text{PV(S)} &= \sum_{i=0}^{\infty} S_{t+i}(1+r)^{-(i+1)} = s\sum_{i=0}^{\infty} Y_{t+i}(1+r)^{-(i+1)} = \frac{sY_t}{1+g_Y}\sum_{i=0}^{\infty}\left(\frac{1+g_Y}{1+r}\right)^{(i+1)} \\ &= \frac{sY_t}{1+g_Y}\frac{1+g_Y}{1+r}\frac{1}{1-\frac{1+g_Y}{1+r}} = \frac{sY_t}{r-g_Y}, \end{aligned}$$

where the second to last equality comes from the rule for the sum of an infinite geometric series. So the right-hand side of (GIBC") becomes $B_t - sY_t/(r-g_Y) \equiv [b_t - s/(r-g_Y)]\, Y_t$.[22]

3. Should a public deficit rule not make a distinction between public consumption and public investment? This issue is taken up in the next section.

6.6 A proper accounting of public investment*

Public investment as a share of GDP has been falling in the EMU countries since the middle of the 1970s, in particular since the run-up to the euro 1993-97. This later development is seen as in part induced by the deficit rule of the Maastrict Treaty (1992) and the Stability and Growth Pact (1997) which, like the customary government budget accounting we have considered up to now, attributes government gross investment as an expense in a single year's operating account instead of just the depreciation of the public capital. Already Musgrave (1939) recommended applying separate capital and operating budgets. Thereby government net investment will be excluded from the definition of the public "budget deficit". And more meaningful deficit rules can be devised.

[22] In a recession where the economy is in a liquidity trap, the non-conventional monetary policy called Quantitative Easing may partly take the form of seigniorage. This is taken up in Chapter 24.

To see the gist of this, we partition G into public consumption, C^g, and public investment, I^g, that is, $G = C^g + I^g$. Public investment produces public capital (infrastructure etc.). Denoting the public capital K^g we may write

$$\Delta K^g = I^g - \delta K^g, \tag{6.24}$$

where δ is a (constant) capital depreciation rate. Let the annual (direct) financial return per unit of public capital be r_g. This is the sum of user fees and the like. Net government revenue, T', now consists of net tax revenue, T, plus the direct financial return $r_g K^g$.[23] In that now only interest payments and the capital depreciation, δK^g, along with C^g, enter the operating account as "true" expenses, the "true" budget deficit is $rB + C^g + \delta K^g - T'$, where $T' = T + r_g K^g$.

We impose a rule requiring balancing the "true structural budget" in the sense that on average over the business cycle

$$T' = rB + C^g + \delta K^g \tag{6.25}$$

should hold. The spending on public investment of course enters the debt accumulation equation which now takes the form

$$\Delta B = rB + C^g + I^g - T'.$$

Substituting (6.25) into this, we get

$$\Delta B = I^g - \delta K^g = \Delta K^g, \tag{6.26}$$

by (6.24). So the balanced "true structural budget" implies that public net investment is financed by an increase in public debt. Other public spending is tax financed.

Suppose public capital keeps pace with trend GDP, Y_t^*, that is, $\Delta K^g/K^g = g_Y > 0$. So the ratio K^g/Y^* remains constant at some level $h > 0$. Then (6.26) implies

$$B_{t+1} - B_t = K_{t+1}^g - K_t^g = g_Y K_t^g = g_Y h Y_t^*. \tag{6.27}$$

What is the implication for the evolution of the debt-to-trend-income ratio, $\hat{b}_t \equiv B_t/Y_t^*$, over time? By (6.27) together with $Y_{t+1}^* = (1 + g_Y)Y_t^*$ follows

$$\hat{b}_{t+1} \equiv \frac{B_{t+1}}{Y_{t+1}^*} = \frac{B_t}{(1+g_Y)Y_t^*} + \frac{g_Y h}{1+g_Y} \equiv \frac{1}{1+g_Y}\hat{b}_t + \frac{g_Y h}{1+g_Y}.$$

[23] There is also an indirect financial return deriving from the fact that better infrastructure may raise efficiency in the supply of public services and increase productivity in the private sector and thereby the tax base. While such expected effects matter for a cost-benefit analysis of a public investment project, from an accounting point of view they will be included in the net tax revenue, T, in the future.

This linear first-order difference equation has the solution

$$\hat{b}_t = (\hat{b}_0 - \hat{b}^*)(1 + g_Y)^{-t} + \hat{b}^*, \qquad \text{where } \hat{b}^* = \frac{1}{1+g_Y}\hat{b}^* + \frac{g_Y h}{1+g_Y} = h,$$

assuming $g_Y > 0$. Then $\hat{b}_t \to h$ for $t \to \infty$. Run-away debt dynamics is precluded.[24] Moreover, the ratio B_t/K_t^g, which equals $\hat{b}_t/h$, approaches 1. Eventually the public debt is in relative terms thus backed by the accumulated public capital.

Fiscal sustainability is here ensured *in spite of* a positive "budget deficit" in the *traditional* sense of Section 6.2 and given by ΔB in (**??**). This result holds even when $r_g < r$, which is perhaps the usual case. Still, the public investment may be worthwhile in view of indirect financial returns as well as non-financial returns in the form of the utility contribution of public goods and services.

Additional remarks

1. The deficit rule described says only that the "true structural budget" should be balanced "on average" over the business cycle. This invites deficits in slumps and surpluses in booms. Indeed, in economic slumps government borrowing is usually cheap. As Harvard economist Lawrence Summers put it: "Idle workers + Low interest rates = Time to rebuild infrastructure" (Summers, 2014).

2. When separating government consumption and investment in budget accounting, a practical as well as theoretical issue arises: where to draw the border between the two? A sizeable part of what is investment in an economic sense is in standard public sector accounting categorized as "public consumption": spending on education, research, and health are obvious examples. Distinguishing between such categories and public consumption in a narrower sense (administration, judicial system, police, defence) may be important when economic growth policy is on the agenda. Apart from noting the issue, we shall not pursue the matter here.

3. That *time lags*, cf. point (iii) in Section 6.1, are a constraining factor for fiscal policy is especially important for macroeconomic stabilization policy aiming at dampening business cycle fluctuations. If the lags are ignored, there is a risk that government intervention comes too late and ends up amplifying the fluctuations instead of dampening them. In particular the *monetarists*, lead by Milton Friedman (1912-2006), warned against this risk, pointing out the "long and variable lags". Other economists find awareness of this potential problem relevant but point to ways to circumvent the problem. During a recession there is for instance the option of reimbursing a part of last year's taxes, a policy that can be quickly implemented. In addition, the ministries of Economic affairs

[24] This also holds if $g_Y = 0$. Indeed, in this case, (6.27) implies $B_{t+1} = B_t = B_0$.

can have plans concerning upcoming public investment ready for implementation and carry them out when expansive fiscal policy is called for. More generally, legislation concerning taxation, transfers, and other spending can be designed with the aim of strengthening the automatic fiscal stabilizers.

6.7 Ricardian equivalence?

Having so far concentrated on the issue of fiscal sustainability, we shall now consider how budget policy affects resource allocation and intergenerational distribution. The role of budget policy for economic activity within a time horizon corresponding to the business cycle is not the issue here. The focus is on the longer run: does it matter for aggregate consumption and aggregate saving in an economy with full capacity utilization whether the government finances its current spending by (lump-sum) taxes or borrowing?

There are two opposite answers in the literature to this question. Some macroeconomists tend to answer the question in the negative. This is the *debt neutrality* view, also called the *Ricardian equivalence* view. The influential American economist Robert Barro is in this camp. Other macroeconomists tend to answer the question in the positive. This is the *debt non-neutrality* view or *absence of Ricardian equivalence* view. The influential French-American economist Olivier Blanchard is in this camp.

The two different views rest on two different models of the economic reality. Yet the two models have a *common* point of departure:

1) $r > g_Y$;

2) fiscal policy satisfies the intertemporal budget constraint with strict equality:
$$\sum_{t=0}^{\infty} \tilde{T}_t(1+r)^{-(t+1)} = \sum_{t=0}^{\infty}(G_t + X_t)(1+r)^{-(t+1)} + B_0, \qquad (6.28)$$
where the initial debt, B_0, and the planned path of $G_t + X_t$ are given;

3) agents have rational (model consistent) expectations;

4) at least some of the taxes are lump sum and only these are varied in the thought experiment to be considered;

5) no financing by money;

6) credit market imperfections are absent.

For a given planned time path of $G_t + X_t$, equation (6.28) implies that a tax cut in any period has to be met by an increase in future taxes of the same present discounted value as the tax cut.

6.7.1 Two differing views

Ricardian equivalence

The *Ricardian equivalence* view is the conception that government debt is *neutral* in the sense that for a given time path of future government spending, aggregate private consumption is unaffected by a temporary tax cut. The temporary tax cut does not make the households feel richer because they expect that the ensuing rise in government debt will lead to higher taxes in the future. The essential claim is that the timing of (lump-sum) taxes does not matter. The name *Ricardian equivalence* comes from a – seemingly false – association of this view with the early nineteenth-century British economist David Ricardo. It is true that Ricardo articulated the possible logic behind debt neutrality. But he suggested several reasons that debt neutrality would not hold in practice and in fact he warned against high public debt levels (Ricardo, 1969, pp. 161-164). Therefore it is doubtful whether Ricardo was a Ricardian.

Debt neutrality was rejuvenated, however, by Robert Barro in a paper entitled "Are government bonds net wealth [of the private sector]?", a question which Barro answered in the negative (Barro 1974). Barro's debt neutrality view rests on a representative agent model, that is, a model where the household sector is described as consisting of a fixed number of infinitely-lived forward-looking "dynasties". With perfect financial markets, a change in the timing of taxes does not change the PV of the infinite stream of taxes imposed on the individual dynasty. A cut in current taxes is offset by the expected higher future taxes. Though current government saving $(T - G - rB)$ goes down, private saving and bequests left to the members of the next generation go up equally much.

More precisely, the logic of the debt neutrality view is as follows. Suppose, for simplicity, that the government waits only 1 period to increase taxes and then does so in one stroke. Then, for each unit of account current taxes are reduced, taxes next period are increased by $(1+r)$ units of account. The PV as seen from the end of the current period of this future tax increase is $(1+r)/(1+r) = 1$. As $1-1 = 0$, the change in the time profile of taxation will make the dynasty feel neither richer nor poorer. Consequently, its current and planned future consumption will be unaffected. That is, its current saving goes up just as much as its current taxation is reduced. In this way the altruistic parents make sure that the next generation is fully compensated for the higher future taxes. Current private consumption in

Ricardian non-equivalence The old saying that "in life only death and tax are certain" fits the Ricardian non-equivalence view well. Many economists dissociate themselves from representative agent models because of their problematic description of the household sector. Instead attention is drawn to overlapping generations models which emphasize finite lifetime and life-cycle behavior of human beings and lead to a refutation of Ricardian equivalence. The essential point is that those individuals who benefit from lower taxes today will only be a fraction of those who bear the higher tax burden in the future. As taxes levied at different times are thereby levied at partly different sets of agents, the timing of taxes generally matters. The current tax cut makes current tax payers feel wealthier and so they increase their consumption and decrease their saving. The present generations benefit and future tax payers (partly future generations) bear the cost in the form of access to less national wealth than otherwise. With another formulation: under full capacity utilization government deficits have a crowding-out effect because they compete with private investment for the allocation of saving.

The next subsection provides an example showing in detail how a change in the timing of taxes affects aggregate private consumption in an overlapping generations life-cycle framework.

6.7.1 A small open OLG economy with a temporary budget deficit

We consider a Diamond-style overlapping generations (OLG) model of a small open economy (henceforth named SOE) with a government sector. The relationship between SOE and international markets is described by the same four assumptions as in Chapter 5.3:

(a) Perfect mobility of goods and financial capital across borders.

(b) No uncertainty and domestic and foreign financial claims are perfect substitutes.

(c) No need for means of payment, hence no need for a foreign exchange market.

(d) No labor mobility across borders.

The assumptions (a) and (b) imply *real interest rate equality.* That is, in equilibrium the real interest rate in SOE must equal the real interest rate, r, in the world financial market. By saying that SOE is "small" we mean it is small enough to not affect the world market interest rate as well as other world market factors. We imagine that all countries trade one and the same homogeneous good. International trade will then be only *intertemporal* trade, i.e., international borrowing and lending of this good.

We assume that r is constant over time and that $r > n \geq 0$. We let L_t denote the size of the young generation and assume $L_t = L_{-1}(1+n)^{t+1}$, $t = 0, 1, 2, \ldots$. Each young supplies one unit of labor inelastically, hence L_t is aggregate labor supply. Assuming full employment and ignoring technical progress, gross domestic product, GDP, is $Y_t = F(K_t, L_t)$.

Firms' behavior and the equilibrium real wage

GDP is produced by an aggregate neoclassical production function with CRS:

$$Y_t = F(K_t, L_t) = L_t F(k_t, 1) \equiv L_t f(k_t),$$

where K_t and L_t are input of capital and labor, respectively, and $k_t \equiv K_t/L_t$. Technological change is ignored. Imposing perfect competition, profit maximization gives $\partial Y_t/\partial K_t = f'(k_t) = r+\delta$, where δ is a constant capital depreciation rate, $0 \leq \delta \leq 1$. When f satisfies the condition $\lim_{k\to 0} f'(k) > r + \delta > \lim_{k\to\infty} f'(k)$, there is always a solution for k_t in this equation and it is unique (since $f'' < 0$) and constant over time (as long as r and δ are constant). Thus,

$$k_t = f'^{-1}(r + \delta) \equiv k, \text{ for all } t \geq 0, \tag{6.29}$$

where k is the desired capital-labor ratio, given r. The endogenous stock of capital, K_t, is determined by the equation $K_t = kL_t$, where, in view of clearing in the labor market, L_t can be interpreted as both employment and labor supply (exogenous).

The desired capital-labor ratio, k, also determines the equilibrium real wage before tax:

$$w_t = \frac{\partial Y_t}{\partial L_t} = f(k_t) - f'(k_t)k_t = f(k) - f'(k)k \equiv w, \tag{6.30}$$

a constant. GDP will evolve over time according to

$$Y_t = f(k)L_t = f(k)L_0(1 + n)^t = Y_0(1 + n)^t.$$

The growth rate of Y thus equals the growth rate of the labor force, i.e., $g_Y = n$.

Some national accounting for an open economy with a public sector

Since we ignore labor mobility across borders, gross national product (= gross national income) in SOE is

$$GNP_t = GDP_t + r \cdot NFA_t = Y_t + r \cdot NFA_t,$$

where NFA_t is net foreign assets at the beginning of period t. If $NFA_t > 0$, SOE has positive net claims on resources in the rest of the world, it may be in the form of direct ownership of production assets or in the form of net financial claims. If $NFA_t < 0$, the reason may be that part of the capital stock, K_t, in SOE is directly owned by foreigners or these have on net financial claims on the citizens of SOE (in practice usually a combination of the two).

Gross national saving is

$$S_t = Y_t + rNFA_t - C_t - G_t = Y_t + rNFA_t - (c_{1t}L_t + c_{2t}L_{t-1}) - G_t, \quad (6.31)$$

where G_t is government consumption in period t, and c_{1t} and c_{2t} are consumption by a young and an old in period t, respectively. In the open economy, generally, gross investment, I_t, differs from gross saving.

National wealth, V_t, of SOE at the beginning of period t is, by definition, national assets minus national liabilities,

$$V_t \equiv K_t + NFA_t.$$

National wealth is also, by definition, the sum of private financial (net) wealth, A_t, and government financial (net) wealth, $-B_t$. We assume the government has no physical assets and B_t is government (net) debt. Thus,

$$V_t \equiv A_t + (-B_t). \quad (6.32)$$

We may also view *national* wealth from the perspective of national *saving. First*, when the young save, they accumulate *private* financial wealth. The private financial wealth at the start of period $t+1$ must in our Diamond framework equal the (net) saving by the young in the previous period, S_{1t}^N, and the latter must equal *minus* the (net) saving by the old in the next period, S_{2t+1}^N :

$$A_{t+1} = s_t L_t \equiv S_{1t}^N = -S_{2t+1}^N. \quad (6.33)$$

The notation in this section of the chapter follows the standard notation for the Diamond model, and so s_t stands for the saving by the young individual in period t, not the primary budget surplus as in the previous sections.

Second, the increase in *national* wealth equals by definition net *national* saving, S_t^N, which in turn equals the sum of net saving by the private sector, $S_{1t}^N + S_{2t}^N$, and the net saving by the public sector, S_{gt}^N. So

$$\begin{aligned} V_{t+1} - V_t &= S_t - \delta K_t = S_t^N \equiv S_{1t}^N + S_{2t}^N + S_{gt}^N = A_{t+1} + (-A_t) + (-GBD_t) \\ &= A_{t+1} - A_t - (B_{t+1} - B_t), \end{aligned}$$

where the second to last equality comes from (6.33) and the identity $S_{gt}^N \equiv -GBD_t$, while the last equality reflects the maintained assumption that budget deficits are fully financed by debt issue.

Government and household behavior

We assume that the role of the government sector is to deliver public goods and services in the amount G_t in period t. Think of non-rival goods like "rule of law", TV-transmitted theatre, and other public services free of charge. Suppose G_t grows at the same rate as Y_t :

$$G_t = G_0(1+n)^t,$$

where G_0 is given, $0 < G_0 < F(K_0, L_0)$. We may think of G_t as being produced by the same technology as the other components of GDP, thus involving the same unit production costs. We ignore that the public good may affect productivity in the private sector (otherwise G should in principle appear as a third argument in the production function F).

To get explicit solutions, we specify the period utility function to be CRRA: $u(c) = (c^{1-\theta} - 1)/(1-\theta)$, where $\theta > 0$. To keep things simple, the utility of the public good enters individuals' life-time utility additively. Thereby it does not affect marginal utilities of private consumption. There is a tax on the young as well as the old in period t, τ_1 and τ_2, respectively. These taxes are *lump sum* (levied on individuals irrespective of their economic behavior). Until further notice, the taxes are time-independent. Possibly, τ_1 or τ_2 is negative, in which case there is a transfer to either the young or the old.

The consumption-saving decision of the young will be the solution to the following problem:

$$\begin{aligned}
\max U(c_{1t}, c_{2t+1}) &= \frac{c_{1t}^{1-\theta}-1}{1-\theta} + v(G_t) + (1+\rho)^{-1}\left[\frac{c_{2t+1}^{1-\theta}-1}{1-\theta} + v(G_{t+1})\right] \text{ s.t.} \\
c_{1t} + s_t &= w - \tau_1, \\
c_{2t+1} &= (1+r)s_t - \tau_2, \\
c_{1t} &\geq 0, c_{1t+1} \geq 0,
\end{aligned}$$

where the function v represents the utility contribution of the public good. The implied Euler equation can be written

$$\frac{c_{2t+1}}{c_{1t}} = \left(\frac{1+r}{1+\rho}\right)^{1/\theta}.$$

Inserting the two budget constraints and solving for s_t, we get

$$s_t = \frac{w - \tau_1 + \left(\frac{1+\rho}{1+r}\right)^{1/\theta}\tau_2}{1 + (1+\rho)\left(\frac{1+r}{1+\rho}\right)^{(\theta-1)/\theta}} \equiv s_0 = s(w, r, \tau_1, \tau_2), \qquad t = 0, 1, 2, \ldots,$$

This shows how saving by the young depends on the preference parameters θ and ρ and on labor income and the interest rate. Further, saving by the young is constant over time.

Before considering the solution for c_{1t} and c_{2t+1}, it is convenient to introduce the *intertemporal* budget constraint of an individual belonging to generation t and consider the value of the individual's after-tax *human wealth*, h_t, evaluated at the end of period t. This is the present (discounted) value, as seen from the end of period t, of *disposable lifetime income* (the "endowment").obtainable by a member of generation t. In the present case we get

$$c_{1t} + \frac{c_{2t+1}}{1+r} = w_t - \tau_1 - \frac{\tau_2}{1+r} \equiv h, \tag{6.34}$$

where h on the right-hand side is the time independent value of h_t under the given circumstances.[26] To ensure that $h > 0$, we must assume that τ_1 and τ_2 in combination are of "moderate" size.

The solutions for consumption in the first and the second period, respectively, can then be written

$$c_{1t} = w - \tau_1 - s_t = \hat{c}_1(r)h \tag{6.35}$$

and

$$c_{2t+1} = \hat{c}_2(r)h, \tag{6.36}$$

where

$$\hat{c}_1(r) \equiv \frac{1+\rho}{1+\rho+\left(\frac{1+r}{1+\rho}\right)^{(1-\theta)/\theta}} \in (0,1) \text{ and} \tag{6.37}$$

$$\hat{c}_2(r) \equiv \left(\frac{1+r}{1+\rho}\right)^{1/\theta} \hat{c}_1(r) = \frac{1+r}{1+(1+\rho)\left(\frac{1+r}{1+\rho}\right)^{(\theta-1)/\theta}} \tag{6.38}$$

are the marginal (= average) propensities to consume out of wealth.[27]

Given r, both in the first and the second period of life is individual consumption proportional to individual human wealth. This is as expected in view of the homothetic lifetime utility function. If $\rho = r$, then $\hat{c}_1(r) = \hat{c}_2(r) = (1+r)/(2+r)$, that is, there is complete consumption smoothing.

The tax revenue in period t is $T_t = \tau_1 L_t + \tau_2 L_{t-1} = (\tau_1 + \tau_2/(1+n))L_t$. Let $B_0 = 0$ and let the "benchmark path" be a path along which the budget is and remains *balanced* for all t, i.e.,

$$T_t = (\tau_1 + \frac{\tau_2}{1+n})L_0(1+n)^t = G_t = G_0(1+n)^t.$$

[26] With technical progress, the real wage would be rising over time and so would h_t.

[27] By calculating backwards from (6.38) to (6.37) to (**??**), the reader will be able to confirm that the calculated s, c_{1t} and c_{2t+1} are consistent.

In this "benchmark policy regime" the tax code (τ_1, τ_2) thus satisfies $(\tau_1+\tau_2/(1+n))L_0 = G_0$. Given L_0, consistency with $h > 0$ in (6.34) requires a "not too large" G_0.

Along the benchmark path, aggregate private consumption grows at the same constant rate as GDP and public consumption, the rate n. Indeed,

$$C_t = c_{1t}L_t + \frac{c_{2t}}{1+n}L_t = (c_{1t} + \frac{c_{2t}}{1+n})L_0(1+n)^t = C_0(1+n)^t.$$

In view of (6.33) and the absence of government debt, also *national wealth* grows at the rate n :

$$V_t = A_t - B_t = A_t - 0 = s_{t-1}L_{t-1} = s_0 L_{t-1} = s_0 L_{-1}(1+n)^t = V_0(1+n)^t, \qquad t = 0, 1, \ldots. \tag{6.39}$$

Consequently, national wealth per old, V_t/L_{t-1}, is constant over time (recall, we have ignored technical progress).

6.7.2 A one-off tax cut

As an alternative to the benchmark path, consider the case where an unexpected one-off cut in taxation by z units of account takes place in period 0 for every individual, whether young or old. What are the consequences of this? The tax cut amounts to creating a budget deficit in period 0 equal to

$$GBD_0 = rB_0 + G_0 - T_0' = G_0 - T_0' = T_0 - T_0' = (L_0 + L_{-1})z,$$

where the value taken by a variable along this *alternative path* is marked with a prime. At the start of period 1, there is now a government debt $B_1' = (L_0+L_{-1})z$. In the benchmark path we had $B_1 = 0$. Since we assume $r > n = g_Y$, government solvency requires that the present value of future taxes, as seen from the beginning of period 1, rises by $(L_0 + L_{-1})z$, cf. (6.28). Suppose this is accomplished by raising the tax on all individuals from period 1 onward by m. Then

$$\Delta T_t = (L_t + L_{t-1})m = (L_0 + L_{-1})(1+n)^t, \qquad t = 1, 2, \ldots.$$

Suppose the government in period 0 credibly announces that the way it will tackle the arisen debt is by his policy. So also the young in period 0 are aware of the future tax rise.

As solvency requires that the present value of future taxes, as seen from the beginning of period 1, rises by $(L_0 + L_{-1})z$, the required value of m will satisfy

$$\sum_{t=1}^{\infty} \Delta T_t(1+r)^{-t} = \sum_{t=1}^{\infty}(L_0 + L_{-1})(1+n)^t m(1+r)^{-t} = (L_0 + L_{-1})z.$$

This gives

$$m\sum_{t=1}^{\infty}\left(\frac{1+n}{1+r}\right)^t = z.$$

As $r > n$, from the rule for the sum of an infinite geometric series follows that

$$m = \frac{r-n}{1+n}z \equiv \bar{m}. \tag{6.40}$$

As an example, let $r = 0,02$ and $n = 0.005$ per year. Then $\bar{m} \simeq 0.015 \cdot z$.

The needed rise in future taxes is thus higher the higher is the interest rate r. This is because the interest burden of the debt will be higher. On the other hand, a higher population growth rate, n, reduces the needed rise in future taxes. This is because the interest burden per capita is mitigated by population growth. Finally, a greater tax cut, z, in the first period implies greater tax rises in future periods. (It is assumed throughout that z is of "moderate" size in the sense of not causing $\bar{m}$ to violate the condition $h_t' > 0$. The requirement is $0 < z < (1+r)(1+n)h/\left[(2+r)(r-n)\right]$.)

Effect on the consumption path

In period 0 the tax cut unambiguously benefits the old. Their increase in consumption equals the saved tax:

$$c_{20}' - c_{20} = z > 0. \tag{6.41}$$

The young in period 0 know that per capita taxes next period will be increased by $\bar{m}$. In view of the tax cut in period 0, the young nevertheless experience an increase in after-tax human wealth equal to

$$\begin{aligned} h_0' - h_0 &= \left(w - \tau_1 + z - \frac{\tau_2 + \bar{m}}{1+r}\right) - \left(w - \tau_1 - \frac{\tau_2}{1+r}\right) \\ &= \left(1 - \frac{r-n}{(1+r)(1+n)}\right)z \qquad \text{(by (6.40))} \\ &= \frac{1+(2+r)n}{(1+r)(1+n)}z > 0. \end{aligned} \tag{6.42}$$

Consequently, through the *wealth effect* this generation enjoys increases in consumption through life equal to

$$c_{10}' - c_{10} = \hat{c}_1(r)(h_0' - h_0) > 0, \qquad \text{and} \tag{6.43}$$

$$c_{21}' - c_{21} = \hat{c}_2(r)(h_0' - h_0) > 0, \tag{6.44}$$

by (6.35) and (6.36), respectively. The two generations alive in period 0 thus gain from the temporary budget deficit.

All *future* generations are worse off, however. These generations do not benefit from the tax relief in period 0, but they have to bear the future cost of the tax relief by a *reduction* in individual after-tax human wealth. Indeed, for $t = 1, 2, \ldots,$

$$\begin{aligned} h_t' - h_t &= h_1' - h = w - \tau_1 - \bar{m} - \frac{\tau_2 + \bar{m}}{1+r} - \left(w - \tau_1 - \frac{\tau_2}{1+r}\right) \\ &= -\left(\bar{m} + \frac{\bar{m}}{1+r}\right) = -\frac{2+r}{1+r}\bar{m} < 0. \end{aligned} \tag{6.45}$$

All things considered, since both the young and the old in period 0 increase their consumption, aggregate consumption in period 0 rises. Ricardian equivalence thus *fails*.

Effect on wealth accumulation*

How does aggregate *private* saving in period 0 respond to the temporary tax cut? Consider first the old in period 0. Along both the benchmark path and the alternative path the old entered period 0 with the financial wealth A_0 and they leave the period with zero financial wealth. So their aggregate net saving is $S_{20}^N = -A_0$ in both fiscal regimes. The young in period 0 increase their consumption in response to the temporary tax cut. At the same time they *increase* their period-0 saving. Indeed, from (6.44) and the period budget constraint as old follows

$$\begin{aligned} 0 &< c_{21}' - c_{21} = (1+r)s_0' - (\tau_2 + \bar{m}) - ((1+r)s_0 - \tau_2) \\ &= (1+r)(s_0' - s_0) - \bar{m} < (1+r)(s_0' - s_0), \end{aligned}$$

thus implying $s_0' - s_0 > 0$. The explanation is that the individuals have a preference for consumption smoothing in that $\theta > 0$. So the young in period 0 want to smooth out the increased consumption possibilities resulting from the increase in their human wealth. To be able to increase consumption as old, their extra saving, with interest, must exceed what is needed to pay the extra tax $\bar{m}$ in period 1. It is the tax cut that makes it possible for the young to increase both consumption and saving in period 0.

The impact on national wealth in period 1 The higher saving by the young in period 0 implies higher aggregate *private* financial wealth per old at the beginning of period 1, since $A_1'/L_0 = s_0' > s_0 = A_1/L_0$. Nevertheless, gross *national* saving, cf. (6.31), is clearly lower than in the benchmark case. Indeed, $C_0' > C_0$ implies

$$S_0' = F(K_0, L_0) + r \cdot NFA_0 - C_0' - G_0 < F(K_0, L_0) + r \cdot NFA_0 - C_0 - G_0 = S_0.$$

That gross national saving is lower is not inconsistent with the just mentioned rise in *private* saving in period 0 compared to the benchmark path. A counterpart of the increased *private* saving is the *public dissaving*, reflecting that the tax cut in period 0 creates a budget deficit one-to-one. Since the increased disposable income implied by the tax cut is used partly to increase private saving *and* partly to increase private consumption, the rise in private saving is *smaller* than the public dissaving. So *total* or *national* saving in period 0 is reduced.

Consequently, we have:

(i) *National wealth* at the start of period 1 is lower in the debt regime than in the no-debt regime.

By how much? In the benchmark regime the national wealth at the start of period 1 is $V_1 = V_0 + S_0^N = V_0 + S_0 - \delta K_0$. This exceeds national wealth in the debt regime by

$$\begin{aligned} V_1 - V_1' &= S_0 - S_0' = C_0' - C_0 = c_{10}' L_0 + c_{20}' L_{-1} - (c_{10} L_0 + c_{20} L_{-1}) \\ &= (c_{10}' - c_{10}) L_0 + (c_{20}' - c_{20}) L_{-1} \\ &= \hat{c}_1(r)(h_0' - h_0) L_0 + z L_{-1} \quad \text{(by (6.43) and (6.41))} \\ &= \left(\hat{c}_1(r) \frac{1 + (2+r)n}{1+r} + 1 \right) \frac{1}{1+n} L_0 z > 0. \quad \text{(by (6.42))} \end{aligned} \tag{6.46}$$

Later consequences As revealed by (6.45), all future generations (those born in period $1, 2, \ldots$) are worse off along the alternative path. This gives rise to two further claims:

(ii) *National wealth per old* along the alternative path, V_t'/L_{t-1}, will remain constant from period 2 onward at a level below that along the path without government debt.

(iii) The constant level along the alternative path from period 2 onward will even be below the level in period 1.

To substantiate these two claims, consider $V_t' \equiv A_t' - B_t'$. In Appendix A it is shown that *government debt* per old will from period 1 onward satisfy

$$\frac{B_t'}{L_{t-1}} = \frac{B_1'}{L_0} = \frac{(L_0 + L_{-1})z}{L_0} = \frac{2+n}{1+n} z, \qquad t = 1, 2, \ldots,$$

and thus be constant. So government debt grows at the rate of population growth. In addition, Appendix A shows that *private* financial wealth per old is constant from period 2 onward and satisfies

$$\frac{A'_t}{L_{t-1}} = s'_{t-1} = s_0 - \left(1 - \hat{c}_1(r)\frac{2+r}{1+r}\right)\frac{r-n}{1+n}z, \qquad t = 2, 3, \ldots .$$

It follows that *national* wealth per old from period 2 onward will be

$$\begin{aligned}\frac{V'_t}{L_{t-1}} &\equiv \frac{A'_t}{L_{t-1}} - \frac{B'_t}{L_{t-1}} = s'_{t-1} - \frac{2+n}{1+n}z = s_0 - \left(1 - \hat{c}_1(r)\frac{2+r}{1+r}\right)\frac{r-n}{1+n}z - \frac{2+n}{1+n}z \\ &= s_0 - \left(1 - \hat{c}_1(r)\frac{r-n}{1+r}\right)\frac{2+r}{1+n}z = \frac{V'_2}{L_1} < s_0 = \frac{V_2}{L_1} = \frac{V_1}{L_0} \qquad t = 2, 3, \ldots \end{aligned} \tag{6.47}$$

where the last two equalities follow from (6.39). This proves our claim (ii).

National wealth per old in period 1 of the debt path is, by (6.46),

$$\begin{aligned}\frac{V'_1}{L_0} &= \frac{V_1}{L_0} - \left(\hat{c}_1(r)\frac{1+(2+r)n}{1+r} + 1\right)\frac{z}{1+n} \\ &= s_0 - \left(\hat{c}_1(r)\frac{1+(2+r)n}{1+r} + 1\right)\frac{z}{1+n} > \frac{V'_2}{L_1},\end{aligned}$$

where the inequality follows by comparison with (6.47). This proves our claim (iii).

Period 1 is special compared to the subsequent periods. While there is a per capita tax increase by $\bar{m}$ like in the subsequent periods, period 1's old generation still benefits from the higher disposable income in period 0. Hence, in period 2 national wealth per old is even lower than in period 1 but remains constant henceforth.

A closed economy Also in a closed economy would a temporary lump-sum tax cut make the future generations worse off. Indeed, in view of reduced national saving in period 0, national wealth (which in the closed economy equals K) would from period 1 onward be smaller than along the no-debt path. The precise calculations are more complicated because the rate of interest will no longer be a constant.

6.7.3 Widening the perspective

The fundamental point underlined by OLG models is that there is a difference between the public sector's future tax base, including the resources of individuals yet to be born, and the future tax base emanating from individuals alive today.

This may be called the *composition-of-tax-base argument* for a tendency to non-neutrality of shifting the timing of (lump-sum) taxation.[28]

The conclusion that under full capacity utilization budget deficits imply a burden for future generations may be seen in a somewhat different light if persistent technological progress is included in the model. In that case, everything else equal, future generations will generally be better off than current generations. Then it might seem less unfair if the former carry some public debt forward to the latter. In particular this is so if a part of G_t represents spending on infrastructure, education, research, health, and environmental protection. As future generations directly benefit from such investment, it seems fair that they also contribute to the financing. This is the "benefits received principle" known from public finance theory.

A further concern is whether the economy is in a state of full capacity utilization or serious unemployment and idle capital. The above analysis assumes the first. What if the economy in period 0 is in economic depression with high unemployment due to insufficient aggregate demand? Some economists maintain that also in this situation is a cut in (lump-sum) taxes to stimulate aggregate demand futile because it has no real effect. The argument is again that foreseeing the higher taxes needed in the future, people will save more to prepare themselves (or their descendants through higher bequests) for paying the higher taxes in the future. The opposite view is, first, that the composition-of-tax-base argument speaks against this as usual. Second, there is in a depression an additional and quantitatively important factor. The "first-round" increase in consumption due to the temporary tax cut raises aggregate demand. Thereby production and income is stimulated and a further (but smaller) rise in consumption occurs in the "second round" and so on (the Keynesian multiplier process).

This Keynesian mechanism is important for the debate about effects of budget deficits because there are limits to how *large* deviations from Ricardian equivalence the composition-of-tax-base argument can deliver in the long-run life-cycle perspective of OLG models. Indeed, taking into account the sizeable life expectancy of the average citizen, Poterba and Summers (1987) point out that the composition-of-tax-base argument by itself delivers only modest deviations if the issue is timing of taxes over the business cycle. They find that to comply with the data on private saving responses to supposedly exogenous shifts in taxation should be combined with the hypothesis that households are "myopic" than what standard OLG models assume.

Another concern is that in the real world, taxes tend to be distortionary and

[28]In Exercise 6.?? the reader is asked how the burden of the public debt is distributed across generations if the debt should be completely wiped out through a tax increase in only periods 1 and 2.

not lump sum. On the one hand, this should not be seen as an argument against the possible *theoretical* validity of the Ricardian equivalence proposition. The reason is that Ricardian equivalence (in its strict meaning) claims absence of allocational effects of changes in the timing of *lump-sum* taxes.

On the other hand, in a wider perspective the interesting question is, of course, how changes in the timing of *distortionary* taxes is likely to affect resource allocation. Consider first *income taxes.* When taxes are proportional to income or progressive (average tax rate rising in income), they provide insurance through reducing the volatility of after-tax income. The fall in taxes in a recession thus helps stimulating consumption through *reduced precautionary saving* (the phenomenon that current saving tends to rise in response to increased uncertainty, cf. Chapter ??). In this way, replacing lump-sum taxation by income taxation underpins the positive wealth effect on consumption, arising from the composition-of-tax-base channel, of a debt-financed tax-cut in an economic recession.

What about *consumption taxes*? A debt-financed temporary cut in consumption taxes stimulates consumption through a positive wealth effect, arising from the composition-of-tax-base channel. On top of this comes a positive intertemporal substitution effect on current consumption caused by the changed consumer price time profile.

The question whether Ricardian non-equivalence is important from a quantitative and empirical point of view pops up in many contexts within macroeconomics. We shall therefore return to the issue several times later in this book.

6.8 Concluding remarks

(incomplete)

Point (iv) in Section 6.1 hints at the fact that when outcomes depend on forward-looking expectations in the private sector, governments may face a time-inconsistency problem. In this context *time inconsistency* refers to the possible temptation of the government to deviate from its previously announced course of action once the private sector has acted. An example: With the purpose of stimulating private saving, the government announces that it will not tax financial wealth. Nevertheless, when financial wealth has reached a certain level, it constitutes a tempting base for taxation and so a tax on wealth might be levied. To the extent the private sector anticipates this, the attempt to affect private saving in the first place fails. This raises issues of *commitment* and *credibility.* We return to this kind of problems in later chapters.

Finally, point (v) in Section 6.1 alludes to the fact that political processes, bureaucratic self-interest, rent seeking, and lobbying by powerful interest groups interferes with fiscal policy.[29] This is a theme in the branch of economics called *political economy* and is outside the focus of this chapter.

6.9 Literature notes

(incomplete)

Sargent and Wallace (1981) study consequences of – and limits to – a shift from debt financing to money financing of sustained government budget deficits in response to threatening increases in the government debt-income ratio.

How the condition $r > g_Y$, for prudent debt policy to be necessary, is modified when the assumption of no uncertainty is dropped is dealt with in Abel et al. (1989), Bohn (1995), Ball et al. (1998), and Blanchard and Weil (2001). On self-fulfilling sovereign debt crises, see, e.g., Cole and Kehoe (2000).

Readers wanting to go more into detail with the policy-oriented debate about the design of the EMU and the Stability and Growth Pact is referred to the discussions in for example Buiter (2003), Buiter and Grafe (2004), Fogel and Saxena (2004), Schuknecht (2005), and Wyplosz (2005). As to discussions of the actual functioning of monetary and fiscal policy in the Eurozone in response to the Great Recession, see for instance the opposing views by De Grauwe and Ji (2013) and Buti and Carnot (2013). Blanchard and Giavazzi (2004) discuss how proper accounting of public investment would modify the deficit and debt rules of the EMU. Beetsma and Giuliodori (2010) survey recent research of costs and benefits of the EMU.

On the theory of *optimal currency areas,* see Krugman, Obstfeld, and Melitz (2012).

In addition to the hampering of Keynesian stabilization policy discussed in Section 6.4.2, also demographic staggering (due to baby booms succeeded by baby busts) may make rigid deficit rules problematic. In Denmark for instance demographic staggering is prognosticated to generate considerable budget deficits during several decades after 2030 where younger and smaller generations will succeed older and larger ones in the labor market. This is prognosticated to take place, however, without challenging the long-run sustainability of current fiscal policy as assessed by the Danish Economic Council (see the English Summary in De Økonomiske Råd, 2014). This phenomenon is in Danish known as "hængekøjeproblemet" (the "hammock problem").

Sources for last part of Section 6.7 ....

[29] *Rent seeking* refers to attempts to gain by increasing one's share of existing wealth, instead of trying to *produce* wealth.

6.10 Appendix A

In Section 6.7.2 we asserted that along the alternative path the government debt will grow at the same rate as the population. The proof is as follows.

The law of motion of the debt is, for $t = 1, 2, \ldots,$

$$\begin{aligned} B'_{t+1} &= (1+r)B'_t + G_t - T'_t = (1+r)B'_t + G_t - \left(\tau_1 + \frac{\tau_2}{1+n} + \bar{m} + \frac{\bar{m}}{1+n}\right) L_t \\ &= (1+r)B'_t - \left(\bar{m} + \frac{\bar{m}}{1+n}\right) L_t = (1+r)B'_t - \frac{2+n}{1+n}\bar{m}L_t, \end{aligned}$$

where the second line follows from $G_t - (\tau_1 + \tau_2(1+n))L_t = 0$ in view of the balanced budget along the benchmark path. It is convenient to rewrite the law of motion in terms of $x_t \equiv B'_t/L_{t-1}$, i.e., government debt per old. We get

$$x_{t+1} \equiv \frac{B'_{t+1}}{L_t} = \left(\frac{1+r}{1+n}\right) x_t - \frac{2+n}{1+n}\bar{m}, \qquad t = 1, 2, \ldots,$$

where we have used that $L_t = (1+n)L_{t-1}$. The solution of this first-order difference equation with constant coefficients is

$$x_t = (x_1 - x^*) \left(\frac{1+r}{1+n}\right)^{t-1} + x^*,$$

with

$$\begin{aligned} x_1 &= \frac{B'_1}{L_0} = \frac{(L_0 + L_{-1})z}{L_0} = \frac{2+n}{1+n}z, \quad \text{and} \\ x^* &= -\frac{2+n}{1+n}\bar{m}\left(1 - \frac{1+r}{1+n}\right)^{-1} = \frac{2+n}{r-n}\bar{m} = \frac{2+n}{1+n}z, \end{aligned}$$

using the solution (6.40) for the tax rise $\bar{m}$. It follows that x_t is constant over time and equals x^*. Hence, from period 1 onward $B'_t/L_{t-1} = (2+n)z/(1+n)$ where z is the per capita tax cut in period 0. □

In Section 6.7.2 we also asserted that along the alternative path the private financial wealth per old will from period 2 onward be constant. The proof is as follows:

For $t = 2, 3, \ldots,$

$$\begin{aligned} \frac{A'_t}{L_{t-1}} &= s'_{t-1} = w - (\tau_1 + \bar{m}) - c'_{1t-1} = w - (\tau_1 + \bar{m}) - \hat{c}_1(r)\left(w - \tau_1 - \bar{m} - \frac{\tau_2 + \bar{m}}{1+r}\right) \\ &= w - \tau_1 - \hat{c}_1(r)(w - \tau_1 - \frac{\tau_2}{1+r}) - \bar{m} + \hat{c}_1(r)\bar{m} + \hat{c}_1(r)\frac{\bar{m}}{1+r} \\ &= s_0 - \left(1 - \hat{c}_1(r)\left(1 + \frac{1}{1+r}\right)\right)\bar{m} = s_0 - \left(1 - \hat{c}_1(r)\frac{2+r}{1+r}\right)\frac{r-n}{1+n}z, \end{aligned}$$

where we have used (6.33), the period budget constraint of the young along the alternative path, (6.35), (6.34), the period budget constraint of the young along the benchmark path, the constancy of saving by the young along the benchmark path, and finally the solution for the tax rise $\bar{m}$. We see that private financial wealth per old is constant from period 2 onward. □

6.11 Exercises

6.? Consider the OLG model of Section 6.7. a) Show that if the temporary per capita tax cut, z, is sufficiently small, the debt can be completely wiped out through a per capita tax increase in only periods 1 and 2. b) Investigate how in this case the burden of the debt is distributed across generations. Compare with the alternative debt policy described in the text.

Chapter 9

The intertemporal consumption-saving problem in discrete and continuous time

In the next two chapters we shall discuss – and apply – the continuous-time version of the basic representative agent model, the Ramsey model. As a preparation for this, the present chapter gives an account of the transition from discrete time to continuous time analysis and of the application of optimal control theory to set up and solve the household's consumption/saving problem in continuous time.

There are many fields in economics where a setup in continuous time is preferable to one in discrete time. One reason is that continuous time formulations expose the important distinction in dynamic theory between stock and flows in a much clearer way. A second reason is that continuous time opens up for application of the mathematical apparatus of differential equations; this apparatus is more powerful than the corresponding apparatus of difference equations. Similarly, optimal control theory is more developed and potent in its continuous time version than in its discrete time version, considered in Chapter 8. In addition, many formulas in continuous time are simpler than the corresponding ones in discrete time (cf. the growth formulas in Appendix A).

As a vehicle for comparing continuous time modeling with discrete time modeling we consider a standard household consumption/saving problem. How does the household assess the choice between consumption today and consumption in the future? In contrast to the preceding chapters we allow for an arbitrary number of periods within the time horizon of the household. The period length may thus be much shorter than in the previous models. This opens up for capturing additional aspects of economic behavior and for undertaking the transition to continuous time in a smooth way.

We first specify the market environment in which the optimizing household operates.

9.1 Market conditions

In the Diamond OLG model no loan market was active and wealth effects on consumption or saving through changes in the interest rate were absent. It is different in a setup where agents live for many periods and realistically have a hump-shaped income profile through life. This motivates a look at the financial market and more refined notions related to intertemporal choice.

A perfect loan market Consider a given household or, more generally, a given *contractor*. Suppose the contractor at a given date t wants to take a loan or provide loans to others at the going interest rate, i_t, measured in money terms. So two contractors are involved, a *borrower* and a *lender*. Let the market conditions satisfy the following four criteria:

(a) the contractors face the same interest rate whether borrowing or lending (that is, monitoring, administration, and other transaction costs are absent);

(b) there are many contractors on each side and none of them believe to be able to influence the interest rate (the contractors are price takers in the loan market);

(c) there are no borrowing restrictions other than the requirement on the part of the borrower to comply with her financial commitments;

(d) the lender faces no default risk (contracts can always be enforced, i.e., the borrower can somehow cost-less be forced to repay the debt with interest on the conditions specified in the contract).

A loan market satisfying these idealized conditions is called a *perfect loan market*. In such a market,

1. various payment streams can be subject to comparison in a simple way; if they have the same present value (PV for short), they are equivalent;

2. any payment stream can be converted into another one with the same present value;

3. payment streams can be compared with the value of stocks.

Consider a payment stream $\{x_t\}_{t=0}^{T-1}$ over T periods, where x_t is the payment in currency at the *end* of period t. Period t runs from time t to time $t+1$ for $t = 0, 1, ..., T-1$. We *ignore uncertainty* and so i_t is the interest rate on a risk-less loan from time t to time $t+1$. Then the present value, PV_0, as seen from the beginning of period 0, of the payment stream is defined as[1]

$$PV_0 = \frac{x_0}{1+i_0} + \frac{x_1}{(1+i_0)(1+i_1)} + \cdots + \frac{x_{T-1}}{(1+i_0)(1+i_1)\cdots(1+i_{T-1})}. \qquad (9.1)$$

If Ms. Jones is entitled to the income stream $\{x_t\}_{t=0}^{T-1}$ and at time 0 wishes to buy a durable consumption good of value PV_0, she can borrow this amount and use a part of the income stream $\{x_t\}_{t=0}^{T-1}$ to repay the debt with interest over the periods $t = 0, 1, 2, ..., T-1$. In general, when Jones wishes to have a time profile on the payment stream different from the income stream, she can attain this through appropriate transactions in the loan market, leaving her with any stream of payments of the same present value as the given income stream.

Real versus nominal rate of return In this chapter we maintain the assumption of perfect competition in all markets, i.e., households take all prices as given from the markets. In the *absence of uncertainty,* the various assets (real capital, stocks, loans etc.) in which households invest give the same rate of return in equilibrium. The good which is traded in the loan market can be interpreted as a (risk-less) *bond.* The borrower issues bonds and the lender buys them. In this chapter all bonds are assumed to be short-term, i.e., one-period bonds. For every unit of account borrowed at the end of period $t-1$, the borrower pays back with certainty (1 + short-term interest rate) units of account at the end of period t. If a borrower wishes to maintain debt through several periods, new bonds are issued at the end of the current period and the obtained loans are spent rolling over the older loans at the going market interest rate. For the lender, who lends in several periods, this is equivalent to offering a variable-rate demand deposit like in a bank.[2]

Our analysis will be in real terms, that is, inflation-corrected terms. In principle the unit of account is a fixed bundle of consumption goods. In the simple macroeconomic models to be studied in this and most subsequent chapters, such

[1] We use “present value” as synonymous with “present discounted value”. As usual our timing convention is such that PV_0 denotes the time-0 value of the payment stream, including the discounted value of the payment (or dividend) indexed by 0.

[2] Unless otherwise specified, this chapter uses terms like “loan market”, “credit market”, and “bond market” interchangeably. As uncertainty is ignored, it does not matter whether we have personalized loan contracts or tradable securities in mind.

a bundle is reduced to *one* consumption good. The models simply assume there *is* only one consumption good in the economy. In fact, there will only be *one produced good*, "the" output good, which can be used for both consumption and capital investment. Whether our unit of account is seen as the consumption good or the output good is thus immaterial.

The *real* (net) rate of return on an investment is the rate of return in units of the output good. More precisely, the *real rate of return* in period t, r_t, is the (proportionate) rate at which the *real* value of an investment, made at the end of period $t-1$, has grown after one period.

The link between this rate of return and the more commonplace concept of a nominal rate of return is the following. Imagine that at the end of period $t-1$ you make a bank deposit of value V_t euro. The *real value* of the deposit when you invest is then V_t/P_{t-1}, where P_{t-1} is the price in euro of the output good at the end of period $t-1$. If the nominal short-term interest rate is i_t, the deposit is worth $V_{t+1} = V_t(1+i_t)$ euro at the end of period t. By definition of r_t, the factor by which the deposit in real terms has expanded is

$$1 + r_t = \frac{V_{t+1}/P_t}{V_t/P_{t-1}} = \frac{V_{t+1}/V_t}{P_t/P_{t-1}} = \frac{1+i_t}{1+\pi_t}, \tag{9.2}$$

where $\pi_t \equiv (P_t - P_{t-1})/P_{t-1}$ is the inflation rate in period t. So the real (net) *rate* of return on the investment is $r_t = (i_t - \pi_t)/(1+\pi_t) \approx i_t - \pi_t$ for i_t and π_t "small". The number $1 + r_t$ is called the real interest *factor* and measures the rate at which current units of output can be traded for units of output one period later.

In the remainder of this chapter we will think in terms of *real* values and completely ignore monetary aspects of the economy.

9.2 Maximizing discounted utility in discrete time

As mentioned, the consumption/saving problem faced by the household is assumed to involve only one consumption good. The composition of consumption in each period is not part of the problem. What remains is the question how to distribute consumption over time.

The intertemporal utility function

A plan for consumption in the periods $0, 1, ..., T-1$ is denoted $\{c_t\}_{t=0}^{T-1}$, where c_t is the consumption in period t. We say the plan has *time horizon* T. Period 0 ("the initial period") need not refer to the "birth" of the household but is just an arbitrary period within the lifetime of the household.

We assume the preferences of the household can be represented by a time-separable intertemporal utility function with a constant utility discount rate and no utility from leisure. The latter assumption implies that the labor supply of the household in each period is inelastic. The time-separability itself just means that the intertemporal utility function is additive, i.e., $U(c_0, c_1, \ldots, c_{T-1}) = u^{(0)}(c_0) + u^{(1)}(c_1) + \ldots + u^{(T-1)}(c_{T-1})$, where $u^{(t)}(c_t)$ is the utility contribution from period-t consumption, $t = 0, 1, \ldots, T-1$. In addition we assume *geometric utility discounting,* meaning that utility obtained t periods ahead is converted into a present equivalent by multiplying by the *discount factor* $(1+\rho)^{-t}$, where ρ is a constant utility *discount rate.* So $u^{(t)}(c_t) = u(c_t)(1+\rho)^{-t}$, where $u(c)$ is a time-independent period utility function. Together, these two assumptions amount to

$$U(c_0, c_1, \cdots, c_{T-1}) = u(c_0) + \frac{u(c_1)}{1+\rho} + \ \ldots \ + \frac{u(c_{T-1})}{(1+\rho)^{T-1}} = \sum_{t=0}^{T-1} \frac{u(c_t)}{(1+\rho)^t}. \quad (9.3)$$

The period utility function is assumed to satisfy $u'(c) > 0$ and $u''(c) < 0$. As explained in Box 9.1, only *linear* positive transformations of the period utility function are admissible.

As (9.3) indicates, the number $1+\rho$ tells how many units of utility in the next period the household insists on "in return" for a decrease of one unit of utility in the current period. So, a $\rho > 0$ will reflect that if the chosen level of consumption is the same in two periods, then the individual always appreciates a marginal unit of consumption higher the earlier it arrives. This explains why ρ is named the *rate of time preference* or, even more to the point, the *rate of impatience.* The utility discount factor, $1/(1+\rho)^t$, indicates how many units of utility the household is at most willing to give up in period 0 to get one additional unit of utility in period t.[3] Admittedly, this assumption that the utility discount rate is a *constant* is questionable. There is a growing body of evidence suggesting that the utility discount rate is usually *not* a constant, but declining with the time distance from the current period to the future periods within the horizon. This phenomenon is referred to as "present bias" or "hyperbolic discounting". Justified or not, macroeconomics often, as a first approach, ignores the problem and assumes a constant ρ to keep things simple. Most of the time we will follow this practice.

It is generally believed that human beings are impatient and that realism

[3]Multiplying through in (9.3) by $(1+\rho)^{-1}$ would make the objective function appear in a way similar to (9.1) in the sense that also the first term in the sum becomes discounted. At the same time the ranking of all possible alternative consumption paths would remain unaffected. For ease of notation, however, we use the form (9.3) which is more standard. Economically, there is no difference.

therefore speaks for assuming ρ positive.[4] The results to be derived in this chapter do not require a positive ρ, however. So we just impose the definitional constraint in discrete time: $\rho > -1$.

Box 9.1. Admissible transformations of the period utility function

When preferences, as assumed here, can be represented by *discounted utility*, the concept of utility appears at two levels. The function U in (9.3) is defined on the set of alternative feasible consumption paths and corresponds to an ordinary utility function in general microeconomic theory. That is, U will express the same ranking between alternative consumption paths as any increasing transformation of U. The period utility function, u, defined on the consumption in a single period, is a less general concept, requiring that reference to "utility of period utility units" is legitimate. That is, the *size* (not just the sign) of the difference in terms of period utility between two outcomes has significance for choices. Indeed, the essence of the discounted utility hypothesis is that we have, for example,

$$u(c_0) - u(c_0') > 0.95 \left[u(c_1') - u(c_1)\right] \Leftrightarrow (c_0, c_1) \succ (c_0', c_1'),$$

meaning that the household, having a utility discount factor $1/(1+\rho) = 0.95$, strictly prefers consuming (c_0, c_1) to (c_0', c_1') in the first two periods, if and only if the utility differences satisfy the indicated inequality. (The notation $x \succ y$ means that x is strictly preferred to y.)

Only a *linear* positive transformation of the utility function u, that is, $v(c) = au(c) + b$, where $a > 0$, leaves the ranking of all possible alternative consumption paths, $\{c_t\}_{t=0}^{T-1}$, unchanged. This is because a linear positive transformation does not affect the *ratios* of marginal utilities (the marginal rates of substitution across time).

The saving problem in discrete time

Suppose the household considered has income from two sources: work and financial wealth. Let a_t denote the real value of (net) financial wealth held by the household at the beginning of period t (a for "assets"). We treat a_t as predetermined at time t and in this respect similar to a variable-interest deposit with a bank. The initial financial wealth, a_0, is thus *given*, independently of what in

[4] If uncertainty were included in the model, $(1+\rho)^{-1}$ might be interpreted as (roughly) reflecting the probability of surviving to the next period. In this perspective, $\rho > 0$ is definitely a plausible assumption.

interest rate is formed in the loan market. And a_0 can be positive as well as negative (in the latter case the household is initially in debt).

The labor income of the household in period t is denoted $w_t \geq 0$ and may follow a typical life-cycle pattern, first rising, then more or less stationary, and finally vanishing due to retirement. Thus, in contrast to previous chapters where w_t denoted the real wage per unit of labor, here a broader interpretation of w_t is allowed. Whatever the time profile of the amount of labor delivered by the household through life, in this chapter, where the focus is on individual saving, we regard this time profile, as well as the hourly wage as exogenous. The present interpretation of w_t will coincide with the one in the other chapters if we imagine that the household in each period delivers one unit of labor.

To avoid corner solutions, we impose the No Fast Assumption $\lim_{c\to 0} u'(c) = \infty$. Since uncertainty is by assumption ruled out, the problem is to choose a plan $(c_0, c_1, \ldots, c_{T-1})$ so as to maximize

$$U = \sum_{t=0}^{T-1} u(c_t)(1+\rho)^{-t} \qquad \text{s.t.} \tag{9.4}$$

$$c_t \geq 0, \tag{9.5}$$

$$a_{t+1} = (1+r_t)a_t + w_t - c_t, \qquad a_0 \text{ given}, \tag{9.6}$$

$$a_T \geq 0, \tag{9.7}$$

where r_t is the interest rate. The control region (9.5) reflects the definitional non-negativity of the control variable, consumption. The dynamic equation (9.6) is an accounting relation telling how financial wealth moves over time. Indeed, income in period t is $r_t a_t + w_t$ and saving is then $r_t a_t + w_t - c_t$. Since saving is by definition the same as the increase in financial wealth, $a_{t+1} - a_t$, we obtain (9.6). Finally, the terminal condition (9.7) is a solvency requirement that no financial debt be left over at the terminal date, T. We may call this decision problem the *standard discounted utility maximization problem* with a *perfect loan market* and *no uncertainty.*

Solving the problem

To solve the problem, let us use the *substitution method.*[5] From (9.6) we have $c_t = (1+r_t)a_t + w_t - a_{t+1}$, for $t = 0, 1, \ldots, T-1$. Substituting this into (9.4), we obtain a function of $a_1, a_2, \ldots, a_T$. Since $u' > 0$, saturation is impossible and so an optimal solution cannot have $a_T > 0$. Hence we can put $a_T = 0$ and the problem is reduced to an essentially unconstrained problem of maximizing a function $\tilde{U}$

[5] Alternative methods include the *Maximum Principle* as described in the previous chapter or *Dynamic Programming* as described in Math Tools.

w.r.t. $a_1, a_2, \ldots, a_{T-1}$. Thereby we indirectly choose $c_0, c_1, \ldots, c_{T-2}$. Given a_{T-1}, consumption in the last period is trivially given as

$$c_{T-1} = (1 + r_{T-1})a_{T-1} + w_{T-1},$$

ensuring

$$a_T = 0, \tag{9.8}$$

the terminal optimality condition, necessary when $u'(c) > 0$ for all $c \geq 0$ (saturation impossible).

To obtain first-order conditions we put the partial derivatives of $\tilde{U}$ w.r.t. a_{t+1}, $t = 0, 1, \ldots, T - 2$, equal to 0:

$$\frac{\partial \tilde{U}}{\partial a_{t+1}} = (1 + \rho)^{-t} \left[u'(c_t) \cdot (-1) + (1 + \rho)^{-1} u'(c_{t+1})(1 + r_{t+1}) \right] = 0.$$

Reordering gives the Euler equations describing the trade-off between consumption in two succeeding periods,

$$u'(c_t) = (1 + \rho)^{-1} u'(c_{t+1})(1 + r_{t+1}), \qquad t = 0, 1, 2, ..., T - 2. \tag{9.9}$$

One of the implications of this condition is that

$$\rho \lesseqqgtr r_{t+1} \text{ causes } u'(c_t) \gtreqqless u'(c_{t+1}), \text{ i.e., } c_t \lesseqqgtr c_{t+1} \tag{9.10}$$

in the optimal plan (due to $u'' < 0$). Absent uncertainty the optimal plan entails either increasing, constant, or decreasing consumption over time depending on whether the rate of time preference is below, equal to, or above the rate of return on saving.

Interpretation The interpretation of (9.9) is as follows. Let the consumption path $(c_0, c_1, \ldots, c_{T-1})$ be our "reference path". Imagine an alternative path which coincides with the reference path except for the periods t and $t + 1$. If it is possible to obtain a higher total discounted utility than in the reference path by varying c_t and c_{t+1} within the constraints (9.5), (9.6), and (9.7), at the same time as consumption in the other periods is kept unchanged, then the reference path cannot be optimal. That is, "local optimality" is a necessary condition for "global optimality". So the optimal plan must be such that the current utility loss by decreasing consumption c_t by one unit equals the discounted expected utility gain next period by having $1 + r_{t+1}$ extra units available for consumption, namely the gross return on saving one more unit in the current period.

A more concrete interpretation, avoiding the notion of "utility units", is obtained by rewriting (9.9) as

$$\frac{u'(c_t)}{(1+\rho)^{-1}u'(c_{t+1})} = 1 + r_{t+1}. \tag{9.11}$$

The left-hand side indicates the marginal rate of substitution, MRS, of period-$(t+1)$ consumption for period-t consumption, namely the increase in period-$(t+1)$ consumption needed to compensate for a one-unit marginal decrease in period-t consumption:

$$MRS_{t+1,t} = -\frac{dc_{t+1}}{dc_t}|_{U=\bar{U}} = \frac{u'(c_t)}{(1+\rho)^{-1}u'(c_{t+1})}. \tag{9.12}$$

And the right-hand side of (9.11) indicates the marginal rate of transformation, MRT, which is the rate at which the loan market allows the household to shift consumption from period t to period $t+1$.

So, in an optimal plan MRS must equal MRT. This has implications for the time profile of optimal consumption as indicated by the relationship in (9.10). The Euler equations, (9.9), can also be seen in a comparative perspective. Consider two alternative values of r_{t+1}. The higher interest rate will induce a *negative substitution effect* on current consumption, c_t. There is also an *income effect*, however, and this goes in the *opposite* direction. The higher interest rate makes the present value of a given consumption plan lower. This allows more consumption in all periods for a given total wealth. Moreover, there is generally a third effect of the rise in the interest rate, a *wealth effect.* As indicated by the intertemporal budget constraint in (9.20) below, total wealth includes the present value of expected future after-tax labor earnings and this present value depends *negatively* on the interest rate, cf. (9.15) below.

From the formula (9.12) we see one of the reasons that the assumption of a *constant* utility discount rate is *convenient* (but also restrictive). The marginal rate of substitution between consumption this period and consumption next period is independent of the level of consumption as long as this level is the same in the two periods.

The formula for MRS between consumption this period and consumption *two* periods ahead is

$$MRS_{t+2,t} = -\frac{dc_{t+2}}{dc_t}|_{U=\bar{U}} = \frac{u'(c_t)}{(1+\rho)^{-2}u'(c_{t+2})}.$$

This displays one of the reasons that the time-separability of the intertemporal utility function is a *strong* assumption. It implies that the trade-off between consumption this period and consumption two periods ahead is independent of consumption in the interim.

Deriving the consumption function when utility is CRRA The first-order conditions (9.9) tell us about the relative consumption levels over time, not the absolute level. The latter is determined by the condition that initial consumption, c_0, must be highest possible, given that the first-order conditions *and* the constraints (9.6) and (9.7) must be satisfied.

To find an explicit solution we have to specify the period utility function. As an example we choose the CRRA function $u(c) = c^{1-\theta}/(1-\theta)$, where $\theta > 0$.[6] Moreover we simplify by assuming $r_t = r$, a constant > -1. Then the Euler equations take the form $(c_{t+1}/c_t)^\theta = (1+r)(1+\rho)^{-1}$ so that

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+r}{1+\rho}\right)^{1/\theta} \equiv \gamma, \tag{9.13}$$

and thereby $c_t = \gamma^t c_0$, $t = 0, 1, \ldots, T-1$. Substituting into the accounting equation (9.6), we thus have $a_{t+1} = (1+r)a_t + w_t - \gamma^t c_0$. By backward substitution we find the solution of this difference equation to be

$$a_t = (1+r)^t \left[a_0 + \sum_{i=0}^{t-1} (1+r)^{-(i+1)}(w_i - \gamma^i c_0) \right].$$

Optimality requires that the left-hand side of this equation vanishes for $t = T$. So we can solve for c_0 :

$$c_0 = \frac{1+r}{\sum_{i=0}^{T-1}\left(\frac{\gamma}{1+r}\right)^i} \left[a_0 + \sum_{i=0}^{T-1}(1+r)^{-(i+1)} w_i \right] = \frac{1+r}{\sum_{i=0}^{T-1}\left(\frac{\gamma}{1+r}\right)^i}(a_0 + h_0), \tag{9.14}$$

where we have inserted the human wealth of the household (present value of expected lifetime labor income) as seen from time zero:

$$h_0 = \sum_{i=0}^{T-1}(1+r)^{-(i+1)} w_i. \tag{9.15}$$

Thus (9.14) says that initial consumption is proportional to initial *total wealth*, the sum of financial wealth and human wealth at time 0. To allow for positive consumption we need $a_0 + h_0 > 0$.

[6] In later sections of this chapter we let the time horizon of the decision maker go to infinity. To ease convergence of an infinite sum of discounted utilities, it is an advantage not to have to bother with additive constants in the period utilities and therefore we write the CRRA function as $c^{1-\theta}/(1-\theta)$ instead of the form, $(c^{1-\theta}-1)/(1-\theta)$, introduced in Chapter 3. As implied by Box 9.1, the two forms represent the same preferences.

In (9.14) γ is not one of the original parameters, but a derived parameter. To express the consumption function only in terms of the original parameters, note that, by (9.14), the propensity to consume out of total wealth depends on:

$$\sum_{i=0}^{T-1}\left(\frac{\gamma}{1+r}\right)^i = \begin{cases} \frac{1-\left(\frac{\gamma}{1+r}\right)^T}{1-\frac{\gamma}{1+r}} & \text{when } \gamma \neq 1+r, \\ T & \text{when } \gamma = 1+r, \end{cases} \tag{9.16}$$

where the result for $\gamma \neq 1+r$ follows from the formula for the sum of a finite geometric series. Inserting this together with (9.13) into (9.14), we end up with the expression

$$c_0 = \begin{cases} \frac{(1+r)\left[1-(1+\rho)^{-1/\theta}(1+r)^{(1-\theta)/\theta}\right]}{1-(1+\rho)^{-T/\theta}(1+r)^{(1-\theta)T/\theta}}(a_0+h_0) & \text{when } \left(\frac{1+r}{1+\rho}\right)^{1/\theta} \neq 1+r, \\ \frac{1+r}{T}(a_0+h_0) & \text{when } \left(\frac{1+r}{1+\rho}\right)^{1/\theta} = 1+r. \end{cases} \tag{9.17}$$

This, together with (9.14), thus says:

Result 1: Consumption is proportional to total wealth, and the factor of proportionality, often called the *marginal propensity to consume out of wealth*, depends on the interest rate r, the time horizon T, and the preference parameters ρ and θ, where ρ is the impatience rate and θ is the strength of the preference for consumption smoothing, respectively.

For the subsequent periods we have from (9.13) that

$$c_t = c_0\left(\left(\frac{1+r}{1+\rho}\right)^{1/\theta}\right)^t, \qquad t = 1,\ldots,T-1. \tag{9.18}$$

EXAMPLE 1 Consider the special case $\theta = 1$ (i.e., $u(c) = \ln c$) together with $\rho > 0$. The upper case in (9.17) is here the relevant one and period-0 consumption will be

$$c_0 = \frac{(1+r)(1-(1+\rho)^{-1})}{1-(1+\rho)^{-T}}(a_0+h_0) \qquad \text{for } \theta = 1.$$

We see that $c_0 \to (1+r)\rho(1+\rho)^{-1}(a_0+h_0)$ for $T \to \infty$, assuming the right-hand side of (9.15) converges for $T \to \infty$.

We have assumed that payment for consumption occurs at the end of the period at the price 1 per consumption unit. To compare with the corresponding result in continuous time with continuous compounding (see Section 9.4), we might want to have initial consumption in the same present value terms as a_0 and h_0. That is, we consider $\tilde{c}_0 \equiv c_0(1+r)^{-1} = \rho(1+\rho)^{-1}(a_0+h_0)$ for $T \to \infty$. □

So far the expression (9.17) is only a *candidate* consumption function. But in view of strict concavity of the objective function, (9.17) is indeed the unique optimal solution when $a_0 + h_0 > 0$.

The conclusion from (9.17) and (9.18) is that *consumers look beyond current income.* More precisely:

Result 2: Under the idealized conditions assumed, including a perfect loan market and perfect foresight, and given the marginal propensity to consume out of total wealth shown in (9.17), the time profile of consumption is determined by the total wealth and the interest rate (relative to impatience corrected for the preference for consumption smoothing). The time profile of *income* does not matter because consumption can be smoothed over time by drawing on the loan market.

EXAMPLE 2 Consider the special case $\rho = r > 0$ (and still $\theta = 1$). Again the upper case in (9.17) is the relevant one and period-0 consumption will be

$$c_0 = \frac{r}{1-(1+r)^{-T}}(a_0 + h_0).$$

We see that $c_0 \to r(a_0 + h_0)$ for $T \to \infty$, assuming the right-hand side of (9.15) converges for $T \to \infty$. So, with an infinite time horizon current consumption equals the interest on total current wealth. By consuming this the individual or household maintains total wealth intact. This consumption function provides an interpretation of Milton Friedman's *permanent income hypothesis.* Friedman defined "permanent income" as "the amount a consumer unit could consume (or believes it could) while maintaining its wealth intact" (Friedman, 1957). The key point of Friedman's theory was the idea that a random change in current income only affects current consumption to the extent that it affects "permanent income". Replacing Friedman's awkward term "permanent income" by the straightforward "total wealth", this feature is a general aspect of all consumption functions considered in this chapter. In contrast to the theory in this chapter, however, Friedman emphasized credit market imperfections and thought of a "subjective income discount rate" of as much as 33% per year. His interpretation of the empirics was that households adopt a much shorter "horizon" than the remainder of their expected lifetimes (Friedman, 1963, Carroll 2001). □

If the real interest rate varies over time, the discount factor $(1 + r)^{-(i+1)}$ for a payment made at the end of period i is replaced by $\Pi_{j=0}^{i}(1 + r_j)^{-1}$.

Alternative approach based on the intertemporal budget constraint

There is another approach to the household's saving problem. With its choice of consumption plan the household must act in conformity with its intertemporal budget constraint (IBC for short). The present value of the consumption plan $(c_1, ..., c_{T-1})$, as seen from time zero, is

$$PV(c_0, c_1, ..., c_{T-1}) \equiv \sum_{t=0}^{T-1} \frac{c_t}{\Pi_{\tau=0}^{t}(1 + r_\tau)}. \tag{9.19}$$

This value cannot exceed the household's total initial wealth, $a_0 + h_0$. So the household's *intertemporal budget constraint* is

$$\sum_{t=0}^{T-1} \frac{c_t}{\Pi_{\tau=0}^{t}(1 + r_\tau)} \leq a_0 + h_0. \tag{9.20}$$

In this setting the household's problem is to choose its consumption plan so as to maximize U in (9.4) subject to this budget constraint.

This way of stating the problem is equivalent to the approach above based on the dynamic budget condition (9.6) and the solvency condition (9.7). Indeed, given the accounting equation (9.6), the consumption plan of the household will satisfy the intertemporal budget constraint (9.20) if and only if it satisfies the solvency condition (9.7). And there will be strict equality in the intertemporal budget constraint if and only if there is strict equality in the solvency condition (the proof is similar to that of a similar claim relating to the government sector in Chapter 6.2).

Moreover, since in our specific saving problem saturation is impossible, an optimal solution must imply strict equality in (9.20). So it is straightforward to apply the substitution method also within the IBC approach. Alternatively one can introduce the *Lagrange function* associated with the problem of maximizing $U = \sum_{t=0}^{T-1}(1 + \rho)^{-t}u(c_t)$ s.t. (9.20) with strict equality.

Infinite time horizon In the Ramsey model of the next chapter, the idea is used that households may have an *infinite* time horizon. One interpretation of this is that parents care about their children's future welfare and leave bequests accordingly. This gives rise to a series of intergenerational links. The household is then seen as a family dynasty with a time horizon beyond the lifetime of the current members of the family. Barro's bequest model in Chapter 7 is an application of this idea. Given a sufficiently large rate of time preference, it is ensured that the sum of achievable discounted utilities over an infinite horizon is bounded from above.

One could say, of course, that infinity is a long time. The sun will eventually, in some billion years, burn out and life on earth become extinct. Nonetheless, there are several reasons that an infinite time horizon may provide a convenient substitute for finite but remote horizons. First, in many cases the solution to an optimization problem for T "large" is in a major part of the time horizon close to the solution for $T \rightarrow \infty$.[7] Second, an infinite time horizon tends to ease aggregation because at any future point in time, remaining time is still infinite. Third, an infinite time horizon may be a convenient notion when in any given period there is a always a positive probability that there will be a next period to be concerned about. This probability may be low, but this can be reflected in a high effective utility discount rate. This idea will be applied in chapters 12 and 13.

We may perform the transition to infinite horizon by letting $T \rightarrow \infty$ in the objective function, (9.4) and the intertemporal budget constraint, (9.20). On might think that, in analogy of (9.8) for the case of finite T, the terminal optimality condition for the case of infinite horizon is $\lim_{T\rightarrow\infty} a_T = 0$. This is generally not so, however. The reason is that with infinite horizon there is no final date where all debt must be settled. The terminal optimality condition in the present problem with a perfect loan market is simply equivalent to the condition that the intertemporal budget constraint should hold with strict equality.

Like with finite time horizon, the saving problem with infinite time horizon may alternatively be framed in terms of a series of dynamic period-by-period budget identities, in the form (9.6), together with the borrowing limit known as the No-Ponzi-Game condition:

$$\lim_{t\rightarrow\infty} a_t \Pi_{i=0}^{t-1}(1+r_i)^{-1} \geq 0.$$

As we saw in Chapter 6.5.2, such a "flow" formulation of the problem is equivalent to the formulation based on the intertemporal budget constraint. We also recall from Chapter 6 that the name Ponzi refers to a guy, Charles Ponzi, who in Boston in the 1920s temporarily became very rich by a loan arrangement based on the chain letter principle. The fact that debts grow without bounds is irrelevant for the lender *if* the borrower can always find new lenders and use their outlay to pay off old lenders with the contracted interest. In the real world, endeavours to establish this sort of financial eternity machine sooner or later break down because the flow of new lenders dries up. Such financial arrangements, in everyday speech known as pyramid companies, are universally illegal.[8] It is exactly such

[7] The turnpike proposition in Chapter 8 exemplifies this.

[8] A related Danish instance, though on a modest scale, could be read in the Danish newpaper *Politiken* on the 21st of August 1992. "A twenty-year-old female student from Tylstrup in Northern Jutland is charged with fraud. In an ad she offered to tell the reader, for 200 DKK, how to make easy money. Some hundred people responded and received the reply: do like me".

A more serious present day example is the Wall Street stockbroker, Bernard Madoff, who admitted a Ponzi scheme that is considered to be the largest financial fraud in U.S. history. In 2009 Madoff was sentenced to 150 years in prison. Other examples of large-scale Ponzi games appeared in Albania 1995-97 and Ukraine 2008.

arrangements the No-Ponzi-Game condition precludes.

The terminal optimality condition, known as a *transversality condition*, can be shown[9] to be

$$\lim_{t\to\infty}(1+\rho)^{-(t-1)}u'(c_{t-1})a_t = 0.$$

9.3 Transition to continuous time analysis

In the formulation of a model we have a choice between putting the model in period terms or in continuous time. In the former case, denoted period analysis or discrete time analysis, the run of time is divided into successive periods of equal length, taken as the time-unit. We may index the periods by $i = 0, 1, 2, ...$ Thus, in period analysis financial wealth accumulates according to

$$a_{i+1} - a_i = s_i, \qquad a_0 \text{ given},$$

where s_i is (net) saving in period i.

Multiple compounding per year

With time flowing continuously, we let $a(t)$ refer to financial wealth at time t. Similarly, $a(t + \Delta t)$ refers to financial wealth at time $t + \Delta t$. To begin with, let Δt equal one time unit. Then $a(i\Delta t)$ equals $a(i)$ and is of the same value as a_i. Consider the *forward* first difference in a, $\Delta a(t) \equiv a(t+\Delta t)-a(t)$. It makes sense to consider this change in a in relation to the length of the time interval involved, that is, to consider the *ratio* $\Delta a(t)/\Delta t$.

Now, *keep the time unit unchanged*, but let the length of the time interval $[t, t + \Delta t)$ approach zero, i.e., let $\Delta t \to 0$. When a is a differentiable function of t, we have

$$\lim_{\Delta t\to 0}\frac{\Delta a(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{a(t+\Delta t)-a(t)}{\Delta t} = \frac{da(t)}{dt},$$

where $da(t)/dt$, often written $\dot{a}(t)$, is known as the *time derivative of* a at the point t. Wealth accumulation in continuous time can then be written

$$\dot{a}(t) = s(t), \qquad a(0) = a_0 \text{ given}, \tag{9.21}$$

where $s(t)$ is the saving flow (saving intensity) at time t. For Δt "small" we have the approximation $\Delta a(t) \approx \dot{a}(t)\Delta t = s(t)\Delta t$. In particular, for $\Delta t = 1$ we have $\Delta a(t) = a(t + 1) - a(t) \approx s(t)$.

[9] The proof is similar to that given in Chapter 8, Appendix C.

As time unit choose one year. Going back to discrete time, if wealth grows at a constant rate g per year, then after i periods of length one year, with annual compounding, we have

$$a_i = a_0(1+g)^i, \qquad i = 0, 1, 2, \dots . \tag{9.22}$$

If instead compounding (adding saving to the principal) occurs n times a year, then after i periods of length $1/n$ year and a growth rate of g/n per such period, we have

$$a_i = a_0(1+\frac{g}{n})^i. \tag{9.23}$$

With t still denoting time measured in years passed since date 0, we have $i = nt$ periods. Substituting into (9.23) gives

$$a(t) = a_{nt} = a_0(1+\frac{g}{n})^{nt} = a_0\left[(1+\frac{1}{m})^m\right]^{gt}, \qquad \text{where } m \equiv \frac{n}{g}.$$

We keep g and t fixed, but let $n \to \infty$. Thus $m \to \infty$. In the limit there is continuous compounding and we get

$$a(t) = a_0 e^{gt}, \tag{9.24}$$

where e is a mathematical constant called the base of the natural logarithm and defined as $e \equiv \lim_{m\to\infty}(1+1/m)^m \simeq 2.7182818285....$

The formula (9.24) is the continuous-time analogue to the discrete time formula (9.22) with annual compounding. A geometric growth factor, $(1+g)^i$, is replaced by an exponential growth factor, e^{gt}, and this growth factor is valid for any t in the time interval $(-\tau_1, \tau_2)$ for which the growth rate of a equals the constant g (τ_1 and τ_2 being some positive real numbers).

We can also view the formulas (9.22) and (9.24) as the solutions to a difference equation and a differential equation, respectively. Thus, (9.22) is the solution to the linear difference equation $a_{i+1} = (1+g)a_i$, given the initial value a_0. And (9.24) is the solution to the linear differential equation $\dot{a}(t) = ga(t)$, given the initial condition $a(0) = a_0$. Now consider a time-dependent growth rate, $g(t)$, a continuous function of t. The corresponding differential equation is $\dot{a}(t) = g(t)a(t)$ and it has the solution

$$a(t) = a(0)e^{\int_0^t g(\tau)d\tau}, \tag{9.25}$$

where the exponent, $\int_0^t g(\tau)d\tau$, is the definite integral of the function $g(\tau)$ from 0 to t. The result (9.25) is called the *accumulation formula* in continuous time and the factor $e^{\int_0^t g(\tau)d\tau}$ is called the *growth factor* or the *accumulation factor*.[10]

[10] Sometimes the accumulation factor with time-dependent growth rate is written in a different way, see Appendix B.

Compound interest and discounting in continuous time

Let $r(t)$ denote the *short-term real interest rate in continuous time* at time t. To clarify what is meant by this, consider a deposit of $V(t)$ euro in a bank at time t. If the general price level in the economy at time t is $P(t)$ euro, the *real* value of the deposit is $a(t) = V(t)/P(t)$ at time t. By definition the *real rate of return* on the deposit in continuous time (with continuous compounding) at time t is the (proportionate) instantaneous rate at which the real value of the deposit expands per time unit when there is no withdrawal from the account. Thus, if the instantaneous nominal interest rate is $i(t)$, we have $\dot{V}(t)/V(t) = i(t)$ and so, by the fraction rule in continuous time (cf. Appendix A),

$$r(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\dot{V}(t)}{V(t)} - \frac{\dot{P}(t)}{P(t)} = i(t) - \pi(t), \tag{9.26}$$

where $\pi(t) \equiv \dot{P}(t)/P(t)$ is the instantaneous inflation rate. In contrast to the corresponding formula in discrete time, this formula is exact. Sometimes $i(t)$ and $r(t)$ are referred to as the nominal and real *force of interest.*

Calculating the terminal value of the deposit at time $t_1 > t_0$, given its value at time t_0 and assuming no withdrawal in the time interval $[t_0, t_1]$, the accumulation formula (9.25) immediately yields

$$a(t_1) = a(t_0)e^{\int_{t_0}^{t_1} r(t)dt}.$$

When calculating *present values* in continuous time, we use compound discounting. We reverse the accumulation formula and go from the compounded or terminal value to the present value, $a(t_0)$. Similarly, given a consumption plan $(c(t))_{t=t_0}^{t_1}$, the present value of this plan as seen from time t_0 is

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-rt} dt, \tag{9.27}$$

presupposing a constant interest rate, r. Instead of the geometric discount factor, $1/(1+r)^t$, from discrete time analysis, we have here an exponential discount factor, $1/(e^{rt}) = e^{-rt}$, and instead of a sum, an integral. When the interest rate varies over time, (9.27) is replaced by

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-\int_{t_0}^{t} r(\tau)d\tau} dt.$$

In (9.27) $c(t)$ is discounted by $e^{-rt} \approx (1 + r)^{-t}$ for r "small". This might not seem analogue to the discrete-time discounting in (9.19) where it is c_{t-1} that is discounted by $(1 + r)^{-t}$, assuming a constant interest rate. When taking into account the timing convention that payment for c_{t-1} in period $t - 1$ occurs at the end of the period (= time t), there is no discrepancy, however, since the continuous-time analogue to this payment is $c(t)$.

The range for particular parameter values

The allowed range for parameters may change when we go from discrete time to continuous time with continuous compounding. For example, the usual equation for aggregate capital accumulation in continuous time is

$$\dot{K}(t) = I(t) - \delta K(t), \qquad K(0) = K_0 \text{ given}, \tag{9.28}$$

where $K(t)$ is the capital stock, $I(t)$ is the gross investment at time t and $\delta \geq 0$ is the (physical) capital depreciation rate. Unlike in period analysis, now $\delta > 1$ is conceptually allowed. Indeed, suppose for simplicity that $I(t) = 0$ for all $t \geq 0$; then (9.28) gives $K(t) = K_0 e^{-\delta t}$. This formula is meaningful for any $\delta \geq 0$. Usually, the time unit used in continuous time macro models is one year (or, in business cycle theory, rather a quarter of a year) and then a realistic value of δ is of course < 1 (say, between 0.05 and 0.10). However, if the time unit applied to the model is large (think of a Diamond-style OLG model), say 30 years, then $\delta > 1$ may fit better, empirically, if the model is converted into continuous time with the same time unit. Suppose, for example, that physical capital has a half-life of 10 years. With 30 years as our time unit, inserting into the formula $1/2 = e^{-\delta/3}$ gives $\delta = (\ln 2) \cdot 3 \simeq 2$.

In many simple macromodels, where the level of aggregation is high, the relative price of a unit of physical capital in terms of the consumption good is 1 and thus constant. More generally, if we let the relative price of the capital good in terms of the consumption good at time t be $p(t)$ and allow $\dot{p}(t) \neq 0$, then we have to distinguish between the physical depreciation of capital, δ, and the *economic depreciation*, that is, the loss in economic value of a machine per time unit. The economic depreciation will be $d(t) = p(t)\delta - \dot{p}(t)$, namely the economic value of the physical wear and tear (and technological obsolescence, say) minus the capital gain (positive or negative) on the machine.

Other variables and parameters that by definition are bounded from below in discrete time analysis, but not so in continuous time analysis, include rates of return and discount rates in general.

Stocks and flows

An advantage of continuous time analysis is that it forces the analyst to make a clear distinction between *stocks* (say wealth) and *flows* (say consumption or saving). Recall, a *stock* variable is a variable measured as a quantity at a given point in time. The variables $a(t)$ and $K(t)$ considered above are stock variables. A *flow* variable is a variable measured as quantity *per time unit* at a given point in time. The variables $s(t)$, $\dot{K}(t)$, and $I(t)$ are flow variables.

One can not add a stock and a flow, because they have *different denominations.* What is meant by this? The elementary measurement units in economics are *quantity units* (so many machines of a certain kind or so many liters of oil or so many units of payment, for instance) and *time units* (months, quarters, years). On the basis of these elementary units we can form *composite measurement units.* Thus, the capital stock, K, has the denomination "quantity of machines", whereas investment, I, has the denomination "quantity of machines per time unit" or, shorter, "quantity/time". A growth rate or interest rate has the denomination "(quantity/time)/quantity" = "time^{-1}". If we change our time unit, say from quarters to years, the value of a flow variable as well as a growth rate is changed, in this case quadrupled (presupposing annual compounding).

In continuous time analysis expressions like $K(t) + I(t)$ or $K(t) + \dot{K}(t)$ are thus illegitimate. But one can write $K(t + \Delta t) \approx K(t) + (I(t) - \delta K(t))\Delta t$, or $\dot{K}(t)\Delta t \approx (I(t) - \delta K(t))\Delta t$. In the same way, suppose a bath tub at time t contains 50 liters of water and that the tap pours $\frac{1}{2}$ liter per second into the tub for some time. Then a sum like $50\ \ell + \frac{1}{2}\ (\ell/\text{sec})$ does not make sense. But the *amount* of water in the tub after one minute is meaningful. This amount would be $50\ \ell + \frac{1}{2} \cdot 60\ ((\ell/\text{sec})\times\text{sec}) = 80\ \ell$. In analogy, economic flow variables in continuous time should be seen as *intensities* defined for every t in the time interval considered, say the time interval $[0, T)$ or perhaps $[0, \infty)$. For example, when we say that $I(t)$ is "investment" at time t, this is really a short-hand for "investment intensity" at time t. The actual investment in a time interval $[t_0, t_0 + \Delta t)$, i.e., the invested amount *during* this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} I(t)dt \approx I(t_0)\Delta t$. Similarly, the flow of individual saving, $s(t)$, should be interpreted as the saving *intensity* (or saving *density*), at time t. The actual saving in a time interval $[t_0, t_0 + \Delta t)$, i.e., the saved (or accumulated) amount during this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} s(t)dt$. If Δt is "small", this integral is approximately equal to the product $s(t_0) \cdot \Delta t$, cf. the hatched area in Fig. 9.1.

The notation commonly used in discrete time analysis blurs the distinction between stocks and flows. Expressions like $a_{i+1} = a_i + s_i$, without further comment, are usual. Seemingly, here a stock, wealth, and a flow, saving, are added. In fact, however, it is wealth at the beginning of period i and the saved *amount during* period i that are added: $a_{i+1} = a_i + s_i \cdot \Delta t$. The tacit condition is that the period length, Δt, is the time unit, so that $\Delta t = 1$. But suppose that, for example in a business cycle model, the period length is one quarter, but the time unit is one year. Then saving in quarter i is $s_i = (a_{i+1} - a_i) \cdot 4$ per year.

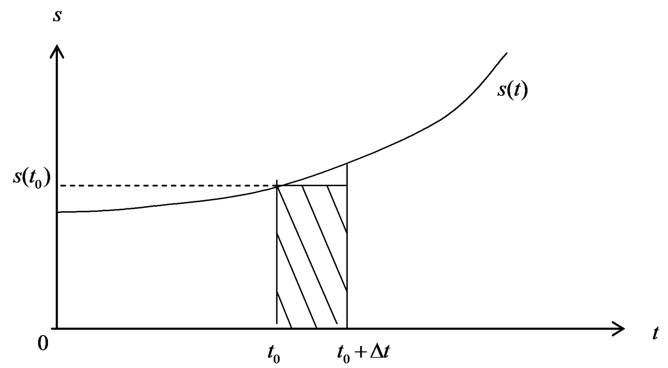


Figure 9.1: With Δt *small* the integral of $s(t)$ from t_0 to $t_0 + \Delta t \approx$ the hatched area.

The choice between discrete and continuous time formulation

In empirical economics, data typically come in discrete time form and data for flow variables typically refer to periods of constant length. One could argue that this discrete form of the data speaks for period analysis rather than continuous time modelling. And the fact that economic actors often think, decide, and plan in period terms, may seem a good reason for putting at least microeconomic analysis in period terms. Nonetheless real time is continuous. Moreover, as for instance Allen (1967) argued, it can hardly be said that the *mass* of economic actors think and decide with the same time distance between successive decisions and actions. In macroeconomics we consider the *sum* of the actions. In this perspective the continuous time approach has the advantage of allowing variation *within* the usually artificial periods in which the data are chopped up. In addition, centralized asset markets equilibrate very fast and respond almost immediately to new information. For such markets a formulation in continuous time seems a good approximation.

There is also a risk that a discrete time model may generate *artificial* oscillations over time. Suppose the "true" model of some mechanism is given by the differential equation

$$\dot{x} = \alpha x, \qquad \alpha < -1. \tag{9.29}$$

The solution is $x(t) = x(0)e^{\alpha t}$ which converges in a monotonic way toward 0 for $t \to \infty$. However, the analyst takes a discrete time approach and sets up the seemingly "corresponding" discrete time model

$$x_{t+1} - x_t = \alpha x_t.$$

This yields the difference equation $x_{t+1} = (1+\alpha)x_t$, where $1+\alpha < 0$. The solution is $x_t = (1+\alpha)^t x_0$, $t = 0, 1, 2, \ldots$. As $(1+\alpha)^t$ is positive when t is even and negative when t is odd, oscillations arise (together with divergence if $\alpha < -2$) in spite of the "true" model generating monotonous convergence towards the steady state $x^* = 0$.

This potential problem can always be avoided, however, by choosing a sufficiently *short* period length in the discrete time model. The solution to a differential equation can always be obtained as the limit of the solution to a corresponding difference equation for the period length approaching zero. In the case of (9.29), the approximating difference equation is $x_{i+1} = (1 + \alpha\Delta t)x_i$, where Δt is the period length, $i = t/\Delta t$, and $x_i = x(i\Delta t)$. By choosing Δt small enough, the solution comes arbitrarily close to the solution of (9.29). It is generally more difficult to go in the opposite direction and find a differential equation that approximates a given difference equation. But the problem is solved as soon as a differential equation has been found that has the initial difference equation as an approximating difference equation.

From the point of view of the economic contents, the choice between discrete time and continuous time may be a matter of taste. Yet, everything else equal, the clearer distinction between stocks and flows in continuous time than in discrete time speaks for the former. From the point of view of mathematical convenience, the continuous time formulation, which has worked so well in the natural sciences, is preferable. At least this is so in the absence of uncertainty. For problems where uncertainty is important, discrete time formulations are easier to work with unless one is familiar with stochastic calculus.[11]

9.4 Maximizing discounted utility in continuous time

9.4.1 The saving problem in continuous time

In continuous time the analogue to the intertemporal utility function, (9.3), is

$$U_0 = \int_0^T u(c(t))e^{-\rho t}dt. \tag{9.30}$$

In this context it is common to name the utility flow, u, the *instantaneous utility function*. We still assume that $u' > 0$ and $u'' < 0$. The analogue in continuous time to the intertemporal budget constraint (9.20) is

$$\int_0^T c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq a_0 + h_0, \tag{9.31}$$

[11] In the latter case, Nobel laureate Robert C. Merton argues in favor of a continuous time formulation (Merton, 1975).

where, as before, a_0 is the historically given initial financial wealth, while h_0 is the given human wealth,

$$h_0 = \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt. \tag{9.32}$$

The household's problem is then to choose a consumption plan $(c(t))_{t=0}^T$ so as to maximize discounted utility, U_0, subject to the budget constraint (9.31).

Infinite time horizon Transition to infinite horizon is performed by letting $T \to \infty$ in (9.30), (9.31), and (9.32). In the limit the household's, or dynasty's, problem becomes one of choosing a plan, $(c(t))_{t=0}^\infty$, which maximizes

$$U_0 = \int_0^\infty u(c(t))e^{-\rho t}dt \qquad \text{s.t.} \tag{9.33}$$

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq a_0 + h_0, \tag{IBC}$$

where h_0 emerges by letting T in (9.32) approach ∞. With an infinite horizon there may exist technically feasible paths along which the integrals in (9.30), (9.31), and (9.32) go to ∞ for $T \to \infty$. In that case maximization is not well-defined. However, the assumptions we are going to make when working with infinite horizon will guarantee that the integrals converge as $T \to \infty$ (or at least that *some* feasible paths have $-\infty < U_0 < \infty$, while the remainder have $U_0 = -\infty$ and are thus clearly inferior). The essence of the matter is that the rate of time preference, ρ, must be assumed sufficiently high.

Generally we define a person as *solvent* if she is able to meet her financial obligations as they fall due. Each person is considered "small" relative to the economy as a whole. As long as all agents in an economy with a perfect loan market remain "small", they will in general equilibrium remain solvent if and only if their net debt does not exceed the present value of future primary saving.[12] Denoting by d_0 net debt at time 0, i.e., $d_0 \equiv -a_0$, the solvency requirement as seen from time 0 is

$$d_0 \leq \int_0^\infty (w(t) - c(t))e^{-\int_0^t r(\tau)d\tau}dt,$$

where the right-hand side is the present value of future primary saving. By the definition in (9.32), we see that this requirement is identical to the intertemporal budget constraint (IBC) which consequently expresses solvency.

[12] By *primary* saving is meant the difference between current non-interest income and current consumption, where non-interest income means labor income and transfers after tax.

The budget constraint in flow terms

The method which is particularly apt for solving intertemporal decision problems in continuous time is based on the mathematical discipline *optimal control theory*. To apply the method, we have to convert the household's budget constraint from the present-value formulation considered above into flow terms.

By mere accounting, in every short time interval $(t,\, t+\Delta t)$ the household's consumption plus saving equals the household's total income, that is,

$$(c(t)+\dot{a}(t))\Delta t = (r(t)a(t)+w(t))\Delta t.$$

Here, $\dot{a}(t) \equiv da(t)/dt$ is the increase per time unit in financial wealth, and thereby the saving intensity, at time t (assuming no robbery). If we divide through by Δt and rearrange, we get for all $t \geq 0$

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t), \qquad a(0) = a_0 \text{ given.} \tag{9.34}$$

This equation in itself is just a dynamic budget identity. It tells how much and in which direction the financial wealth is changing due to the difference between current income and current consumption. The equation *per se* does not impose any restriction on consumption over time. If this equation were the only "restriction", one could increase consumption indefinitely by incurring an increasing debt without limits. It is not until we add the requirement of solvency that we get a *constraint*. When $T < \infty$, the relevant solvency requirement is $a(T) \geq 0$ (that is, no debt is left over at the terminal date). This is equivalent to satisfying the intertemporal budget constraint (9.31).

When $T \to \infty$, the relevant solvency requirement is the No-Ponzi-Game condition

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0. \tag{NPG}$$

This condition says that the present value of net debt, measured as $-a(t)$, infinitely far out in the future, is not permitted to be positive. We have the following equivalency:

PROPOSITION 1 *(equivalence of NPG condition and intertemporal budget constraint)* Let the time horizon be infinite and assume that the integral (9.32) remains finite for $T \to \infty$. Then, given the accounting relation (9.34), we have:
(i) the requirement (NPG) is satisfied if and only if the intertemporal budget constraint, (IBC), is satisfied; and
(ii) there is strict equality in (NPG) if and only if there is strict equality in (IBC).

Proof. See Appendix C.

The condition (NPG) does not preclude that the household, or family dynasty, can remain in debt. This would also be an unnatural requirement as the dynasty is infinitely-lived. The condition does imply, however, that there is an upper bound for the speed whereby debt can increase in the long term. The NPG condition says that in the long term, debts are not allowed to grow at a rate as high as (or higher than) the interest rate.

To understand the implication, consider the case with a constant interest rate $r > 0$. Assume that the household at time t has net debt $d(t) > 0$, i.e., $a(t) \equiv -d(t) < 0$. If $d(t)$ were persistently growing at a rate equal to or greater than the interest rate, (NPG) would be violated.[13] Equivalently, one can interpret (NPG) as an assertion that lenders will only issue loans if the borrowers in the long run cover their interest payments by other means than by taking up new loans. In this way, it is avoided that $\dot{d}(t) \geq rd(t)$ in the long run. In brief, the borrowers are not allowed to run a Ponzi Game.

9.4.2 Solving the saving problem

The household's consumption/saving problem is one of choosing a path for the *control variable* $c(t)$ so as to maximize a *criterion function,* in the form of an integral, subject to constraints that include a first-order differential equation where the control variable enters, namely (9.34). Choosing a time path for the control variable, this differential equation determines the evolution of the *state variable*, $a(t)$. Optimal control theory, which in Chapter 8 was applied to a related discrete time problem, offers a well-suited apparatus for solving this kind of optimization problems. We will make use of a special case of Pontryagin's *Maximum Principle* (the basic tool of optimal control theory) in its continuous time version. We shall consider both the finite and the infinite horizon case. The only regularity condition required is that the exogenous variables, here $r(t)$ and $w(t)$, are piecewise continuous and that the control variable, here $c(t)$, is piecewise continuous and take values within some given set $\mathbb{C} \subset \mathbb{R}$, called the *control region.*

For a fixed $T < \infty$ the problem is: choose a plan $(c(t))_{t=0}^{T}$ that maximizes

$$U_0 = \int_0^T u(c(t))e^{-\rho t}dt \qquad \text{s.t.} \tag{9.35}$$

$$c(t) \geq 0, \qquad \text{(control region)} \tag{9.36}$$

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t), \quad a(0) = a_0 \text{ given,} \tag{9.37}$$

$$a(T) \geq 0. \qquad \text{(solvency requirement)} \tag{9.38}$$

With an infinite time horizon, the upper limit of integration, T, in (9.35) is

[13] Starting from a given initial positive debt, d_0, when $\dot{d}(t)/d(t) \geq r > 0$, we have $d(t) \geq d_0 e^{rt}$ so that $d(t)e^{-rt} \geq d_0 > 0$ for all $t \geq 0$. Consequently, $a(t)e^{-rt} = -d(t)e^{-rt} \leq -d_0 < 0$ for all $t \geq 0$, that is, $\lim_{t\to\infty} a(t)e^{-rt} < 0$, which violates (NPG).

interpreted as going to infinity, and the solvency condition (9.38) is replaced by

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0. \tag{NPG}$$

Let I denote the time interval $[0,T]$ if $T<\infty$ and the time interval $[0,\infty)$ if $T\to\infty$. If $c(t)$ and the corresponding evolution of $a(t)$ fulfil (9.36) and (9.37) for all $t\in I$ as well as the relevant solvency condition, we call $(a(t),c(t))_{t=0}^T$ an *admissible path.* If a given admissible path $(a(t),c(t))_{t=0}^T$ solves the problem, it is referred to as an *optimal path.*[14] We assume that $w(t)>0$ for all t. No condition on the impatience parameter ρ is imposed (in this chapter).

Necessary conditions for an optimal plan

The solution procedure for this problem is as follows:[15]

1. We set up the *current-value Hamiltonian function* (often just called the *current-value Hamiltonian* or even just the *Hamiltonian)*:

$$H(a,c,\lambda,t)\equiv u(c)+\lambda(ra+w-c),$$

where λ is the *adjoint variable* (also called the *co-state variable*) associated with the dynamic constraint (9.37).[16] That is, λ is an auxiliary variable which is a function of t and is analogous to the Lagrange multiplier in static optimization.

2. At every point in time, we maximize the Hamiltonian w.r.t. the *control variable.* Focusing on an *interior* optimal path,[17] we calculate

$$\frac{\partial H}{\partial c}=u'(c)-\lambda=0.$$

For every $t\in I$ we thus have the condition

$$u'(c(t))=\lambda(t). \tag{9.39}$$

[14] The term "path", sometimes "trajectory", is common in the natural sciences for a solution to a differential equation because one may think of this solution as the path of a particle moving in two- or three-dimensional space.

[15] The four-step solution procedure below is applicable to a large class of dynamic optimization problems in continuous time, see Math tools.

[16] The explicit dating of the time-dependent variables a, c, and λ is omitted where not needed for clarity.

[17] A path, $(a_t,c_t)_{t=0}^T$, is an *interior* path if for no $t\in[0,T)$, the pair (a_t,c_t) is at a boundary point of the set of admissible such pairs. In the present case where a_t is not constrained, except at $t=T$, $(a_t,c_t)_{t=0}^T$, is an *interior* path if $c_t>0$ for all $t\in[0,T)$.

3. We calculate the partial derivative of H with respect to the *state variable* and put it equal to minus the time derivative of λ plus the discount rate (as it appears in the integrand of the criterion function) multiplied by λ, that is
$$\frac{\partial H}{\partial a} = \lambda r = -\dot{\lambda} + \rho\lambda.$$
This says that, for all $t \in I$, the adjoint variable λ should fulfil the differential equation
$$\dot{\lambda}(t) = (\rho - r(t))\lambda(t). \tag{9.40}$$

4. We now apply the *Maximum Principle* which in the present context says: an interior optimal path $(a(t), c(t))_{t=0}^{T}$ will satisfy that there exits a continuous function $\lambda = \lambda(t)$ such that for all $t \in I$, (9.39) and (9.40) hold along the path, and the *transversality condition,*
$$\begin{aligned} a(T)\lambda(T) &= 0, \text{ if } T < \infty, \text{ and} \\ \lim_{t\to\infty} a(t)\lambda(t)e^{-\rho t} &= 0, \text{ if "}T = \infty\text{"}, \end{aligned} \tag{TVC}$$
is satisfied.

The transversality condition stated for the *finite* horizon case is general and simply a part of the Maximum Principle. But for the *infinite* horizon case no such *general* mathematical theorem exists. Fortunately, for the household's intertemporal consumption-saving problem, the condition stated in the second row of (TVC) turns out to *be* the true *necessary* transversality condition, as we shall see at the end of this section.

Let us provide an interpretation of the set of optimality conditions. Overall, the Maximum Principle characterizes an optimal path as a path that for every t implies maximization of the Hamiltonian associated with the problem and at the same time satisfies a certain terminal condition. Intuitively, maximization of the Hamiltonian at every t is needed because the Hamiltonian weighs the direct contribution of the marginal unit of the control variable to the criterion function in the "right" way relative to the indirect contribution, which comes from the generated change in the state variable (here financial wealth). In the present context "right" means that the trade-off between consuming or saving the marginal unit of account is described in accordance with the opportunities offered by the rate of return vis-a-vis the time preference rate, ρ. Indeed, the optimality condition (9.39) can be seen as a $MC = MB$ condition in utility terms: on the margin one unit of account (here the consumption good) must be equally valuable in its two uses: consumption and wealth accumulation.

Together with the optimality condition (9.40), this signifies that the adjoint variable λ can be interpreted as the *shadow price* (measured in units of current utility) of financial wealth along the optimal path.[18] Reordering the differential equation (9.40) gives

$$\frac{r\lambda + \dot{\lambda}}{\lambda} = \rho. \tag{9.41}$$

This can be interpreted as a no-arbitrage condition. The left-hand side gives the *actual* rate of return, measured in utility units, on the marginal unit of saving. Indeed, $r\lambda$ can be seen as a dividend and $\dot{\lambda}$ as a capital gain. The right-hand side is the *required* marginal rate of return in utility units, ρ. Along an optimal path the two must coincide. The household is willing to save the marginal unit of income only up to the point where the actual return on saving equals the required return.

We may alternatively write the no-arbitrage condition as

$$r = \rho - \frac{\dot{\lambda}}{\lambda}. \tag{9.42}$$

On the left-hand-side appears the actual *real* rate of return on saving and on the right-hand-side the *required real* rate of return. The intuition behind this condition can be seen in the following way. Suppose Mr. Jones makes a deposit of V utility units in a "bank" that offers a proportionate rate of expansion of the utility value of the deposit equal to i (assuming no withdrawal occurs), i.e.,

$$\frac{\dot{V}}{V} = i.$$

This is the actual *utility* rate of return, a kind of "nominal interest rate". To calculate the corresponding "real interest rate" let the "nominal price" of a consumption good be λ utility units. Dividing the number of invested utility units, V, by λ, we get the *real* value, $m = V/\lambda$, of the deposit at time t. The actual *real* rate of return on the deposit is therefore

$$r = \frac{\dot{m}}{m} = \frac{\dot{V}}{V} - \frac{\dot{\lambda}}{\lambda} = i - \frac{\dot{\lambda}}{\lambda}. \tag{9.43}$$

Mr. Jones is just willing to save the marginal unit of income if this actual real rate of return on saving equals the required real rate, that is, the right-hand side of (9.42). In turn, this necessitates that the "nominal interest rate", i, in (9.43) equals the required nominal rate, ρ. The formula (9.43) is analogue to the discrete-time formula (9.2) except that the unit of account in (9.43) is current utility while in (9.2) it is currency.

[18] Recall, a *shadow price* (measured in some unit of account) of a good is, from the point of view of the buyer, the maximum number of units of account that the optimizing buyer is willing to offer for one extra unit of the good.

The first-order conditions (9.39) and (9.40) thus say in what local direction the pair $(a(t), \lambda(t))_{t=0}^{T}$ must move in the (a, λ) plane to be optimal. This "local optimality" condition, which must hold at every t, is generally satisfied by infinitely many admissible paths. In addition, we need "overall optimality". This requires that the "general level" of the path $(a(t), \lambda(t))_{t=0}^{T}$ is such that a specific *terminal* condition holds, namely the transversality condition (TVC).

Let us see how close to an intuitive understanding of (TVC) as a necessary terminal optimality condition we can come. Consider first the *finite horizon* case $T < \infty$. The solvency requirement is $a(T) \geq 0$. Here (TVC) claims that $a(T)\lambda(T) = 0$ is needed for optimality. This condition is equivalent to

$$a(T)\lambda(T)e^{-\rho T} = 0, \tag{9.44}$$

since multiplying by a positive constant, $e^{-\rho T}$, does not change the condition. The form (9.44) has the advantage of being "parallel" to the transversality condition for the case $T \to \infty$. Transparency is improved if we insert (9.39) to get

$$a(T)u'(c(T))e^{-\rho T} = 0. \tag{9.45}$$

Note that the factor $u'(c(T))e^{-\rho T} > 0$.

Could an optimal plan have "$>$" in (9.45) instead of "$=$"? That would require $a(T) > 0$. But then, within some positive time interval, consumption could be increased without the ensuing decrease in $a(T)$ violating the solvency requirement $a(T) \geq 0$. Thereby, discounted utility, U_0, would be increased, which contradicts that the considered plan is an optimal plan.

Could an optimal plan have "$<$" in (9.45) instead of "$=$"? That would require $a(T) < 0$. Thereby the solvency requirement $a(T) \geq 0$ would be violated. So the plan is not admissible and therefore cannot be optimal.

From these two observations we conclude that with finite horizon, (9.45) *is* necessary for optimality. This is then also true for the original $a(T)\lambda(T) = 0$ condition in (TVC). So far so good.

Consider now the *infinite horizon* case $T \to \infty$. Here our intuitive reasoning will be more shaky, hence a precise proposition with a formal proof will finally be presented. Anyway, in (9.45) let $T \to \infty$ so as to give the condition

$$\lim_{T\to\infty} a(T)u'(c(T))e^{-\rho T} = 0. \tag{9.46}$$

By (9.39), this says the same as (TVC) for "$T = \infty$" does. In analogy with the finite horizon case, a plan that violates the condition (9.46), by having "$>$" instead of "$=$", reveals scope for improvement. Such a plan would amount to "postponing possible consumption *forever*", which cannot be optimal. The postponed consumption possibilities could be transformed to consumption on earth in real time by reducing $a(T)$ somewhat, without violating (NPG).

Finally, our intuitive belief is again that an optimal plan, violating (9.46) by having "<" instead "=", can be ruled out because it will defy the solvency requirement, that is, the (NPG) condition.

These intuitive considerations do not settle the issue whether (TVC) is really necessary for optimality, and if so, why? Luckily, regarding the household's intertemporal consumption-saving problem a simple formal proof, building on the household's intertemporal budget constraint, exists.

PROPOSITION 2 *(the household's necessary transversality condition with infinite time horizon)* Let $T \to \infty$ in the criterion function (9.35) and assume that the human wealth integral (9.32) converges (and thereby remains bounded) for $T \to \infty$. Assume further that the adjoint variable, $\lambda(t)$, satisfies the first-order conditions (9.39) and (9.40). Then:
(i) In an optimal plan, (NPG) is satisfied with strict equality.
(ii) (TVC) is satisfied if and only if (NPG) is satisfied with strict equality.

Proof. (i) By definition, an admissible plan must satisfy (NPG) and, in view of Proposition 1 in Section 9.4.1, thereby also the household's intertemporal budget constraint, (IBC). To be optimal, an admissible plan must satisfy (IBC) with strict equality; otherwise there would be scope for improving the plan by raising $c(t)$ a little in some time interval without decreasing $c(t)$ in other time intervals. By Proposition 1 strict equality in (IBC) is equivalent to strict equality in (NPG). Hence, in an optimal plan, (NPG) is satisfied with strict equality. (ii) See Appendix D. □

In view of this proposition, we can write the transversality condition for $T \to \infty$ as the NPG condition with strict equality:

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} = 0. \qquad \text{(TVC')}$$

This offers a nice economically interpretable formulation of the necessary transversality condition. In view of the equivalence of the NPG condition with strict equality and the IBC with strict equality, established in Proposition 1, the transversality condition for $T \to \infty$ can also be written this way:

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt = a_0 + h_0. \qquad \text{(IBC')}$$

Above we dealt with the difficult problem of stating necessary transversality conditions when the time horizon is infinite. The approach was to try a simple straightforward extension of the necessary transversality condition for the finite horizon case to the infinite horizon case. Such an extension is often valid (as it was here), but not always. There is no completely general all-embracing mathematical theorem about the necessary transversality condition for infinite-horizon optimal control problems. So, with infinite horizon problems care must be taken.

The current-value Hamiltonian versus the present-value Hamiltonian The prefix "current-value" is used to distinguish the current-value Hamiltonian from what is known as the *present-value Hamiltonian.* The latter is defined as $\hat{H} \equiv He^{-\rho t}$ with $\lambda e^{-\rho t}$ substituted by μ, which is the associated (discounted) adjoint variable. The solution procedure is similar except that step 3 is replaced by $\partial \hat{H}/\partial a = -\dot{\mu}$ and $\lambda(t)e^{-\rho t}$ in the transversality condition is replaced by $\mu(t)$. The two methods are equivalent (and if the discount rate is nil, the formulas for the optimality conditions coincide). But for many economic problems the *current-value* Hamiltonian has the advantage that it makes both the calculations and the interpretation slightly simpler. The adjoint variable, $\lambda(t)$, which as mentioned acts as a shadow price of the state variable, becomes a *current* price along with the other prices in the problem, $w(t)$ and $r(t)$. This is in contrast to $\mu(t)$ which is a *discounted* price.

9.4.3 The Keynes-Ramsey rule

The first-order conditions have interesting implications. Differentiate both sides of (9.39) w.r.t. t to get $u''(c)\dot{c} = \dot{\lambda}$. This equation can be written $u''(c)\dot{c}/u'(c) = \dot{\lambda}/\lambda$ by drawing on (9.39) again. Applying (9.40) now gives

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta(c(t))}(r(t) - \rho), \tag{9.47}$$

where $\theta(c)$ is the (absolute) *elasticity of marginal utility* w.r.t. consumption,

$$\theta(c) \equiv -\frac{c}{u'(c)}u''(c) > 0. \tag{9.48}$$

As in discrete time, $\theta(c)$ indicates the strength of the consumer's preference for consumption smoothing. The inverse of $\theta(c)$ measures the *instantaneous intertemporal elasticity of substitution* in consumption, which in turn indicates the willingness to change the time profile of consumption over time when the interest rate changes, see Appendix F.

The result (9.47) says that an optimal consumption plan is characterized in the following way. The household will completely smooth − even out − consumption over time if the rate of time preference equals the real interest rate. The household will choose an upward-sloping time path for consumption if and only if the rate of time preference is less than the real interest rate. In this case the household will have to accept a relatively low level of current consumption with the purpose of enjoying higher consumption in the future. The higher the real interest rate relative to the rate of time preference, the more favorable is it to defer consumption − *everything else equal.* The proviso is important. In addition to

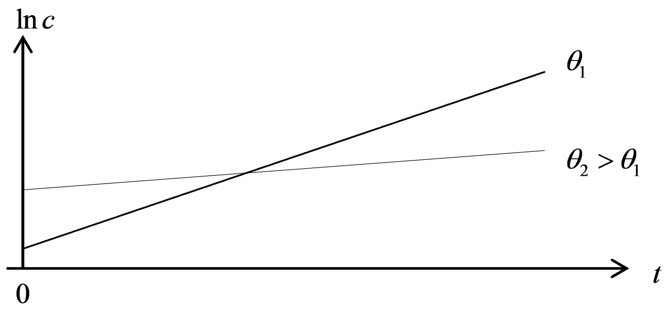


Figure 9.2: Optimal consumption paths for a low and a high constant θ, given a constant $r > \rho$.

the *negative* substitution effect on current consumption of a higher interest rate, there is a *positive* income effect due to the present value of a given intertemporal consumption plan being reduced by a higher interest rate (see (IBC)). On top of this comes a *negative* wealth effect due to a higher interest rate causing a lower present value of expected future labor earnings (again see (IBC)). The special case of a CRRA utility function provides a convenient agenda for sorting these details out, see Example 1 in Section 9.5.

By (9.47) we also see that the greater the elasticity of marginal utility (that is, the greater the curvature of the utility function), the greater the incentive to smooth consumption for a given value of $r(t) - \rho$. The reason for this is that a strong curvature means that the marginal utility will drop sharply if consumption increases, and will rise sharply if consumption decreases. Fig. 9.2 illustrates this in the CRRA case where $\theta(c) = \theta$, a positive constant. For a given constant $r > \rho$, the consumption path chosen when θ is high has lower slope, but starts from a higher level, than when θ is low.

The condition (9.47), which holds for all t within the time horizon whether this is finite or infinite, is referred to as the *Keynes-Ramsey rule*. The name springs from the English mathematician Frank Ramsey who derived the rule in 1928, while his mentor, John Maynard Keynes, suggested a simple and intuitive way of presenting it. The rule is the continuous-time counterpart to the consumption Euler equation in discrete time.

The Keynes-Ramsey rule reflects the general microeconomic principle that the consumer equates the marginal rate of substitution between any two goods to the corresponding price ratio. In the present context the principle is applied to a situation where the "two goods" refer to the same consumption good delivered at two different dates. In Section 9.2 we used the principle to interpret the optimal saving behavior in discrete time. How can the principle be translated into a continuous time setting?

Local optimality in continuous time* Let $(t, t+\Delta t)$ and $(t+\Delta t, t+2\Delta t)$ be two short successive time intervals. The marginal rate of substitution, $MRS_{t+\Delta t,t}$, of consumption in the second time interval for consumption in the first, is[19]

$$MRS_{t+\Delta t,t} \equiv -\frac{dc(t+\Delta t)}{dc(t)}|_{U=\bar{U}} = \frac{u'(c(t))}{e^{-\rho \Delta t}u'(c(t+\Delta t))}, \tag{9.49}$$

approximately. On the other hand, by saving $-\Delta c(t)$ more per time unit (where $\Delta c(t) < 0$) in the short time interval $(t, t+\Delta t)$, one can, via the market, transform $-\Delta c(t) \cdot \Delta t$ units of consumption in this time interval into

$$\Delta c(t+\Delta t) \cdot \Delta t \approx -\Delta c(t) \Delta t \; e^{\int_t^{t+\Delta t} r(\tau) d\tau} \tag{9.50}$$

units of consumption in the time interval $(t+\Delta t, t+2\Delta t)$. The marginal rate of transformation is therefore

$$\begin{aligned} MRT_{t+\Delta t,t} &\equiv -\frac{dc(t+\Delta t)}{dc(t)}|_{U=\bar{U}} \approx \\ &= e^{\int_t^{t+\Delta t} r(\tau) d\tau}. \end{aligned}$$

In the optimal plan we must have $MRS_{t+\Delta t,t} = MRT_{t+\Delta t,t}$ which gives

$$\frac{u'(c(t))}{e^{-\rho \Delta t}u'(c(t+\Delta t))} = e^{\int_t^{t+\Delta t} r(\tau) d\tau}, \tag{9.51}$$

approximately. When $\Delta t = 1$ and ρ and $r(t)$ are small, this relation can be approximated by (9.11) from discrete time (generally, by a first-order Taylor approximation, we have $e^x \approx 1 + x$, when x is close to 0).

Taking logs on both sides of (9.51), dividing through by Δt, inserting (9.50), and letting $\Delta t \to 0$, we get (see Appendix E)

$$\rho - \frac{u''(c(t))}{u'(c(t))}\dot{c}(t) = r(t). \tag{9.52}$$

With the definition of $\theta(c)$ in (9.48), this is exactly the same as the Keynes-Ramsey rule (9.47) which, therefore, is merely an expression of the general optimality condition $MRS = MRT$. When $\dot{c}(t) > 0$, the household is willing to sacrifice some consumption today for more consumption tomorrow only if it is compensated by an interest rate sufficiently above ρ. Naturally, the required compensation is higher, the faster marginal utility declines with rising consumption, i.e., the larger is $(-u''/u')\dot{c}$ already. Indeed, a higher c_t in the future than today implies a lower marginal utility of consumption in the future than of consumption today. Saving of the marginal unit of income today is thus only warranted if the rate of return is sufficiently above ρ, and this is what (9.52) indicates.

[19] The underlying analytical steps can be found in Appendix E.

9.4.4 Mangasarian's sufficient conditions

For dynamic optimization problems with one state variable, one control variable, and finite or infinite horizon, the present version of the Maximum Principle delivers a set of first-order conditions and suggests a terminal optimality condition, the transversality condition. The first-order conditions are *necessary* conditions for an interior path to be optimal. With infinite horizon, the necessity of the suggested transversality condition in principle requires a verification in each case. In the present case the verification is implied by Proposition 2. So, up to this point we have only shown that if the consumption/saving problem has an interior solution, then this solution satisfies the Keynes-Ramsey rule and a transversality condition, (TVC').

But are these conditions also *sufficient*? The answer is yes in the present case. This follows from *Mangasarian's sufficiency theorem* (see Math tools) which, applied to the present problem, tells us that if the Hamiltonian is *jointly concave* in (a, c) for every t within the time horizon, then the listed first-order conditions, together with the transversality condition, are also sufficient. Because the instantaneous utility function (the first term in the Hamiltonian) is here strictly concave in c and the second term is linear in (a, c), the Hamiltonian *is* jointly concave in (a, c).

To sum up: if we have found a path satisfying the Keynes-Ramsey rule and (TVC'), we have a *candidate solution.* Applying the Mangasarian theorem, we check whether our candidate *is* an optimal solution. In the present optimization problem it is. In fact the strict concavity of the Hamiltonian with respect to the control variable in this problem ensures that the optimal solution is *unique* (Exercise 9.?).

9.5 The consumption function

We have not yet fully solved the saving problem. The Keynes-Ramsey rule gives only the optimal rate of *change* of consumption over time. It says nothing about the *level* of consumption at any given time. In order to determine, for instance, the level $c(0)$, we implicate the solvency condition which limits the amount the household can borrow in the long term. Among the infinitely many consumption paths satisfying the Keynes-Ramsey rule, the household will choose the "highest" one that also fulfils the solvency requirement (NPG). Thus, the household acts so that strict equality in (NPG) obtains. As we saw in Proposition 2, this is equivalent to the transversality condition being satisfied.

EXAMPLE 1 *(constant elasticity of marginal utility; infinite time horizon).* In the problem in Section 9.4.2 with $T \to \infty$, we consider the case where the elasticity of marginal utility $\theta(c)$, as defined in (9.48), is a constant $\theta > 0$. From Appendix A of Chapter 3 we know that this requirement implies that up to a positive linear transformation the utility function must be of the form:

$$u(c) = \begin{cases} \frac{c^{1-\theta}}{1-\theta}, & \text{when } \theta > 0, \theta \neq 1, \\ \ln c, & \text{when } \theta = 1. \end{cases} \tag{9.53}$$

This is our familiar CRRA utility function. In this case the Keynes-Ramsey rule implies $\dot{c}(t) = \theta^{-1}(r(t) - \rho)c(t)$. Solving this linear differential equation yields

$$c(t) = c(0)e^{\frac{1}{\theta}\int_0^t (r(\tau)-\rho)d\tau}, \tag{9.54}$$

cf. the general accumulation formula, (9.25).

We know from Proposition 2 that the transversality condition is equivalent to the NPG condition being satisfied with strict equality, and from Proposition 1 we know that this condition is equivalent to the intertemporal budget constraint being satisfied with strict equality, i.e.,

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt = a_0 + h_0, \tag{IBC'}$$

where h_0 is the human wealth,

$$h_0 = \int_0^\infty w(t)e^{-\int_0^t r(\tau)d\tau}dt. \tag{9.55}$$

This result can be used to determine $c(0)$.[20] Substituting (9.54) into (IBC') gives

$$c(0)\int_0^\infty e^{\int_0^t [\frac{1}{\theta}(r(\tau)-\rho)-r(\tau)]d\tau}dt = a_0 + h_0.$$

The consumption function is thus

$$\begin{aligned} c(0) &= \beta_0(a_0 + h_0), \qquad \text{where} \\ \beta_0 &\equiv \frac{1}{\int_0^\infty e^{\int_0^t [\frac{1}{\theta}(r(\tau)-\rho)-r(\tau)]d\tau}dt} = \frac{1}{\int_0^\infty e^{\frac{1}{\theta}\int_0^t [(1-\theta)r(\tau)-\rho]d\tau}dt} \end{aligned} \tag{9.56}$$

is the marginal propensity to consume out of wealth. We have here assumed that these improper integrals over an infinite horizon are bounded from above for all admissible paths. We see that consumption is proportional to total wealth. The factor of proportionality, often called the *marginal propensity to consume out of wealth*, depends on the expected future interest rates and on the preference

[20] The method also applies if instead of $T = \infty$, we have $T < \infty$.

parameters ρ and θ, that is, the impatience rate and the strength of the preference for consumption smoothing, respectively.

Generally, an increase in the interest rate level, for given total wealth, $a_0 + h_0$, can effect $c(0)$ both positively and negatively.[21] On the one hand, such an increase makes future consumption cheaper in present value terms. This change in the trade-off between current and future consumption entails a negative *substitution effect* on $c(0)$. On the other hand, the increase in the interest rates decreases the present value of a given consumption plan, allowing for higher consumption both today and in the future, for given total wealth, cf. (IBC'). This entails a positive *pure income effect* on consumption today as consumption is a normal good. If $\theta < 1$ (small curvature of the utility function), the substitution effect will dominate the pure income effect, and if $\theta > 1$ (large curvature), the reverse will hold. This is because the larger is θ, the stronger is the propensity to smooth consumption over time.

In the intermediate case $\theta = 1$ (the logarithmic case) we get from (9.56) that $\beta_0 = \rho$, hence

$$c(0) = \rho(a_0 + h_0). \tag{9.57}$$

In this special case the marginal propensity to consume is time independent and equal to the rate of time preference. For a given *total* wealth, $a_0 + h_0$, current consumption is thus independent of the expected path of the interest rate. That is, in the logarithmic case the *substitution* and *pure income effects* on current consumption exactly offset each other. Yet, on top of this comes the negative *wealth effect* on current consumption of an increase in the interest rate level. The present value of future wage incomes becomes lower (similarly with expected future dividends on shares and future rents in the housing market in a more general setup). Because of this, h_0 (and so $a_0 + h_0$) becomes lower, which adds to the negative substitution effect. Thus, even in the logarithmic case, and *a fortiori* when $\theta < 1$, the *total effect* of an increase in the interest rate level is unambiguously negative on $c(0)$.

If, for example, $r(t) = r$ and $w(t) = w$ (positive constants), we get

$$\begin{aligned} \beta_0 &= [(\theta - 1)r + \rho]/\theta, \\ a_0 + h_0 &= a_0 + w/r. \end{aligned}$$

When $\theta = 1$, the negative effect of a higher r on h_0 is decisive. When $\theta < 1$, a higher r reduces both β_0 and h_0, hence the total effect on $c(0)$ is even "more

[21] By an increase in the interest rate *level* we mean an upward shift in the time-profile of the interest rate. That is, there is at least one time interval within $[0, \infty)$ where the interest rate is higher than in the original situation and no time interval within $[0, \infty)$ where the interest rate is lower.

negative". When $\theta > 1$, a higher r implies a higher β_0 which more or less offsets the lower h_0, so that the total effect on $c(0)$ becomes ambiguous. As referred to in Chapter 3, available empirical studies generally suggest a value of θ somewhat above 1. □

A remark on *fixed-rate loans* and *positive net debt* is appropriate here. Suppose $a_0 < 0$ and assume that this net debt is *not* in the form of a variable-rate loan (as hitherto assumed), but for instance a fixed-rate mortgage loan. Then a rise in the interest rate level implies a lowering of the present value of the debt and thereby *raises* financial wealth and *possibly total wealth.* If so, the rise in the interest rate level implies a *positive* wealth effect on current consumption, thereby "joining" the positive *pure* income effect in *counterbalancing* the negative substitution effect.

EXAMPLE *2* (*constant absolute semi-elasticity of marginal utility; infinite time horizon*). In the problem in Section 9.4.2 with $T \to \infty$, we consider the case where the sensitivity of marginal utility, measured by the absolute value of the semi-elasticity of marginal utility, $-u''(c)/u'(c) \approx -(\Delta u'/u')/\Delta c$, is a positive constant, α. The utility function must then, up to a positive linear transformation, be of the form,

$$u(c) = -\alpha^{-1}e^{-\alpha c}, \alpha > 0. \tag{9.58}$$

This is known as the CARA utility function (where the name CARA comes from "Constant Absolute Risk Aversion"). The Keynes-Ramsey rule now becomes $\dot{c}(t) = \alpha^{-1}(r(t) - \rho)$. When the interest rate is a constant $r > 0$, we find, through (IBC') and partial integration, $c(0) = r(a_0 + h_0) - (r - \rho)/(\alpha r)$, presupposing $r \geq \rho$ and $a_0 + h_0 > (r - \rho)/(ar^2)$.

This hypothesis of a "constant absolute variability aversion" implies that the degree of *relative* variability aversion is $\theta(c) = \alpha c$ and thus greater, the larger is c. The CARA function has been popular in the theory of behavior under uncertainty. One of the theorems of expected utility theory is that the degree of absolute risk aversion, $-u''(c)/u'(c)$, is proportional to the risk premium which the economic agent will require to be willing to exchange a specified amount of consumption received with certainty for an uncertain amount having the same mean value. Empirically this risk premium seems to be a decreasing function of the level of consumption. Therefore the CARA function is generally considered less realistic than the CRRA function of the previous example. □

EXAMPLE 3 *(logarithmic utility; finite time horizon; retirement).* We consider a life-cycle saving problem. A worker enters the labor market at time 0 with a financial wealth of 0, has finite lifetime T (assumed known), retires at time $t_1 \in (0, T]$, and does not wish to pass on bequests. For simplicity we assume that $r_t = r > 0$ for all $t \in [0, T]$ and labor income is $w(t) = w > 0$ for $t \in [0, t_1]$, while

$w(t) = 0$ for $t > t_1$. The decision problem is

$$\begin{aligned}
\max_{(c(t))_{t=0}^T} U_0 &= \int_0^T (\ln c(t))e^{-\rho t}dt \quad \text{s.t.} \\
c(t) &\geq 0, \\
\dot{a}(t) &= ra(t) + w(t) - c(t), \qquad\qquad a(0) = 0, \\
a(T) &\geq 0.
\end{aligned}$$

The Keynes-Ramsey rule becomes $\dot{c}_t/c_t = r - \rho$. A solution to the problem will thus fulfil

$$c(t) = c(0)e^{(r-\rho)t}. \tag{9.59}$$

Inserting this into the differential equation for a, we get a first-order linear differential equation the solution of which (for $a(0) = 0$) can be reduced to

$$a(t) = e^{rt}\left[\frac{w}{r}(1 - e^{-rz}) - \frac{c_0}{\rho}(1 - e^{-\rho t})\right], \tag{9.60}$$

where $z = t$ if $t \leq t_1$, and $z = t_1$ if $t > t_1$. We need to determine $c(0)$. The transversality condition implies $a(T) = 0$. Having $t = T$, $z = t_1$ and $a_T = 0$ in (9.60), we get

$$c(0) = (\rho w/r)(1 - e^{-rt_1})/(1 - e^{-\rho T}). \tag{9.61}$$

Substituting this into (9.59) gives the optimal consumption plan.[22]

If $r = \rho$, consumption is constant over time at the level given by (9.61). If, in addition, $t_1 < T$, this consumption level is less than the wage income per year up to t_1 (in order to save for retirement); in the last years the level of consumption is maintained although there is no wage income; the retired person uses up both the return on financial wealth and this wealth itself. □

The examples illustrate the importance of *forward-looking expectations*, here expectations about future wage income and interest rates. The expectations affect $c(0)$ both through their impact on the marginal propensity to consume (cf. β_0 in Example 1) and through their impact on the present value of expected future labor income (or of expected future dividends on shares or imputed rental income on owner-occupied houses in a more general setup).[23]

To avoid misunderstanding: The examples should *not* be interpreted such that for *any* evolution of wages and interest rates there exists a solution to the

[22]For $t_1 = T$ and $T \to \infty$ we get in the limit $c(0) = \rho w/r \equiv \rho h_0$, which is also what (9.56) gives when $a(0) = 0$ and $\theta = 1$.

[23]There exist cases where, due to new information, a shift in expectations occurs so that a discontinuity in a responding endogenous variable results. How to deal with such cases is treated in Chapter 11.

household's maximization problem with infinite horizon. There is generally no guarantee that integrals converge and thus have an upper bound for $T \to \infty$. The evolution of wages and interest rates which prevails in *general equilibrium* is not arbitrary, however. It is determined by the requirement of equilibrium. In turn, of course *existence* of an equilibrium *imposes restrictions* on the utility discount rate relative to the potential growth in instantaneous utility. We shall return to these aspects in the next chapter.

9.6 Concluding remarks

(incomplete)

...

The examples above – and the consumption theory in this chapter in general – should only be seen as a first, simple approximation to actual consumption/saving behavior. One thing is that we have not made a distinction between durable and non-durable consumption goods. More importantly, real world factors such as uncertainty and narrow credit constraints (absence of perfect loan and insurance markets) have been ignored. Debt limits imposed on borrowers are usually much sharper than the NPG condition. When such circumstances are included, current income and expected income in the *near* future tend to become important co-determinants of current consumption, at least for a large fraction of the population with little financial wealth.

Short- and medium-run macro models typically attempt to take into account different kinds of such "frictions". There is also a growing interest in incorporating psychological factors such as short-sightedness (in one or another form, including present-bias) into macroeconomic theory.

9.7 Literature notes

(incomplete)

In Chapter 6, where the borrower was a "large" agent with fiscal and monetary policy mandates, namely the public sector, satisfying the intertemporal budget constraint was a necessary condition for solvency (when the interest rate exceeds the growth rate of income), but not a sufficient condition. When the modelled borrowers are "small" private agents as in this chapter, the situation is different. Neoclassical models with perfect markets then usually contain equilibrium mechanisms such that the agents' compliance with their intertemporal budget constraint is sufficient for lenders' willingness and ability to supply the demanded finance. See ...

Abel 1990, p. 726-53.

Epstein-Zin preferences. See Poul Schou.

Present-bias and time-inconsistency. Strots (1956). Laibson, QJE 1997: $1, \alpha\beta, \alpha\beta^2, ...$

Loewenstein and Thaler (1989) survey the evidence suggesting that the utility discount rate is generally not constant, but declining with the time distance from the current period to the future periods within the horizon. This is known as *hyperbolic discounting.*

The assumptions regarding the underlying intertemporal preferences which allow them to be represented by the present value of period utilities discounted at a constant rate are dealt with by Koopmans (1960), Fishburn and Rubinstein (1982), and − in summary form − by Heal (1998).

Borovika, WP 2013, Recursive preferences, separation of risk aversion and IES.

Deaton, A., *Understanding Consumption*, OUP 1992.

On continuous-time finance, see for instance Merton (1990).

Goldberg (1958).

Allen (1967).

To Math Tools: Rigorous and more general presentations of the Maximum Principle in continuous time applied in economic analysis are available in, e.g., Seierstad and Sydsæter (1987), Sydsæter et al. (2008) and Seierstad and Sydsæter (Optimization Letters, 2009, 3, 507-12).

9.8 Appendix

A. Growth arithmetic in continuous time

Let the variables z, x, and y be differentiable functions of time t. Suppose $z(t)$, $x(t)$, and $y(t)$ are positive for all t. Then:

PRODUCT RULE $z(t) = x(t)y(t) \Rightarrow \dot{z}(t)/z(t) = \dot{x}(t)/x(t) + \dot{y}(t)/y(t)$.

Proof. Taking logs on both sides of the equation $z(t) = x(t)y(t)$ gives $\ln z(t) = \ln x(t) + \ln y(t)$. Differentiation w.r.t. t, using the chain rule, gives the conclusion. □

The procedure applied in this proof is called “logarithmic differentiation” w.r.t. t : take the log and then the time derivative.

QUOTIENT RULE $z(t) = x(t)/y(t) \Rightarrow \dot{z}(t)/z(t) = \dot{x}(t)/x(t) - \dot{y}(t)/y(t)$.

The proof is similar.

POWER FUNCTION RULE $z(t) = x(t)^{\alpha} \Rightarrow \dot{z}(t)/z(t) = \alpha\dot{x}(t)/x(t)$.

The proof is similar.

In continuous time these simple formulas are exactly true. In discrete time the analogue formulas are only approximately true and the approximation can be quite bad unless the growth rates of x and y are small, cf. Appendix A to Chapter 4.

B. Average growth and interest rates

Sometimes we may want to express the accumulation formula in continuous time,

$$a(t) = a(0)e^{\int_0^t g(\tau)d\tau}, \tag{9.62}$$

in terms of then arithmetic average of the growth rates in the time interval $[0, t]$. This is defined as $\bar{g}_{0,t} = (1/t)\int_0^t g(\tau)d\tau$. So we can rewrite (9.62) as

$$a(t) = a(0)e^{\bar{g}_{0,t}t}, \tag{9.63}$$

which has form similar to (9.24). Similarly, let $\bar{r}_{0,t}$ denote the arithmetic average of the (short-term) interest rates from time 0 to time t, i.e., $\bar{r}_{0,t} = (1/t)\int_0^t r(\tau)d\tau$. Then we can write the present value of the consumption stream $(c(t))_{t=0}^T$ as $PV = \int_0^T c(t)e^{-\bar{r}_{0,t}t}dt$.

The arithmetic average growth rate, $\bar{g}_{0,t}$, coincides with the average *compound* growth rate from time 0 to time t, that is, the number g satisfying

$$a(t) = a(0)e^{gt}, \tag{9.64}$$

for the same $a(0)$ and $a(t)$ as in (9.63).

There is no similar concordance within discrete time modeling.

Discrete versus continuous compounding Suppose the period length is one year so that the given observations, $a_0, a_1 \ldots, a_n$, are annual data. There are two alternative ways of calculating an average compound growth rate (often just called the "average growth rate") for the data. We may apply the geometric growth formula,

$$a_n = a_0(1 + G)^n, \tag{9.65}$$

which is natural if the compounding behind the data *is* discrete and occurs for instance quarterly. Or we may apply the exponential growth formula,

$$a_n = a_0e^{gn}, \tag{9.66}$$

corresponding to continuous compounding. Unless $a_n = a_0$ (in which case $g = 0$), the resulting g will be smaller than the average compound growth rate, G, calculated from (9.65) for the same data. Indeed,

$$g = \frac{\ln \frac{a_n}{a_0}}{n} = \ln(1+G) \lessapprox G$$

for G "small", where "$\lessapprox$" means "close to" (by a first-order Taylor approximation about $G = 0$) but "less than" except if $G = 0$. The intuitive reason for "less than" is that a given growth force is more powerful when compounding is continuous. To put it differently: rewriting $(1+G)^n$ into exponential form gives $(1+G)^n = (e^{\ln(1+G)})^n = e^{gn} < e^{Gn}$, as $\ln(1+G) < G$ for all $G \neq 0$.

On the other hand, the difference between G and g is usually unimportant. If for example G refers to the annual GDP growth rate, it will be a small number, and the difference between G and g immaterial. For example, to $G = 0.040$ corresponds $g \approx 0.039$. Even if $G = 0.10$, the corresponding g is 0.0953. But if G stands for the inflation rate and there is high inflation, the difference between G and g will be substantial. During hyperinflation the monthly inflation rate may be, say, $G = 100\%$, but the corresponding g will be only 69%.[24]

C. Proof of Proposition 1 (about equivalence between the No-Ponzi-Game condition and the intertemporal budget constraint)

We consider the book-keeping relation

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t), \tag{9.67}$$

where $a(0) = a_0$ (given), and the solvency requirement

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0. \tag{NPG}$$

Technical remark. The expression in (NPG) should be understood to include the possibility that $a(t)e^{-\int_0^t r(\tau)d\tau} \to \infty$ for $t \to \infty$. Moreover, if full generality were aimed at, we should allow for infinitely fluctuating paths in both the (NPG) and (TVC) and therefore replace "$\lim_{t\to\infty}$" by "$\liminf_{t\to\infty}$", i.e., the *limit inferior*. The limit inferior for $t \to \infty$ of a function $f(t)$ on $[0, \infty)$ is defined as $\lim_{t\to\infty} \inf \{f(s)| \ s \geq t\}$.[25] As noted in Appendix E of the previous chapter, however, undamped infinitely fluctuating paths do not turn up in "normal" economic

[24] Apart from the discrete compounding instead of continuous compounding, a *geometric* growth factor is equivalent to a "corresponding" *exponential* growth factor. Indeed, we can rewrite the growth factor $(1+g)^t$, $t = 0, 1, 2, \ldots$, into exponential form since $(1+g)^t = (e^{\ln(1+g)})^t = e^{[\ln(1+g)]t}$. Moreover, if g is "small", we have $e^{[\ln(1+g)]t} \approx e^{gt}$.

[25] By "inf" is meant *infimum* of the set, that is, the largest number less than or equal to all numbers in the set.

optimization problems, whether in discrete or continuous time. Hence, we apply the simpler concept "lim" rather than "lim inf". □

On the background of (9.67), Proposition 1 in the text claimed that (NPG) is equivalent to the intertemporal budget constraint,

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq h_0 + a_0, \tag{IBC}$$

being satisfied, where h_0 is defined as in (9.55) and is assumed to be a finite number. In addition, Proposition 1 in Section 9.4 claimed that there is strict equality in (IBC) if and only there is strict equality in (NPG). A plain proof goes as follows.

Proof. Isolate $c(t)$ in (9.67) and multiply through by $e^{-\int_0^t r(\tau)d\tau}$ to obtain

$$c(t)e^{-\int_0^t r(\tau)d\tau} = w(t)e^{-\int_0^t r(\tau)d\tau} - (\dot{a}(t) - r(t)a(t))e^{-\int_0^t r(\tau)d\tau}.$$

Integrate from 0 to $T > 0$ to get $\int_0^T c(t)e^{-\int_0^t r(\tau)d\tau}dt$

$$\begin{aligned}
&= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - \int_0^T \dot{a}(t)e^{-\int_0^t r(\tau)d\tau}dt + \int_0^T r(t)a(t)e^{-\int_0^t r(\tau)d\tau}dt \\
&= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - \left(\left[a(t)e^{-\int_0^t r(\tau)d\tau}\right]_0^T - \int_0^T a(t)e^{-\int_0^t r(\tau)d\tau}(-r(t))dt\right) \\
&+ \int_0^T r(t)a(t)e^{-\int_0^t r(\tau)d\tau}dt \\
&= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - (a(T)e^{-\int_0^T r(\tau)d\tau} - a(0)),
\end{aligned}$$

where the second equality follows from integration by parts. If we let $T \to \infty$ and use the definition of h_0 and the initial condition $a(0) = a_0$, we get (IBC) if and only if (NPG) holds. It follows that when (NPG) is satisfied with strict equality, so is (IBC), and vice versa. □

An alternative proof is obtained by using the general solution to a linear inhomogenous first-order differential equation and then let $T \to \infty$. Since this is a more generally applicable approach, we will show how it works and use it for Claim 1 below (an extended version of Proposition 1) and for the proof of Proposition 2 in the text. Claim 1 will for example prove useful in Exercise 9.1 and in the next chapter.

CLAIM 1 Let $f(t)$ and $g(t)$ be given continuous functions of time, t. Consider the differential equation

$$\dot{x}(t) = g(t)x(t) + f(t), \tag{9.68}$$

with $x(t_0) = x_{t_0}$, a given initial value. Then the inequality

$$\lim_{t\to\infty} x(t)e^{-\int_{t_0}^{t} g(s)ds} \geq 0 \tag{9.69}$$

is equivalent to

$$-\int_{t_0}^{\infty} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds} d\tau \leq x_{t_0}. \tag{9.70}$$

Moreover, if and only if (9.69) is satisfied with strict equality, then (9.70) is satisfied with strict equality.

Proof. The linear differential equation (9.68) has the solution

$$x(t) = x(t_0)e^{\int_{t_0}^{t} g(s)ds} + \int_{t_0}^{t} f(\tau)e^{\int_{\tau}^{t} g(s)ds} d\tau. \tag{9.71}$$

Multiplying through by $e^{-\int_{t_0}^{t} g(s)ds}$ yields

$$x(t)e^{-\int_{t_0}^{t} g(s)ds} = x(t_0) + \int_{t_0}^{t} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds} d\tau.$$

By letting $t \to \infty$, it can be seen that if and only if (9.69) is true, we have

$$x(t_0) + \int_{t_0}^{\infty} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds} d\tau \geq 0.$$

Since $x(t_0) = x_{t_0}$, this is the same as (9.70). We also see that if and only if (9.69) holds with strict equality, then (9.70) also holds with strict equality. □

COROLLARY Let n be a given constant and let

$$h_{t_0} \equiv \int_{t_0}^{\infty} w(\tau)e^{-\int_{t_0}^{\tau}(r(s)-n)ds} d\tau, \tag{9.72}$$

which we assume is a finite number. Then, given the flow budget identity

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t), \text{ where } a(t_0) = a_{t_0}, \tag{9.73}$$

it holds that

$$\lim_{t\to\infty} a(t)e^{-\int_{t_0}^{t}(r(s)-n)ds} \geq 0 \Leftrightarrow \int_{t_0}^{\infty} c(\tau)e^{-\int_{t_0}^{\tau}(r(s)-n)ds} d\tau \leq a_{t_0} + h_{t_0}, \tag{9.74}$$

where a strict equality on the left-hand side of "⇔" implies a strict equality on the right-hand side, and vice versa.

Proof. In (9.68), (9.69), and (9.70), let $x(t) = a(t)$, $g(t) = r(t) - n$ and $f(t) = w(t) - c(t)$. Then the conclusion follows from Claim 1. □

By setting $t_0 = 0$ in the corollary and replacing τ by t and n by 0, we have hereby provided an alternative proof of Proposition 1.

D. Proof of Proposition 2 (the household's necessary transversality condition with an infinite time horizon)

In the differential equation (9.68) we let $x(t) = \lambda(t)$, $g(t) = -(r(t) - \rho)$, and $f(t) = 0$. This gives the linear differential equation $\dot{\lambda}(t) = (\rho - r(t))\lambda(t)$, which is identical to the first-order condition (9.40) in Section 9.4. The solution is

$$\lambda(t) = \lambda(t_0)e^{-\int_{t_0}^{t}(r(s)-\rho)ds}.$$

Substituting this into (TVC) in Section 9.4 yields

$$\lambda(t_0)\lim_{t\to\infty} a(t)e^{-\int_{t_0}^{t}(r(s)-n)ds} = 0. \tag{9.75}$$

From the first-order condition (9.39) in Section 9.4 we have $\lambda(t_0) = u'(c(t_0)) > 0$ so that $\lambda(t_0)$ in (9.75) can be ignored. Thus, (TVC) in Section 9.4 is equivalent to the condition that (NPG) in that section is satisfied with strict equality (let $t_0 = 0 = n$). This proves (ii) of Proposition 2 in the text. □

E. Intertemporal consumption smoothing

We claimed in Section 9.4 that equation (9.49) gives approximately the marginal rate of substitution of consumption in the time interval $(t + \Delta t,\ t + 2\Delta t)$ for consumption in $(t,\ t+\Delta t)$. This can be seen in the following way. To save notation we shall write our time-dependent variables as c_t, r_t, etc., even though they are continuous functions of time. The contribution from the two time intervals to the criterion function is

$$\begin{aligned}\int_t^{t+2\Delta t} u(c_\tau)e^{-\rho\tau}d\tau &\approx e^{-\rho t}\left(\int_t^{t+\Delta t} u(c_t)e^{-\rho(\tau-t)}d\tau + \int_{t+\Delta t}^{t+2\Delta t} u(c_{t+\Delta t})e^{-\rho(\tau-t)}d\tau\right)\\ &= e^{-\rho t}\left(u(c_t)\left[\frac{e^{-\rho(\tau-t)}}{-\rho}\right]_t^{t+\Delta t} + u(c_{t+\Delta t})\left[\frac{e^{-\rho(\tau-t)}}{-\rho}\right]_{t+\Delta t}^{t+2\Delta t}\right)\\ &= \frac{e^{-\rho t}(1-e^{-\rho\Delta t})}{\rho}\left[u(c_t) + u(c_{t+\Delta t})e^{-\rho\Delta t}\right].\end{aligned}$$

Requiring unchanged utility integral $U_0 = \bar{U}_0$ is thus approximately the same as requiring $\Delta[u(c_t) + u(c_{t+\Delta t})e^{-\rho\Delta t}] = 0$, which by carrying through the differentiation and rearranging gives (9.49).

The instantaneous local optimality condition, equation (9.52), can be interpreted on the basis of (9.51). Take logs on both sides of (9.51) to get

$$\ln u'(c_t) + \rho\Delta t - \ln u'(c_{t+\Delta t}) = \int_t^{t+\Delta t} r_\tau d\tau.$$

Dividing by Δt, substituting (9.50), and letting $\Delta t \to 0$ we get

$$\rho - \lim_{\Delta t \to 0} \frac{\ln u'(c_{t+\Delta t}) - \ln u'(c_t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{R_{t+\Delta t} - R_t}{\Delta t}, \tag{9.76}$$

where R_t is the antiderivative of r_t. By the definition of a time derivative, (9.76) can be written

$$\rho - \frac{d \ln u'(c_t)}{dt} = \frac{dR_t}{dt}.$$

Carrying out the differentiation, we get

$$\rho - \frac{1}{u'(c_t)} u''(c_t)\dot{c}_t = r_t,$$

which was to be shown.

F. Elasticity of intertemporal substitution in continuous time

The relationship between the elasticity of marginal utility and the concept of *instantaneous elasticity of intertemporal substitution* in consumption can be exposed in the following way: consider an indifference curve for consumption in the non-overlapping time intervals $(t, t+\Delta t)$ and $(s, s+\Delta t)$. The indifference curve is depicted in Fig. 9.3. The consumption path outside the two time intervals is kept unchanged. At a given point $(c_t\Delta t, c_s\Delta t)$ on the indifference curve, the marginal rate of substitution of s-consumption for t-consumption, MRS_{st}, is given by the absolute slope of the tangent to the indifference curve at that point. In view of $u''(c) < 0$, MRS_{st} is rising along the curve when c_t decreases (and thereby c_s increases).

Conversely, we can consider the ratio c_s/c_t as a function of MRS_{st} along the given indifference curve. The elasticity of this consumption ratio w.r.t. MRS_{st} as we move along the given indifference curve then indicates the *elasticity of substitution* between consumption in the time interval $(t, t+\Delta t)$ and consumption in the time interval $(s, s+\Delta t)$. Denoting this elasticity by $\sigma(c_t, c_s)$, we thus have:

$$\sigma(c_t, c_s) = \frac{MRS_{st}}{c_s/c_t} \frac{d(c_s/c_t)}{dMRS_{st}} \approx \frac{\frac{\Delta(c_s/c_t)}{c_s/c_t}}{\frac{\Delta MRS_{st}}{MRS_{st}}}.$$

At an optimum point, MRS_{st} equals the ratio of the discounted prices of good t and good s. Thus, the elasticity of substitution can be interpreted as approximately equal to the percentage increase in the ratio of the chosen goods, c_s/c_t, generated by a one percentage increase in the inverse price ratio, holding the utility level and the amount of other goods unchanged. If $s = t + \Delta t$ and the

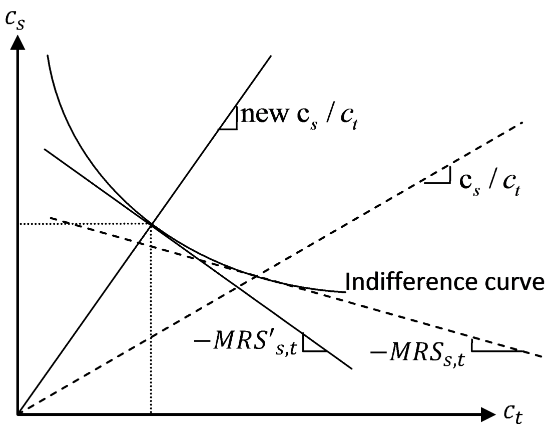


Figure 9.3: Substitution of s-consumption for t-consumption as MRS_{st} increases to MRS'_{st}.

interest rate from date t to date s is r, then (with continuous compounding) this price ratio is $e^{r\Delta t}$, cf. (9.51). Inserting MRS_{st} from (9.49) with $t + \Delta t$ replaced by s, we get

$$\begin{aligned} \sigma(c_t, c_s) &= \frac{u'(c_t)/[e^{-\rho(s-t)}u'(c_s)]}{c_s/c_t} \frac{d(c_s/c_t)}{d\{u'(c_t)/[e^{-\rho(s-t)}u'(c_s)]\}} \\ &= \frac{u'(c_t)/u'(c_s)}{c_s/c_t} \frac{d(c_s/c_t)}{d(u'(c_t)/u'(c_s))}, \end{aligned} \tag{9.77}$$

since the factor $e^{-\rho(t-s)}$ cancels out.

We now interpret the d's in (9.77) as differentials (recall, the differential of a differentiable function $y = f(x)$ is denoted dy and defined as $dy = f'(x)dx$ where dx is some arbitrary real number). Calculating the differentials we get

$$\sigma(c_t, c_s) \approx \frac{u'(c_t)/u'(c_s)}{c_s/c_t} \frac{(c_t dc_s - c_s dc_t)/c_t^2}{[u'(c_s)u''(c_t)dc_t - u'(c_t)u''(c_s)dc_s]/u'(c_s)^2}.$$

Hence, for $s \to t$ we get $c_s \to c_t$ and

$$\sigma(c_t, c_s) \to \frac{c_t(dc_s - dc_t)/c_t^2}{u'(c_t)u''(c_t)(dc_t - dc_s)/u'(c_t)^2} = -\frac{u'(c_t)}{c_t u''(c_t)} \equiv \tilde{\sigma}(c_t).$$

This limiting value is known as the *instantaneous elasticity of intertemporal substitution* of consumption. It reflects the opposite of the preference for consumption smoothing. Indeed, we see that $\tilde{\sigma}(c_t) = 1/\theta(c_t)$, where $\theta(c_t)$ is the elasticity of marginal utility at the consumption level $c(t)$.

9.9 Exercises

9.1 We look at a household (or dynasty) with infinite time horizon. The household's labor supply is inelastic and grows at the constant rate $n > 0$. The household has a constant rate of time preference $\rho > n$ and the individual instantaneous utility function is $u(c) = c^{1-\theta}/(1-\theta)$, where θ is a positive constant. There is no uncertainty. The household maximizes the integral of per capita utility discounted at the rate $\rho - n$. Set up the household's optimization problem. Show that the optimal consumption plan satisfies

$$\begin{aligned} c(0) &= \beta_0(a_0 + h_0), \qquad \text{where} \\ \beta_0 &= \frac{1}{\int_0^\infty e^{\int_0^t (\frac{(1-\theta)r(\tau)-\rho}{\theta}+n)d\tau} dt}, \qquad \text{and} \\ h_0 &= \int_0^\infty w(t) e^{-\int_0^t (r(\tau)-n)d\tau} dt, \end{aligned}$$

where $w(t)$ is the real wage per unit of labor and otherwise the same notation as in this chapter is used. *Hint:* apply the corollary to Claim 1 in Appendix C and the method of Example 1 in Section 9.5. As to h_0, start by considering

$$H_0 \equiv h_0 L_0 = \int_0^\infty w(t) L_t e^{-\int_0^t (r(\tau)-n)d\tau} dt$$

and apply that $L(t) = L_0 e^{nt}$.

Chapter 10

The basic representative agent model: Ramsey

As early as 1928 a sophisticated model of a society's optimal saving was published by the British mathematician and economist Frank Ramsey (1903-1930). Ramsey's contribution was mathematically demanding and did not experience much response at the time. Three decades had to pass until his contribution was taken up seriously (Samuelson and Solow, 1956). His model was merged with the growth model by Solow (1956) and became a cornerstone in neoclassical growth theory from the mid 1960s. The version of the model which we present below was completed by the work of Cass (1965) and Koopmans (1965). Hence the model is also known as the *Ramsey-Cass-Koopmans model*.

The model is one of the basic workhorse models in macroeconomics. As we conclude at the end of the chapter, the model can be seen as placed at one end of a line segment. At the other end appears another basic workhorse model, namely Diamond's overlapping generations model considered in chapters 3 and 4. While in the Diamond model there is an *unbounded* number of households (since in every new period a new generation enters the economy) and these have a *finite* time horizon, in the Ramsey model there is a *finite* number of households with an *unbounded* time horizon. Moreover, in the standard Ramsey model households are completely alike. The model is the main example of a *representative agent* model. In contrast, the Diamond model has heterogeneous agents, young versus old, interacting in every period. There are important economic questions where these differences in the setup lead to salient differences in the answers.

The purpose of this chapter is to describe and analyze the continuous-time version of the Ramsey framework. In the main sections we consider the case of a perfectly competitive market economy. In this context we shall see, for example, that the Solow growth model can be interpreted as a special case of the Ramsey model. toward the end of the chapter we consider the Ramsey framework in a setting with an "all-knowing and all-powerful" social planner.

10.1 Preliminaries

We consider a closed economy. Time is continuous. We assume households own the capital goods and hire them out to firms at a market *rental rate*, $\hat{r}$. This is just to have something concrete in mind. If instead the capital goods were owned by the firms using them in production, and the capital investment by these firms were financed by issuing shares and bonds, then the conclusions would remain the same as long as we ignore uncertainty.

Although time is considered continuous, to save notation, we shall write the time-dependent variables as w_t, $\hat{r}_t$, etc. instead of $w(t)$, $\hat{r}(t)$, etc. In every short time interval $(t, t+\Delta t)$, the individual firm employs labor at the market wage w_t and rents capital goods at the rental rate $\hat{r}_t$. The combination of labor and capital produces the homogeneous output good. This good can be used for consumption as well as investment. So in every short time interval there are at least three active markets, one for the "all-purpose", homogeneous output good, one for labor, and one for capital services (the rental market for capital goods). It may be convenient to imagine that there is also a perfect loan market. As all households are alike, however, the loan market will not be active in general equilibrium.

There is perfect competition in all markets, that is, households and firms are price takers. Any need for means of payment − money − is abstracted away. Prices are measured in units of the homogeneous output good.

There are no stochastic elements in the model. We assume households understand exactly how the economy works and can predict the future path of wages and interest rates. In other words, we assume "rational expectations". In our non-stochastic setting this amounts to *perfect foresight*. The results that emerge from the model are thereby the outcome of economic mechanisms in isolation from expectational errors.

Uncertainty being absent, rates of return on alternative assets are in equilibrium the same. In spite of the not active loan market, it is usual to speak of this common rate of return as the *real interest rate* of the economy. Denoting this rate r_t, for a given rental rate of capital, $\hat{r}_t$, we have

$$r_t = \frac{\hat{r}_t K_t - \delta K_t}{K_t} = \hat{r}_t - \delta, \tag{10.1}$$

where the right-hand side is the rate of return on holding K_t capital goods, δ (≥ 0) being a constant rate of capital depreciation. This relationship may be considered a *no-arbitrage condition* between investing in the loan market and in capital goods.

We describe, first, the households' behavior and next the firms' behavior. Thereafter the interaction between households and firms in general equilibrium and the resulting dynamics will be analyzed.

10.2 The agents

10.2.1 Households

There is a fixed number, N, of identical households with an infinite time horizon. This feature makes aggregation very simple: we just have to multiply the behavior of a single household with the number of households (for simplicity we later normalize N to equal 1). Every household has L_t (adult) members; L_t changes over time at a constant rate, n :

$$L_t = L_0 e^{nt}, \qquad L_0 > 0. \tag{10.2}$$

Indivisibility is ignored.

Each household member supplies inelastically one unit of labor per time unit. Equation (10.2) therefore describes the growth of both the population and the labor force. Since there is only one consumption good, the only decision problem is how to distribute current income between consumption and saving.

Intertemporal utility function

The household's preferences can be represented by an additive intertemporal utility function with a constant rate of time preference, ρ. Seen from time 0, the intertemporal utility function is

$$U_0 = \int_0^\infty u(c_t) L_t e^{-\rho t} dt,$$

where $c_t \equiv C_t/L_t$ is consumption per family member. The instantaneous utility function, $u(c)$, has $u'(c) > 0$ and $u''(c) < 0$, i.e., positive but diminishing marginal utility of consumption. The utility contribution from consumption per family member is weighted by the number of family members, L_t.

The household is seen as an infinitely-lived family, a family dynasty. The current members of the dynasty act in unity and are concerned about the utility from own consumption as well as the utility of the future generations within the dynasty.[1] Births (into adult life) do not amount to emergence of *new* economic agents with independent interests. Births and population growth are seen as just an expansion of the size of the already existing families. In contrast, in the Diamond OLG model births imply entrance of new economic decision makers whose preferences no-one cared about in advance.

In view of (10.2), U_0 can be written as

$$U_0 = \int_0^\infty u(c_t)e^{-(\rho-n)t}dt, \tag{10.3}$$

where the inconsequential positive factor L_0 has been eliminated. Here $\rho - n$ is known as the *effective* rate of time preference while ρ is the *pure* rate of time preference. We later introduce a restriction on $\rho - n$ to ensure boundedness from above of the utility integral in general equilibrium.

The household chooses a consumption-saving plan which maximizes U_0 subject to its budget constraint. Let $A_t \equiv a_t L_t$ be the household's (net) financial wealth in real terms at time t. We have

$$\dot{A}_t \equiv \frac{dA_t}{dt} = r_t A_t + w_t L_t - c_t L_t, \qquad A_0 \text{ given.} \tag{10.4}$$

This equation is a book-keeping relation telling how financial wealth or debt $(-A)$ changes over time depending on how consumption relates to current income. The equation merely says that the increase in financial wealth per time unit equals saving which equals income minus consumption. Income is the sum of the net return on financial wealth, $r_t A_t$, and labor income, $w_t L_t$, where w_t is the real wage.[2] Saving can be negative. In that case the household dissaves and does so simply by selling a part of its stock of capital goods or by taking loans in the loan market. The market prices, w_t and r_t, faced by the household are assumed to be piecewise continuous functions of time.

When the dynamic budget identity (10.4) is combined with a requirement of solvency, we have a budget *constraint*. Given the assumed perfect loan market, the relevant solvency requirement is the No-Ponzi-Game condition (NPG for short):

$$\lim_{t\to\infty} A_t e^{-\int_0^t r_s ds} \geq 0. \tag{10.5}$$

This condition says that financial wealth far out in the future cannot have a negative present value. That is, in the long run, debt is at most allowed to rise

[1] The discrete-time Barro model of Chapter 7 articulated such an altruistic bequest motive. In that chapter we also discussed some of the conceptual difficulties associated with the dynasty setup.

[2] Since the technology exhibits constant returns to scale, in competitive equilibrium the firms make no (pure) profits to pay out to their owners.

at a rate *less* than the real interest rate r. The NPG condition thus precludes permanent financing of the interest payments by new loans.[3]

The decision problem is: choose a plan $(c_t)_{t=0}^{\infty}$ so as to maximize U_0 subject to non-negativity of the control variable, c, and the constraints (10.4) and (10.5). The problem is a slight generalization of the problem studied in Section 9.4 of the previous chapter.

To solve the problem we shall apply the Maximum Principle. This method can be applied directly to the problem as stated above or to an equivalent problem with constraints expressed in per capita terms. Let us follow the latter approach. From the definition $a_t \equiv A_t/L_t$ we get by differentiation w.r.t. t

$$\dot{a}_t = \frac{L_t\dot{A}_t - A_t\dot{L}_t}{L_t^2} = \frac{\dot{A}_t}{L_t} - a_t n.$$

Substitution of (10.4) gives the dynamic budget identity in per capita terms:

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t, \qquad a_0 \text{ given.} \tag{10.6}$$

By inserting $A_t \equiv a_t L_t = a_t L_0 e^{nt}$, the NPG condition (10.5) can be rewritten

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0, \tag{10.7}$$

where the unimportant factor L_0 has been eliminated.

We see that in both (10.6) and (10.7) a kind of corrected interest rate appears, namely the interest rate, r, minus the family size growth rate, n. Although deferring consumption gives a real interest rate of r, this return is diluted on a per capita basis because it will have to be shared with more members of the family when $n > 0$. In the form (10.7) the NPG condition requires that per capita debt, if any, in the long run at most grows at a rate *less* than $r - n$, assuming the interest rate is a constant, r.

Solving the consumption-saving problem

The decision problem is now: choose $(c_t)_{t=\infty}^{0}$ so as to a maximize U_0 subject to the constraints: $c_t \geq 0$, (10.6), and (10.7). To solve the problem we apply the

[3]From the previous chapter we know that the NPG condition, in combination with (10.4), is equivalent to an ordinary *intertemporal* budget constraint which says that the present value of the planned consumption path cannot exceed initial total wealth, i.e., the sum of the initial financial wealth and the present value of expected future labor income.

Violating the NPG condition means running a "Ponzi game", that is, trying to make a fortune through the chain-letter principle where old investors are payed off with money from the new investors.

Maximum Principle. So we follow the same solution procedure as in the alike problem (apart from $n = 0$) of Section 9.4 of the previous chapter:

1) Set up the current-value Hamiltonian

$$H(a, c, \lambda, t) = u(c) + \lambda \left[(r - n) a + w - c\right],$$

where λ is the *adjoint variable* associated with the differential equation (10.6).

2) Differentiate H partially w.r.t. the control variable, c, and put the result equal to zero:

$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0. \tag{10.8}$$

3) Differentiate H partially w.r.t. the state variable, a, and put the result equal to minus the time derivative of λ plus the effective discount rate (appearing in the integrand of the criterion function) multiplied by λ:

$$\frac{\partial H}{\partial a} = \lambda(r - n) = -\dot{\lambda} + (\rho - n)\lambda. \tag{10.9}$$

4) Apply the Maximum Principle: an interior optimal path, $(a_t, c_t)_{t=0}^{\infty}$, will satisfy that there exists a continuous function $\lambda = \lambda_t$ such that for all $t \geq 0$, (10.8) and (10.9) hold along the path, and the transversality condition,

$$\lim_{t\to\infty} a_t \lambda_t e^{-(\rho-n)t} = 0, \tag{10.10}$$

is satisfied.[4]

The interpretation of these optimality conditions is as follows. The condition (10.8) can be considered a $MC = MB$ condition (in utility terms). It illustrates together with (10.9) that the adjoint variable, λ, constitutes the shadow price, measured in current utility, of per head financial wealth along the optimal path. In the differential equation (10.9) λn cancels out, and rearranging (10.9) gives

$$\frac{r\lambda + \dot{\lambda}}{\lambda} = \rho.$$

This can be interpreted as a no-arbitrage condition. The left-hand side gives the *actual* rate of return, measured in utility units, on the marginal unit of saving: $r\lambda$ can be seen as a dividend and $\dot{\lambda}$ as a capital gain. The right-hand side is the *required* rate of return in utility units, ρ. The household is willing to save the

[4] That in the present problem, optimality does indeed require the "standard" condition (10.10) satisfied is true (as shown in Chapter 9.4). It is not a *general* result contained in the Maximum Principle.

marginal unit of income only up to the point where the actual return on saving equals the required return.

The transversality condition (10.10) says that optimality requires that the present shadow value of per capita financial wealth goes to zero for $t \to \infty$. Combined with (10.8), the condition can be written

$$\lim_{t\to\infty} a_t u'(c_t)e^{-(\rho-n)t} = 0. \tag{10.11}$$

This requirement is not surprising if we compare with the alternative case where $\lim_{t\to\infty} a_t u'(c_t)e^{-(\rho-n)t} > 0$. In this case there would be over-saving; U_0 could be increased by reducing the long-run a_t through consuming more and thereby saving less. The opposite inequality, $\lim_{t\to\infty} a_t u'(c_t)e^{-(\rho-n)t} < 0$, will reflect over-consumption and not even satisfy the NPG condition in view of Proposition 2 of the previous chapter. In fact, from that proposition we know that the transversality condition (10.10) is equivalent to the NPG condition (10.7) being satisfied with strict equality, i.e.,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s-n)ds} = 0. \tag{10.12}$$

We should recall that the Maximum Principle gives only *necessary* conditions for an optimal plan. But since the Hamiltonian is jointly concave in (a, c) for every t, the necessary conditions are also *sufficient*, by Mangasarian's sufficiency theorem (Math Tools).

The first-order conditions (10.8) and (10.9) give the Keynes-Ramsey rule:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta(c_t)}(r_t - \rho), \tag{10.13}$$

where $\theta(c_t)$ is the (absolute) elasticity of marginal utility,

$$\theta(c_t) \equiv -\frac{c_t}{u'(c_t)}u''(c_t) > 0. \tag{10.14}$$

As we know from previous chapters, this elasticity measures the consumer's wish to smooth consumption over time. The inverse of $\theta(c_t)$ is the elasticity of intertemporal substitution in consumption. It measures the strength of the willingness to vary consumption over time in response to a change in the interest rate.

Note that the population growth rate, n, does not appear in the Keynes-Ramsey rule. Going from $n = 0$ to $n > 0$ implies that r_t is replaced by $r_t - n$ in the dynamic budget identity (10.6) and ρ is replaced by $\rho - n$ in the criterion function. Hence n cancels out in the Keynes-Ramsey rule. Yet n appears in the transversality condition and is thereby a co-determinant of the *level* of consumption for given wealth, cf. (10.18) below.

CRRA utility

In order that the model can accommodate Kaldor's stylized facts, it should be capable of generating a balanced growth path. When the population grows at the same constant rate as the labor force, here n, by definition balanced growth requires that per capita output, per capita capital, and per capita consumption grow at constant rates. At the same time another of Kaldor's stylized facts is that the general rate of return in the economy tends to be constant. But (10.13) shows that having a constant per capita consumption growth rate at the same time as r is constant, is only possible if the elasticity of marginal utility does *not* vary with c. Hence, it makes sense to assume that the right-hand-side of (10.14) is a positive constant, θ. We thus assume that the instantaneous utility function is of CRRA form:

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \qquad \theta > 0; \tag{10.15}$$

where, for $\theta = 1$, the right-hand side should be interpreted as $\ln c$ as explained in Section 3.3 of Chapter 3.[5]

In later sections of this chapter we let the time horizon of the decision maker go to infinity. To ease convergence of an infinite sum of discounted utilities, it is an advantage not to have to bother with additive constants in the period utilities and therefore we write the CRRA function as $c^{1-\theta}/(1-\theta)$ instead of the form, $(c^{1-\theta}-1)/(1-\theta)$, introduced in Chapter 3. As implied by Box 9.1, the two forms represent the same preferences.

So our Keynes-Ramsey rule simplifies to

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho). \tag{10.16}$$

The consumption function* The Keynes-Ramsey rule characterizes the optimal *rate of change* of consumption. The optimal initial *level* of consumption, c_0, will be the highest feasible c_0 which is compatible with both the Keynes-Ramsey rule and the NPG condition. And for this reason the choice of c_0 will exactly comply with the transversality condition (10.12). Although at this stage an explicit determination of c_0 is not necessary to pin down the equilibrium path of the economy (see below), we note in passing that c_0 can be found by the method described at the end of Chapter 9. Indeed, given the book-keeping relation (10.6), we know from Proposition 1 of that chapter that the transversality condition

[5] As mentioned in the previous chapter, in problems with infinite horizon it is an advantage not to have to bother with additive constants in the instantaneous utilities. Otherwise, convergence of the improper integral (10.3) may go by the board. Hence we write the CRRA function as in (10.15), without subtraction of the constant $1/1-\theta$).

(10.12) is equivalent to satisfying the intertemporal budget constraint with strict equality:

$$\int_0^\infty c_t e^{-\int_0^t (r_s - n)ds} dt = a_0 + h_0. \tag{10.17}$$

Solving the differential equation (10.16), we get $c_t = c_0 e^{\frac{1}{\theta}\int_0^t (r_s-\rho)ds}$ which we substitute for c_t in (10.17). Isolating c_0 now gives[6]

$$\begin{aligned} c_0 &= \beta_0(a_0 + h_0), \quad \text{where} \qquad (10.18)\\ \beta_0 &= \frac{1}{\int_0^\infty e^{\int_0^t (\frac{(1-\theta)r_s - \rho}{\theta} + n)ds} dt}, \quad \text{and} \\ h_0 &= \int_0^\infty w_t e^{-\int_0^t (r_s - n)ds} dt. \end{aligned}$$

Initial consumption is thus proportional to total wealth. The factor of proportionality is β_0, also called the marginal (and average) propensity to consume out of wealth. We see that the entire expected future evolution of wages and interest rates affects c_0 through β_0. Moreover, β_0 is less, the greater is the population growth rate, n.[7] The explanation is that the effective utility discount rate, $\rho - n$, is less, the greater is n. The propensity to save is greater the more mouths to feed in the future. The initial saving level will be $r_0 a_0 + w_0 - c_0 = r_0 a_0 + w_0 - \beta_0(a_0 + h_0)$.

In case $r_t = r$ for all t and $w_t = w_0 e^{gt}$, where $g < r - n$, we get $\beta_0 = [(\theta - 1)r + \rho - \theta n]/\theta$ and $a_0 + h_0 = a_0 + w_0/(r - n - g)$.

In the Solow growth model the saving-income ratio is a parameter, a given constant. The Ramsey model endogenizes the saving-income ratio. Solow's parametric saving-income ratio is replaced by two "deeper" parameters, the rate of impatience, ρ, and the desire for consumption smoothing, θ. As we shall see, the resulting saving-income ratio will not generally be constant outside the steady state of the dynamic system implied by the Ramsey model. But first we need a description of production.

10.2.2 Firms

There is a large number of firms. They have the same neoclassical production function with CRS,

$$Y_t = F(K_t^d, T_t L_t^d) \tag{10.19}$$

where Y_t is supply of output, K_t^d is capital input, and L_t^d is labor input, all measured per time unit, at time t. The superscript d on the two inputs indicates

[6] These formulas can also be derived directly from Example 1 of Chapter 9.5 by replacing $r(\tau)$ and ρ by $r(\tau) - n$ and $\rho - n$, respectively. As to h_0, see the hint in Exercise 9.1.

[7] This holds also if $\theta = 1$, i.e., $u(c) = \ln c$, since in that case $\beta_0 = \rho - n$.

that these inputs are seen from the demand side. The factor T_t represents the economy-wide level of technology as of time t and is exogenous. We assume there is technological progress at a constant rate g (≥ 0) :

$$T_t = T_0 e^{gt}, \qquad T_0 > 0. \tag{10.20}$$

Thus the economy features Harrod-neutral technological progress, as is needed for compliance with Kaldor's stylized facts.

Necessary and sufficient conditions for the factor combination (K_t^d, L_t^d), where $K_t^d > 0$ and $L_t^d > 0$, to maximize profits under perfect competition are that

$$F_1(K_t^d, T_t L_t^d) = \hat{r}_t \equiv r_t + \delta, \tag{10.21}$$

$$F_2(K_t^d, T_t L_t^d) T_t = w_t, \tag{10.22}$$

$\hat{r}_t$ being the rental rate of capital, cf. (10.1).

10.3 General equilibrium and dynamics

We now consider the economy as a whole and thereby the interaction between households and firms in the various markets. For simplicity, we assume that the number of households, N, is the same as the number of firms. We normalize this common number to *one* so that $F(\cdot, \cdot)$ from now on is interpreted as the aggregate production function and C_t as aggregate consumption.

Factor markets

In the short term, i.e., for fixed t, the available quantities of labor, $L_t = L_0 e^{nt}$, and capital, K_t, are predetermined. The factor markets clear at all points in time, that is,

$$K_t^d = K_t, \quad \text{and} \quad L_t^d = L_t, \qquad \text{for all } t \geq 0. \tag{10.23}$$

It is the rental rate, $\hat{r}_t$, and the wage rate, w_t, which adjust (immediately) so that this is achieved for every t. Aggregate output can be written

$$Y_t = F(K_t, T_t L_t) = T_t L_t F(\tilde{k}_t, 1) \equiv T_t L_t f(\tilde{k}_t), \qquad f' > 0, f'' < 0, \tag{10.24}$$

where $\tilde{k}_t \equiv k_t/T_t \equiv K_t/(T_t L_t)$ is the effective capital-labor ratio, also sometimes just called the "capital intensity". Substituting (10.23) into (10.21) and (10.22), we find the equilibrium interest rate and wage rate:

$$r_t = \hat{r}_t - \delta = \frac{\partial(T_t L_t f(\tilde{k}_t))}{\partial K_t} - \delta = f'(\tilde{k}_t) - \delta, \tag{10.25}$$

$$w_t = \frac{\partial(T_t L_t f(\tilde{k}_t))}{\partial(T_t L_t)} T_t = \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \right] T_t \equiv \tilde{w}(\tilde{k}_t) T_t, \tag{10.26}$$

where $\tilde{k}_t$ is at any point in time predetermined and where in (10.25) we have used the no-arbitrage condition (10.1).

Capital accumulation

From now on we leave out the explicit dating of the variables when not needed for clarity. By national product accounting we have

$$\dot{K} = Y - C - \delta K. \tag{10.27}$$

Let us check whether we get the same result from the wealth accumulation equation of the household. Because physical capital is the only asset in the economy, aggregate financial wealth, A, at time t equals the total quantity of capital, K, at time t.[8] With S^N denoting aggregate net saving, we thus have

$$\begin{aligned}
\dot{K} &= \dot{A} = S^N = rK + wL - cL && \text{(by (10.4))} \\
&= (f'(\tilde{k}) - \delta)K + (f(\tilde{k}) - \tilde{k}f'(\tilde{k}))TL - cL && \text{(by (10.25) and (10.26))} \\
&= f(\tilde{k})TL - \delta K - cL && \text{(by rearranging and use of } K \equiv \tilde{k}TL) \\
&= F(K, TL) - \delta K - C = Y - C - \delta K && \text{(by } C \equiv cL).
\end{aligned}$$

Hence the book-keeping is in order (the national product account is consistent with the national income account).

We now face an important difference as compared with models where households have a finite horizon, such as the Diamond OLG model. Current consumption cannot be determined independently of the expected entire future evolution of the economy. Consumption and saving, as we saw in Section 10.2, depend on the expectations of the future path of wages and interest rates. And given the presumption of rational expectations, here in the form of perfect foresight, the households' expectations are identical to the prediction that can be calculated from the model. In this way there is mutual dependence between expectations and the level and evolution of consumption. We can determine the level of consumption only in the context of the overall dynamic analysis. In fact, the economic agents are in some sense in the same situation as the outside analyst. They, too, have to think through the entire dynamics of the economy, including the mutual dependency between expectations and actual evolution, in order to form their rational expectations.

The dynamic system

We get a concise picture of the dynamics by reducing the model to the minimum number of coupled differential equations. This minimum number is two. The key

[8] Whatever financial claims on each other the households might have, they net out for the household sector as a whole.

endogenous variables are $\tilde{k} \equiv K/(TL)$ and $\tilde{c} \equiv C/(TL) \equiv c/T$. Using the rule for the growth rate of a quotient, we get

$$\begin{aligned}
\frac{\dot{\tilde{k}}}{\tilde{k}} &= \frac{\dot{K}}{K} - \frac{\dot{T}}{T} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - (g+n) \qquad \text{(from (10.2) and (10.20))} \\
&= \frac{F(K,TL) - C - \delta K}{K} - (g+n) \qquad \text{(from (10.27))} \\
&= \frac{f(\tilde{k}) - \tilde{c}}{\tilde{k}} - (\delta + g + n) \qquad \text{(from (10.24))}.
\end{aligned}$$

The associated differential equation for $\tilde{c}$ is obtained in a similar way:

$$\begin{aligned}
\frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{\dot{c}}{c} - \frac{\dot{T}}{T} = \frac{1}{\theta}(r_t - \rho) - g \quad \text{(from the Keynes-Ramsey rule)} \\
&= \frac{1}{\theta}\left[f'(\tilde{k}) - \delta - \rho - \theta g\right] \qquad \text{(from (10.25))}.
\end{aligned}$$

We thus end up with the dynamic system

$$\dot{\tilde{k}} = f(\tilde{k}) - \tilde{c} - (\delta + g + n)\tilde{k}, \qquad \tilde{k}_0 > 0 \quad \text{given}, \tag{10.28}$$

$$\dot{\tilde{c}} = \frac{1}{\theta}\left[f'(\tilde{k}) - \delta - \rho - \theta g\right]\tilde{c}. \tag{10.29}$$

There is no given initial value of c. Instead we have the transversality condition (10.12). Using $a_t = K_t/L_t \equiv \tilde{k}_t T_t = \tilde{k}_t T_0 e^{gt}$ and $r_t = f'(\tilde{k}_t) - \delta$, we see that (10.12) is equivalent to

$$\lim_{t \to \infty} \tilde{k}_t e^{-\int_0^t \left(f'(\tilde{k}_s) - \delta - g - n\right) ds} = 0. \tag{10.30}$$

Phase diagram By a *phase diagram* for the dynamic system (10.28) – (10.29) is meant a graph in the $(\tilde{k}, \tilde{c})$ plane showing projections of the time paths, $(\tilde{k}_t, \tilde{c}_t)_{t=0}^{\infty}$, that are consistent with the system for alternative arbitrary initial points, $(\tilde{k}_0, \tilde{c}_0)$. The phase diagram is shown in Fig. 10.2 below.

Fig. 10.1 is an aid for the construction of the phase diagram in Fig. 10.2. The curve OEB in Fig. 10.2 represents the points in the $(\tilde{k}, \tilde{c})$ plane where $\dot{\tilde{k}} = 0$ according to the differential equation (10.28). Such a curve is called a *nullcline* for $\tilde{k}$. We see from (10.28) that

$$\dot{\tilde{k}} = 0 \text{ for } \tilde{c} = f(\tilde{k}) - (\delta + g + n)\tilde{k} \equiv \tilde{c}(\tilde{k}). \tag{10.31}$$

The value of $\tilde{c}(\tilde{k})$ for alternative values of $\tilde{k}$ can be read off in Fig. 10.1 as the vertical distance between the curve $\tilde{y} = f(\tilde{k})$ and the line $\tilde{y} = (\delta + g + n)\tilde{k}$ (to save space, the proportions are somewhat distorted).[9] The maximum value of $\tilde{c}(\tilde{k})$, if it exists, is reached at the point where the tangent to the OEB curve in Fig. 10.2 is horizontal, i.e., where $\tilde{c}'(\tilde{k}) = f'(\tilde{k}) - (\delta + g + n) = 0$ or $f'(\tilde{k}) - \delta = g + n$. The value of $\tilde{k}$ satisfying this is the golden-rule capital intensity, $\tilde{k}_{GR}$:

$$f'(\tilde{k}_{GR}) - \delta = g + n. \tag{10.32}$$

By (10.28) follows that $\partial\dot{\tilde{k}}/\partial\tilde{c} = -1$. For points above the $\dot{\tilde{k}} = 0$ locus we thus have $\dot{\tilde{k}} < 0$, whereas for points below the $\dot{\tilde{k}} = 0$ locus, $\dot{\tilde{k}} > 0$. The horizontal arrows in the figure indicate these directions of movement of $\tilde{k}$ in the different regions.

We see from (10.29) that

$$\dot{\tilde{c}} = 0 \text{ for } f'(\tilde{k}) = \delta + \rho + \theta g \qquad \text{or} \qquad \tilde{c} = 0. \tag{10.33}$$

Let $\tilde{k}^* > 0$ satisfy the equation $f'(\tilde{k}^*) - \delta = \rho + \theta g$. Then the vertical half-line $\tilde{k} = \tilde{k}^*$, $\tilde{c} \geq 0$, represents points where $\dot{\tilde{c}} = 0$, and so does the horizontal half-line $\tilde{c} = 0, \tilde{k} \geq 0$. These two half-lines thus make up *nullclines* for $\tilde{c}$ according to the differential equation (10.29).

By (10.28) follows that for $\tilde{c} > 0$, $\partial\dot{\tilde{c}}/\partial\tilde{k} = \theta^{-1}f''(\tilde{k})\tilde{c} < 0$. For points to the left of the $\tilde{k} = \tilde{k}^*$ line we thus have $\dot{\tilde{c}} > 0$. And for points to the right of the $\tilde{k} = \tilde{k}^*$ line we have $\dot{\tilde{c}} < 0$. The vertical arrows in Fig. 10.2 indicate these directions of movement of $\tilde{c}$ in the different regions. Four illustrative examples of solution curves (*I*, *II*, *III*, and *IV*) are drawn in the figure. Since our dynamic system is "autonomous", the direction of movement depends only on the initial position, not on time. Hence, generally in a phase diagram the time index on $\tilde{k}$ and $\tilde{c}$ is omitted.

[9] As the graph is drawn, $f(0) = 0$, i.e., capital is assumed essential. But none of the conclusions we are going to consider depends on this.

Steady state

The point E in Fig. 10.2 has coordinates ($\tilde{k}^*$, $\tilde{c}^*$) and represents the unique steady state.[10] From (10.33) and (10.31), respectively, follows that

$$\begin{aligned} f'(\tilde{k}^*) &= \delta + \rho + \theta g, \qquad \text{and} \qquad (10.34) \\ \tilde{c}^* &= f(\tilde{k}^*) - (\delta + g + n)\tilde{k}^*. \qquad (10.35) \end{aligned}$$

So, in steady state the real interest rate is

$$r^* = f'(\tilde{k}^*) - \delta = \rho + \theta g. \qquad (10.36)$$

The effective capital-labor ratio satisfying this equation is known as the *modified-golden-rule* capital intensity, $\tilde{k}_{MGR}$. The modified golden rule is the rule saying that for a representative agent economy to be in steady state, the capital intensity must be such that the net marginal productivity of capital equals the required rate of return, taking into account the pure rate of time preference, ρ, and the desire for consumption smoothing, θ.[11]

We show below that the steady state is, in a specific sense, asymptotically stable. First we have to make sure, however, that the steady state is consistent with general equilibrium. This consistency requires that the household's transversality condition (10.30) holds in the point E, where, for all $t \geq 0$, $\tilde{k}_t = \tilde{k}^*$ and $f'(\tilde{k}_t) - \delta = \rho + \theta g$. So the condition (10.30) becomes

$$\lim_{t\to\infty} \tilde{k}^* e^{-(\rho+\theta g-g-n)t} = 0. \qquad (10.37)$$

This is fulfilled if and only if $\rho + \theta g > g + n$, a condition equivalent to

$$\rho - n > (1 - \theta)g. \qquad (A1)$$

This "sufficient impatience" condition also ensures that the improper integral U_0 is bounded from above (see Appendix B). If $\theta \geq 1$, (A1) is fulfilled as soon as

[10] As (10.33) shows, if $\tilde{c}_t = 0$, then $\dot{\tilde{c}} = 0$. Therefore, mathematically, point B (if it exists) in Fig. 10.2 is also a stationary point of the dynamic system. And if $f(0) = 0$, then according to (10.29) and (10.31) also the point $(0, 0)$ in the figure is a stationary point. But these stationary points have zero consumption forever and are therefore not steady states of any *economic* system. That is, they are "trivial" steady states.

[11] The ρ of the Ramsey model corresponds to the intergenerational discount rate R of Barro's dynasty model in Chapter 7. In the discrete time Barro model we have $1+r^* = (1+R)(1+g)^\theta$, which, by taking logs on both sides and using first-order Taylor approximations of $\ln(1 + x)$ around $x = 0$ gives $r^* \approx \ln(1 + r^*) = \ln(1 + R) + \theta \ln(1 + g) \approx R + \theta g$. Recall, however, that in view of the considerable period length (about 25-30 years) of the Barro model, this approximation may not be good.

the effective utility discount rate, $\rho - n$, is positive. (A1) may even hold for a negative $\rho - n$ if not "too" negative. If $\theta < 1$, (A1) requires $\rho - n$ to be "sufficiently positive".

Since the parameter restriction (A1) can be written $\rho + \theta g > g + n$, it implies that the steady-state interest rate, r^*, given in (10.36), is higher than the "natural" growth rate, $g + n$. If this did not hold, the transversality condition (10.12) would fail at the steady-state point E. Indeed, along the steady-state path we have

$$a_t e^{-(r^*-n)t} = k_t e^{-(r^*-n)t} = k_0 e^{gt} e^{-(r^*-n)t} = k_0 e^{(g+n-r^*)t},$$

which would take the constant positive value k_0 for all $t \geq 0$ if $r^* = g + n$ and would go to ∞ for $t \to \infty$ if $r^* < g + n$. The individual households would thus be over-saving. Each household would in this situation alter its behavior and the steady state could not be an equilibrium path.

Another way of seeing that $r^* \leq g + n$ can not be an equilibrium in a Ramsey model is to recognize that this condition would make the infinitely-lived household's human wealth $= \infty$ because wage income, wL, would grow at a rate, $g + n$, at least as high as the real interest rate, r^*. This would motivate an immediate increase in consumption and so the considered steady-state path would again not be an equilibrium.

To have a model of interest, from now on we assume that the preference and technology parameters satisfy the inequality (A1). As an implication, the effective capital-labor ratio in steady state, $\tilde{k}^*$, is less than the golden-rule value $\tilde{k}_{GR}$. Indeed, $f'(\tilde{k}^*) - \delta = \rho + \theta g > g + n = f'(\tilde{k}_{GR}) - \delta$, so that $\tilde{k}^* < \tilde{k}_{GR}$, in view of $f'' < 0$.

So far we have only ensured that *if* the steady state, E, exists, it is consistent with general equilibrium. Existence of a steady state requires that the marginal productivity of capital is sufficiently sensitive to variation in the effective capital-labor ratio:

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) - \delta > \rho + \theta g > \lim_{\tilde{k}\to\infty} f'(\tilde{k}) - \delta. \tag{A2}$$

We could proceed with this assumption. To allow comparison of the steady state of the model with a golden rule allocation, we need that a golden rule allocation exists. This requires that $\lim_{\tilde{k}\to 0} f'(\tilde{k}) - \delta > g + n > \lim_{\tilde{k}\to\infty} f'(\tilde{k}) - \delta$. This together with both (A2) and (A1) gives the "synthesized" condition

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) - \delta > \rho + \theta g > g + n > \lim_{\tilde{k}\to\infty} f'(\tilde{k}) - \delta. \tag{A2'}$$

By continuity of f', these inequalities ensure the existence of both $\tilde{k}^*$ and $\tilde{k}_{GR}$ such that $0 < \tilde{k}^* < \tilde{k}_{GR}$.[12] As illustrated by Fig. 10.1, the inequalities also ensure

[12] The often presumed Inada conditions, $\lim_{\tilde{k}\to 0} f'(\tilde{k}) = \infty$ and $\lim_{\tilde{k}\to\infty} f'(\tilde{k}) = 0$, are stricter than both (A2) and (A2') and are not necessary.

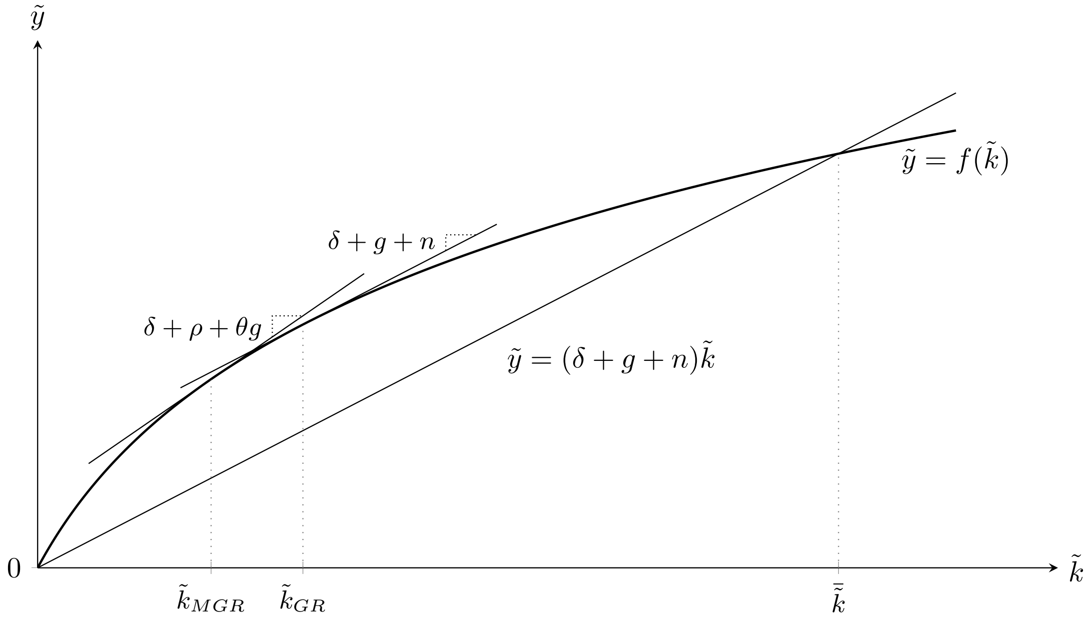


Figure 10.1: Building blocks for the phase diagram.

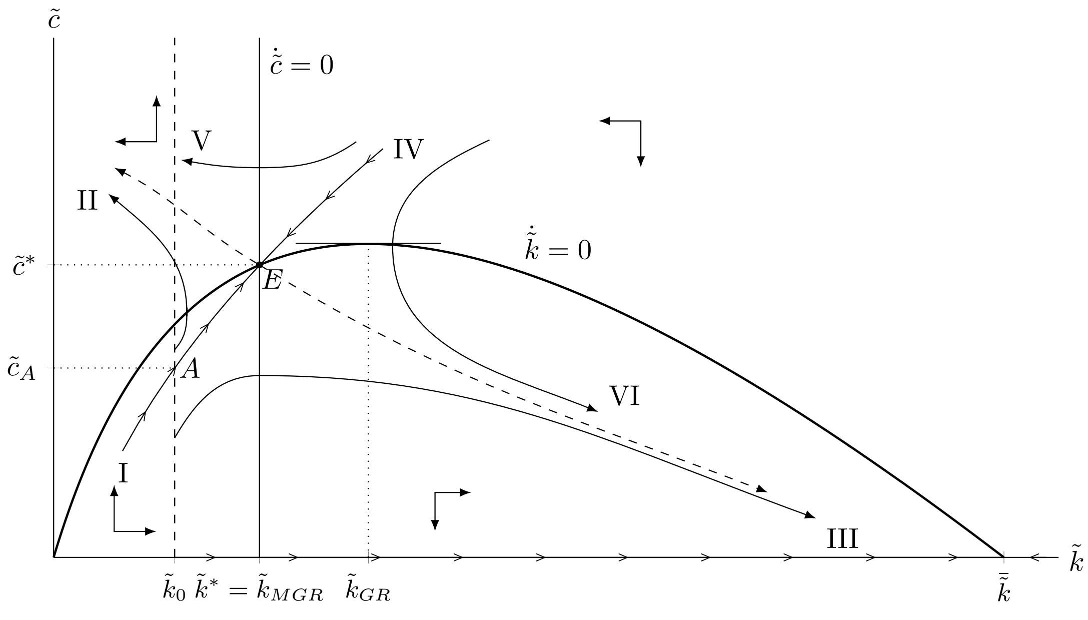


Figure 10.2: Phase diagram for the Ramsey model.

existence of a $\bar{\tilde{k}} > 0$ with the property that $f(\bar{\tilde{k}}) - (\delta + g + n)\tilde{k} = 0$.[13] Because $f'(\tilde{k}) > 0$ for all $\tilde{k} > 0$, it is implied by the assumption (A2') that $\delta + g + n > 0$. Even without deciding on the sign of n (a decreasing workforce should not be ruled out in our days), this inequality seems a plausible presumption.

Trajectories in the phase diagram

A first condition for a path $(\tilde{k}_t,\ \tilde{c}_t)$, with $\tilde{k}_t > 0$ and $\tilde{c}_t > 0$ for all $t \geq 0$, to be a solution to the model is that it satisfies the system of differential equations (10.28)-(10.29). Indeed, to be technically feasible, it must satisfy (10.28) and to comply with the Keynes-Ramsey rule, it must satisfy (10.29). Technical feasibility of the path also requires that the initial value for $\tilde{k}$ equals the historically given value $\tilde{k}_0 \equiv K_0/(T_0 L_0)$. In contrast, for $\tilde{c}$ we have no given initial value. This is because $\tilde{c}_0$ is a *jump variable,* also known as a *forward-looking variable.* These names are used for an endogenous variable which can immediately shift to another value if new information arrives so as to alter expectations about the future. We shall see that the terminal condition (10.30), reflecting the transversality condition of the households, makes up for this lack of an initial condition for c.

In Fig. 10.2 we have drawn some paths that are consistent with our dynamic system (10.28)-(10.29). We are especially interested in the paths which are consistent with the historically given $\tilde{k}_0$, that is, paths starting at some point on the stippled vertical line in the figure. If the economy started out with a "high" value of $\tilde{c}$, it would follow a curve like II in the figure. The low level of saving implies that the capital stock goes to zero in finite time (see Appendix C). If the economy starts out with a "low" level of $\tilde{c}$, it will follow a curve like III in the figure. The high level of saving implies that the effective capital-labor ratio converges toward $\bar{\tilde{k}}$ in the figure.

All in all this suggests the existence of an initial level of consumption somewhere in between, which results in a path like I. Indeed, since the curve II emerged with a high $\tilde{c}_0$, then by lowering this $\tilde{c}_0$ slightly, a path will emerge in which the maximal value of $\tilde{k}$ on the $\dot{\tilde{k}} = 0$ locus is greater than curve II's maximal $\tilde{k}$ value.[14] We continue lowering $\tilde{c}_0$ until the path's maximal $\tilde{k}$ value is exactly equal to $\tilde{k}^*$, where the the path ends. The path which emerges from this, namely the path I, starting at the point A, is special in that it converges toward

[13] We claim that $\bar{\tilde{k}} > \tilde{k}_{GR}$ *must* hold. Indeed, this inequality follows from $f'(\tilde{k}_{GR}) = \delta + n + g \equiv f(\bar{\tilde{k}})/\bar{\tilde{k}} > f'(\bar{\tilde{k}})$, the latter inequality being due to $f'' < 0$ and $f(0) \geq 0$ (consider the graph of $f(\tilde{k})$).

[14] As an implication of the uniqueness theorem for differential equations (see Math Tools), two solution paths in the phase plane cannot intersect.

the steady-state point E. No other path starting at the stippled line, $\tilde{k} = \tilde{k}_0$, has this property. Paths starting above A do not, as we just saw. Neither do paths starting below A, like path III. Either this path never reaches the consumption level $\tilde{c}_A$ in which case it can not converge to E, of course. Or, after a while its consumption level reaches $\tilde{c}_A$, but at the same time it must have $\tilde{k} > \tilde{k}_0$. From then on, as long as $\tilde{k} \leq \tilde{k}^*$, for every $\tilde{c}$-value that path III has in common with path I, path III has a higher $\dot{\tilde{k}}$ and a lower $\dot{\tilde{c}}$ than path I (use (10.28) and (10.29)). Hence, path III diverges from point E.

Had we considered a value of $\tilde{k}_0 > \tilde{k}^*$, there would similarly be a unique value of $\tilde{c}_0$ such that the path starting from $(\tilde{k}_0, \tilde{c}_0)$ would converge to E (see path IV in Fig. 10.2).

The point E is a *saddle point*. By this is meant a steady state with the following property: there exists exactly two paths in the phase plane, one from each side of $\tilde{k}^*$, that converge toward the steady-state point. All other paths (at least if starting in a neighborhood of the steady state) move away from the steady state and asymptotically approach one of the two diverging paths, the stippled North-West and South-East curves in Fig. 10.2.[15] The two converging paths are called *saddle paths*.[16] In combination they make up what is known as the *stable branch* (or *stable arm*). The stippled diverging paths in Fig. 10.2, together, make up the *unstable branch* (or *unstable arm*).

The equilibrium path

A solution to the model is a path which is technically feasible and satisfies a set of equilibrium conditions. In analogy with the definition in discrete time (see Chapter 3) a path $(\tilde{k}_t, \tilde{c}_t)_{t=0}^{\infty}$ is called a *technically feasible path* if (i) the path has $\tilde{k}_t \geq 0$ and $\tilde{c}_t \geq 0$ for all $t \geq 0$; (ii) it satisfies the accounting equation (10.28); and (iii) it starts out, at $t = 0$, with the historically given initial effective capital-labor ratio. An *equilibrium path* with perfect foresight is then a technically feasible path $(\tilde{k}_t, \tilde{c}_t)_{t=0}^{\infty}$ with the properties that the path (a) is consistent with firms' profit maximization and households' optimization given their expectations and budget constraints; (b) is consistent with market clearing for all $t \geq 0$; and (c) has the property that the evolution of the pair (w_t, r_t), where $w_t = \tilde{w}(\tilde{k}_t)T_t$ and $r_t = f'(\tilde{k}_t) - \delta$, is as expected by the households. Among other things these conditions require the (transformed) Keynes-Ramsey rule, (10.29), and the transversality condition, (10.30), to hold for all $t \geq 0$.

Consider the case illustrated in Fig. 10.2, where $0 < \tilde{k}_0 < \tilde{k}^*$. The path which

[15] The algebraic definition of a saddle point, in terms of eigenvalues, is given in Appendix A.

[16] If $\lim_{\tilde{k}\to 0} f(\tilde{k}) = 0$, then the saddle path on the left-hand side of the steady state in Fig. 10.2 will start out infinitely close to the origin, see Appendix A.

starts at point A and follows the saddle path toward the steady state is an equilibrium path because, by construction, it is technically feasible and in addition has the required properties, (a), (b), and (c). More intuitively: if the households expect an evolution of w_t and r_t corresponding to this path (that is, expect a corresponding underlying movement of $\tilde{k}_t$, which we know unambiguously determines r_t and w_t), then these expectations will induce a behavior the aggregate result of which is an actual path for $(\tilde{k}_t, \tilde{c}_t)$ that confirms the expectations. And along this path the households find no reason to correct their behavior because the path allows both the Keynes-Ramsey rule and the transversality condition to be satisfied.

No other path than the saddle path can be an equilibrium path. This is because no other technically feasible path is compatible with the households' individual utility maximization under perfect foresight. An initial point above point A can be excluded because the implied path of type II does not satisfy the household's NPG condition (and, consequently, not at all the transversality condition).[17] If the individual household expected an evolution of r_t and w_t corresponding to path II, then the household would immediately choose a *lower* level of consumption, that is, the household would *deviate* in order not to suffer the same fate as Charles Ponzi. In fact, *all* the households would react in this way. Thus, path II would not be realized and the expectation that it would, can not be a rational expectation.

Likewise, an initial point below point A can be ruled out because the implied path of type III does not satisfy the household's transversality condition but implies over-saving. Indeed, at some point in the future, say at time t_1, the economy's effective capital-labor ratio would pass the golden rule value so that for all $t > t_1$, $r_t < g + n$. But with a rate of interest permanently below the growth rate of wage income of the household, the present value of human wealth is *infinite*. This motivates a *higher* consumption level than that along the path. Thus, if the household expects an evolution of r_t and w_t corresponding to path III, then the household will immediately *deviate* and choose a higher initial level of consumption. But so will *all* the households react and the expectation that the economy will follow path III can not be rational.

We have presumed $0 < \tilde{k}_0 < \tilde{k}^*$. If instead $\tilde{k}_0 > \tilde{k}^*$, the economy would move along the saddle path *from above*. Paths like VI and V in Fig. 10.2 can be ruled out because they violate the transversality condition and the NPG condition, respectively (in fact, violating the NPG implies violating the TVC as well). With this we have shown:

PROPOSITION 1 Assume (A1) and (A2). Let there be a given $\tilde{k}_0 > 0$. Then the Ramsey model exhibits a unique equilibrium path, characterized by $(\tilde{k}_t,\ \tilde{c}_t)$

[17] This is shown in Appendix C.

converging, for $t \to \infty$, toward a unique steady state with an effective capital-labor ratio, $\tilde{k}^*$, satisfying $f'(\tilde{k}^*) - \delta = \rho + \theta g$. In the steady state the real interest rate is given by the modified-golden-rule formula, $r^* = \rho + \theta g$, the per capita consumption path is $c_t^* = \tilde{c}^* T_0 e^{gt}$, where $\tilde{c}^* = f(\tilde{k}^*) - (\delta + g + n)\tilde{k}^*$, and the real wage path is $w_t^* = \tilde{w}(\tilde{k}^*) T_0 e^{gt}$.

A numerical example based on one year as the time unit: $g = 0.02$, $n = 0.01$, $\theta = 2$, and $\rho = 0.01$. Then, $r^* = 0.05 > 0.03 = g + n$.

So output per capita, $y_t \equiv Y_t/L_t \equiv \tilde{y}_t T_t$, tends to grow at the rate of technological progress, g :

$$\frac{\dot{y}_t}{y_t} \equiv \frac{\dot{\tilde{y}}_t}{\tilde{y}_t} + \frac{\dot{T}_t}{T_t} = \frac{f'(\tilde{k}_t)\dot{\tilde{k}}_t}{f(\tilde{k}_t)} + g \to g \quad \text{for} \quad t \to \infty,$$

in view of $\dot{\tilde{k}}_t \to 0$ combined with $\lim_{t\to\infty} f'(\tilde{k}_t)/f(\tilde{k}_t) = f'(\tilde{k}^*)/f(\tilde{k}^*)$. This is also true for the growth rate of consumption per capita and the real wage, since $c_t \equiv \tilde{c}_t T_t$ and $w_t = \tilde{w}(\tilde{k}_t) T_t$.

The intuition behind the convergence lies in the neoclassical principle of *diminishing marginal productivity of capital.* Starting from a *low* effective capital-labor ratio and therefore a high marginal and average productivity of capital, the resulting high aggregate saving[18] will be more than enough to maintain the effective capital-labor ratio which therefore increases. But when this happens, the marginal and average productivity of capital decreases and the resulting saving, as a proportion of the capital stock, declines until eventually it is only sufficient to replace worn-out machines and equip new "effective" workers with enough machines to just maintain the effective capital-labor ratio. If instead we start from a *high* effective capital-labor, a similar story can be told in reverse. In the long run the effective capital-labor ratio settles down at the steady-state level, $\tilde{k}^*$, where the marginal saving and investment yields a return as great as the representative household's willingness to postpone the marginal unit of consumption. Since the adjustment process is based on capital accumulation, the process is slow. The "speed of adjustment", in the sense of the proportionate rate of decline per year of the distance to the steady state, $\left|\tilde{k} - \tilde{k}^*\right|$, is generally assessed to be in the interval (0.02, 0.10), assuming absence of disturbances to the system during the adjustment.

The equilibrium path generated by the Ramsey model is necessarily dynamically efficient and satisfies the modified golden rule in the long run. Why is there this contrast to Diamonds OLG model where equilibrium paths *may* be

[18] Saving will be high because the negative substitution and wealth effects on current consumption of the high interest rate dominate the income effect.

dynamically inefficient? The reason lies in the fact that only a "single infinity", not a "double infinity", is present in the Ramsey model. The time horizon of the economy is infinite but the number of decision makers is finite. Births (into adult life) do not reflect the emergence of new economic agents with separate interests. In the OLG model, however, births imply entrance of new economic decision makers whose preferences no-one cared about in advance. In that model neither is there any final date, nor any final decision maker. Because of this difference, in several respects the two models give different results. A type of equilibria, namely dynamically inefficient ones, can be realized in the Diamond model but not so in the Ramsey model. A rate of time preference low enough to generate a *tendency* to a long-run interest rate below the income growth rate is inconsistent with existence of general equilibrium in the Ramsey model. It was precisely with the aim of ruling out such a low rate of impatience that we imposed the parameter restriction (A1) above.

The concept of saddle-point stability

The steady state of the model is globally asymptotically stable for arbitrary initial values of the effective capital-labor ratio (the phase diagram only verifies local asymptotic stability, but the extension to global asymptotic stability is verified in Appendix A). If $\tilde{k}$ is hit by a shock at time 0 (say by a discrete jump in the technology level T_0), the economy will converge toward the same unique steady state as before. At first glance this might seem peculiar considering that the steady state is a saddle point. Such a steady state is unstable for arbitrary values of *both* coordinates of the initial point $(\tilde{k}_0, \tilde{c}_0)$. But the crux of the matter is that it is only the initial $\tilde{k}$ that *is* arbitrary. The model assumes that the decision variable c_0, and therefore the value of $\tilde{c}_0 \equiv c_0/T_0$, immediately adjusts to the new situation. That is, the model assumes that $\tilde{c}_0$ always takes the value needed for the household's transversality condition under perfect foresight to be satisfied. This ensures that the economy is initially on the saddle path, cf. the point A in Fig. 10.2. In the language of differential equations *conditional* asymptotic stability is present. The condition that transform the conditional stability to actual stability is the transversality condition.

We shall follow the common terminology in macroeconomics and call a steady state of a two-dimensional dynamic system (locally) *saddle-point stable* if:

1. the steady state is a saddle point;

2. one of the two endogenous variables is predetermined while the other is a jump variable;

3. at least close to the steady state, the saddle path is not parallel to the jump-variable axis;

4. there is a boundary condition on the system such that the diverging paths are ruled out as solutions.

To establish saddle-point stability, all four properties must be verified. If for instance point 1 and 2 hold but, contrary to point 3, the saddle path is parallel to the jump variable axis, then saddle-point stability does not obtain. Indeed, given that the predetermined variable initially deviated from its steady-state value, it would not be possible to find any initial value of the jump variable such that the solution of the system would converge to the steady state for $t \to \infty$.

In the present case, we have already verified point 1 and 2. And as the phase diagram indicates, the saddle path is not vertical. So also point 3 holds. The transversality condition ensures that also point 4 holds. Thus, the Ramsey model is saddle-point stable. In Appendix A it is shown that the positively-sloped saddle path in Fig. 10.2 ranges over *all* $\tilde{k} > 0$ (there is nowhere a vertical asymptote to the saddle path). Hence, the steady state is *globally* saddle-point stable. All in all, these characteristics of the Ramsey model are analogue to those of Barro's dynasty model in discrete time when the bequest motive is operative.

10.4 Comparative analysis

10.4.1 The role of key parameters

The conclusion that in the long run the real interest rate is given by the modified golden rule formula, $r^* = \rho + \theta g$, tells us that only three parameters matter: the rate of time preference, the elasticity of marginal utility, and the rate of technological progress. A higher ρ, i.e., more impatience and thereby less willingness to defer consumption, implies less capital accumulation and thus in the long run smaller effective capital-labor ratio, higher interest rate, and lower consumption than otherwise. The long-run growth rate is unaffected.

A higher θ will have a similar effect, when $g > 0$. As θ is a measure of the desire for consumption smoothing, a higher θ implies that a larger part of the greater wage income in the future, reflecting technology growth, will be consumed immediately. This implies less saving and thereby less capital accumulation and so a lower $\tilde{k}^*$ and higher r^*. Similarly, the long-run interest rate will depend positively on the technology growth rate g because the higher g is, the greater is the expected future wage income. Thereby the consumption possibilities in the future are greater even without any current saving. This discourages current saving and we end up with lower capital accumulation and lower effective capital-labor ratio in the long run, hence higher interest rate. It is also true that the higher is g, the higher is the rate of return needed to induce the saving required for maintaining a steady state and resist the desire for more consumption smoothing.

The long-run interest rate is independent of the particular form of the aggregate production function, f. This function matters for *what* effective capital-labor ratio and *what* consumption level per unit of effective labor are compatible with the long-run interest rate. This kind of results are specific to representative agent models. This is because only in these models will the Keynes-Ramsey rule hold not only for the individual household, but also at the aggregate level.

Unlike the Solow growth model, the Ramsey model provides a *theory* of the evolution and long-run level of the saving-income ratio. The endogenous saving-income ratio of the economy is

$$\begin{aligned} s_t &\equiv \frac{Y_t - C_t}{Y_t} = \frac{\dot{K}_t + \delta K_t}{Y_t} = \frac{\dot{K}_t/K_t + \delta}{Y_t/K_t} = \frac{\dot{\tilde{k}}_t/\tilde{k}_t + g + n + \delta}{f(\tilde{k}_t)/\tilde{k}_t} \\ &\to \frac{g + n + \delta}{f(\tilde{k}^*)/\tilde{k}^*} \equiv s^* \quad \text{for} \quad t \to \infty. \end{aligned} \tag{10.38}$$

By determining the path of $\tilde{k}_t$, the Ramsey model determines how s_t moves over time and adjusts to its constant long-run level. Indeed, for any given $\tilde{k} > 0$, the equilibrium value of $\tilde{c}_t$ is uniquely determined by the requirement that the economy must be on the saddle path. Since this defines $\tilde{c}_t$ as a function, $\tilde{c}(\tilde{k}_t)$, of $\tilde{k}_t$, there is a corresponding function for the saving-income ratio in that $s_t = 1 - \tilde{c}(\tilde{k}_t)/f(\tilde{k}_t) \equiv s(\tilde{k}_t)$. So $s(\tilde{k}^*) = s^*$.

We note that the long-run saving-income ratio is a decreasing function of the rate of impatience, ρ, and the desire of consumption smoothing, θ. The ratio is an increasing function of the capital depreciation rate, δ, and the rate of population growth, n.

For an example with an explicit formula for the long-run saving-income ratio, consider:

EXAMPLE 1 Suppose the production function is Cobb-Douglas:

$$\tilde{y} = f(\tilde{k}) = A\tilde{k}^{\alpha}, \qquad A > 0, 0 < \alpha < 1. \tag{10.39}$$

Then $f'(\tilde{k}) = A\alpha\tilde{k}^{\alpha-1} = \alpha f(\tilde{k})/\tilde{k}$. In steady state we get, by use of the steady-state result (10.34),

$$\frac{f(\tilde{k}^*)}{\tilde{k}^*} = \frac{1}{\alpha} f'(\tilde{k}^*) = \frac{\delta + \rho + \theta g}{\alpha}.$$

Substitution in (10.38) gives

$$s^* = \alpha \frac{\delta + g + n}{\delta + \rho + \theta g} < \alpha, \tag{10.40}$$

where the inequality follows from our parameter restriction (A1). Indeed, (A1) implies $\rho + \theta g > g + n$. The long-run saving-income ratio depends positively on the following parameters: the elasticity of production w.r.t. to capital, α, the capital depreciation rate, δ, and the population growth rate, n. The long-run saving-income ratio depends negatively on the rate of impatience, ρ, and the desire for consumption smoothing, θ. The role of the rate of technological progress is ambiguous.[19]

A numerical example (time unit = 1 year): If $n = 0.005$, $g = 0.015$, $\rho = 0.025$, $\theta = 3$, and $\delta = 0.07$, then $s^* = 0.21$. With the same parameter values except $\delta = 0.05$, we get $s^* = 0.19$.

It can be shown (see Appendix D) that if, by coincidence, $\theta = 1/s^*$, then $s'(\tilde{k}) = 0$, that is, the saving-income ratio s_t is also outside of steady state equal to s^*. In view of (10.40), the condition $\theta = 1/s^*$ is equivalent to the "knife-edge" condition $\theta = (\delta + \rho)/\left[\alpha(\delta + g + n) - g\right] \equiv \bar{\theta}$. More generally, assuming $\alpha(\delta + g + n) > g$ (which seems likely empirically), we have that if $\theta \lesseqgtr 1/s^*$ (i.e., $\theta \lesseqgtr \bar{\theta}$), then $s'(\tilde{k}) \lesseqgtr 0$, respectively (and if instead $\alpha(\delta + g + n) \leq g$, then $s'(\tilde{k}) < 0$, unconditionally).[20] Data presented in Barro and Sala-i-Martin (2004, p. 15) indicate no trend for the US saving-income ratio, but a positive trend for several other developed countries since 1870. One interpretation is that whereas the US has for a long time been close to its steady state, the other countries are still in the adjustment process toward the steady state. As an example, consider the parameter values $\delta = 0.05$, $\rho = 0.02$, $g = 0.02$ and $n = 0.01$. In this case we get $\bar{\theta} = 10$ if $\alpha = 0.33$; given $\theta < 10$, these other countries should then have $s'(\tilde{k}) < 0$ which, according to the model, is compatible with a rising saving-income ratio over time only if these countries are approaching their steady state from *above* (i.e., they should have $\tilde{k}_0 > \tilde{k}^*$). It may be argued that α should also reflect the role of education and R&D in production and thus be higher; with $\alpha = 0.75$ we get $\bar{\theta} = 1.75$. Then, if $\theta > 1.75$, these countries would have $s'(\tilde{k}) > 0$ and thus approach their steady state from *below* (i.e., $\tilde{k}_0 < \tilde{k}^*$). □

10.4.2 Special case: Solow's growth model*

The above results give a hint that Solow's growth model, with a given constant saving-income ratio $s \in (0, 1)$ and given δ, g, and n (with $\delta + g + n > 0$), can, under

[19] Partial differentiation w.r.t. g yields $\partial s^*/\partial g = \alpha\left[\rho - \theta n - (\theta - 1)\delta\right]/(\delta + \rho + \theta g)^2$, the sign of which cannot be determined a priori.

[20] See Appendix D.

certain circumstances, be interpreted as a special case of the Ramsey model. The Solow model in continuous time is given by

$$\dot{\tilde{k}}_t = sf(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t.$$

The constant saving-income ratio implies proportionality between consumption and income. In growth-corrected terms per capita consumption is

$$\tilde{c}_t = (1 - s)f(\tilde{k}_t).$$

For the Ramsey model to yield this, the production function must be like in (10.39) (i.e., Cobb-Douglas) with $\alpha > s$. And the elasticity of marginal utility, θ, must satisfy $\theta = 1/s$. Finally, the rate of time preference, ρ, must be such that (10.40) holds with s^* replaced by s, which implies $\rho = \alpha(\delta + g + n)/s - \delta - \theta g$. It remains to show that this ρ satisfies the inequality, $\rho - n > (1 - \theta)g$, which is necessary for existence of an equilibrium in the Ramsey model. Since $\alpha/s > 1$, the chosen ρ satisfies $\rho > \delta + g + n - \delta - \theta g = n + (1 - \theta)g$, which was to be proved. Thus, in this case the Ramsey model generates an equilibrium path which implies an evolution identical to that generated by the Solow model with $s = 1/\theta$.[21]

With this foundation of the Solow model, it will always hold that $s = s^* < s_{GR}$, where s_{GR} is the golden rule saving-income ratio. Indeed, from (10.38) and (10.32), respectively,

$$s_{GR} = \frac{(\delta + g + n)\tilde{k}_{GR}}{f(\tilde{k}_{GR})} = \frac{f'(\tilde{k}_{GR})\tilde{k}_{GR}}{f(\tilde{k}_{GR})} = \alpha > s^*,$$

from the Cobb-Douglas specification and (10.40), respectively.

A point of the Ramsey model vis-a-vis the Solow model is to replace a mechanical saving rule by maximization of discounted utility and thereby, on the one hand, open up for (i) a wider range of possible evolutions; (ii) welfare analysis; and (iii) analysis of incentive effects of economic policy on households' saving. On the other hand, in some respects the Ramsey model narrows down the range of possibilities, for example by unconditionally ruling out over-accumulation (dynamic inefficiency).

10.5 A social planner's problem

Another implication of the Ramsey framework is that the decentralized market equilibrium (within the idealized presumptions of the model) brings about the same allocation of resources as would a social planner facing the same technology and initial resources as described above and having the same criterion function as the representative household.

[21] A proof is given in Appendix D.

10.5.1 The equivalence theorem

As in Chapter 8, by a *social planner* we mean a hypothetical central authority who is "all-knowing and all-powerful" and is constrained only by the limitations arising from technology and initial resources. Within these confines the social planner can fully decide on the resource allocation. Since we consider a closed economy, the social planner has no access to an international loan market.

Let the economy be closed and let the social welfare function be time separable with constant elasticity, $\hat{\theta}$, of marginal utility and a pure rate of time preference $\hat{\rho}$.[22] Then the social planner's optimization problem is

$$\begin{aligned}
\max_{(c_t)_{t=0}^{\infty}} W_0 &= \int_0^{\infty} \frac{c_t^{1-\hat{\theta}}}{1-\hat{\theta}} e^{-(\hat{\rho}-n)t} dt \qquad \text{s.t.} && (10.41)\\
c_t &\geq 0, && (10.42)\\
\dot{\tilde{k}}_t &= f(\tilde{k}_t) - \frac{c_t}{T_t} - (\delta + g + n)\tilde{k}_t, && (10.43)\\
\tilde{k}_t &\geq 0 \qquad \text{for all } t \geq 0. && (10.44)
\end{aligned}$$

We assume $\hat{\theta} > 0$ and $\hat{\rho} - n > (1 - \hat{\theta})g$ in line with the assumption (A1) for the market economy above. In case $\hat{\theta} = 1$, the expression $c_t^{1-\hat{\theta}}/(1-\hat{\theta})$ should be interpreted as $\ln c_t$. No market prices or other elements belonging to the specific market institutions of the economy enter the social planner's problem. The dynamic constraint (10.43) reflects the national product account. Because the economy is closed, the social planner does not have the opportunity of borrowing or lending from abroad. Hence there is no solvency requirement. Instead we just impose the definitional constraint (10.44) of non-negativity of the state variable $\tilde{k}$.

The social planner's problem is to select, within the technically feasible paths, the one that maximizes the value of the social welfare function W_0. The problem is a continuous time analogue of the social planner's problem in discrete time in Chapter 8. Note, however, a minor conceptual difference, namely that in continuous time there is in the short run no *upper* bound on the *flow* variable c_t, that is, no bound like $c_t \leq T_t \left[f(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t \right]$. A consumption intensity c_t which is higher than the right-hand side of this inequality will just be reflected in a negative value of the flow variable $\dot{\tilde{k}}_t$.[23]

[22] Possible reasons for allowing these two preference parameters to deviate from the corresponding parameters in the private sector were discussed in Chapter 8.1.1.

[23] As usual we presume that capital can be "eaten". That is, we consider the capital good to be instantaneously convertible to a consumption good. Otherwise there *would* be at any time

To solve the problem we apply the Maximum Principle. The current-value Hamiltonian is

$$H(\tilde{k}, c, \lambda, t) = \frac{c^{1-\hat{\theta}}}{1-\hat{\theta}} + \lambda \left[f(\tilde{k}) - \frac{c}{T} - (\delta + g + n)\tilde{k} \right],$$

where λ is the adjoint variable associated with the dynamic constraint (10.43). An interior optimal path $(\tilde{k}_t, c_t)_{t=0}^{\infty}$ will satisfy that there exists a continuous function $\lambda = \lambda(t)$ such that, for all $t \geq 0$,

$$\frac{\partial H}{\partial c} = c^{-\hat{\theta}} - \frac{\lambda}{T} = 0, \text{ i.e., } c^{-\hat{\theta}} = \frac{\lambda}{T}, \quad \text{and} \tag{10.45}$$

$$\frac{\partial H}{\partial \tilde{k}} = \lambda(f'(\tilde{k}) - \delta - g - n) = (\hat{\rho} - n)\lambda - \dot{\lambda} \tag{10.46}$$

hold along the path. Finally, in the present problem the "standard" transversality condition,

$$\lim_{t\to\infty} \tilde{k}_t \lambda_t e^{-(\hat{\rho}-n)t} = 0, \tag{10.47}$$

is necessary for optimality, when $\hat{\rho} - n > (1-\hat{\theta})g$, as assumed above.[24]

The condition (10.45) can be seen as a $MC = MB$ condition and illustrates that λ_t is the social planner's shadow price, measured in terms of current utility, of $\tilde{k}_t$ along the optimal path.[25] The differential equation (10.46) tells us how this shadow price evolves over time. The transversality condition, (10.47), together with (10.45), entails the condition

$$\lim_{t\to\infty} \tilde{k}_t c_t^{-\hat{\theta}} e^{gt} e^{-(\hat{\rho}-n)t} = 0,$$

where the unimportant factor T_0 has been eliminated. Imagine the opposite were true, namely that $\lim_{t\to\infty} \tilde{k}_t c_t^{-\hat{\theta}} e^{[g-(\hat{\rho}-n)]t} > 0$. Then, intuitively U_0 could be increased by reducing the long-run value of $\tilde{k}_t$, i.e., consume more and save less.

By taking logs in (10.45) and differentiating w.r.t. t, we get $-\hat{\theta}\dot{c}/c = \dot{\lambda}/\lambda - g$. Inserting (10.46) and rearranging gives the condition

$$\frac{\dot{c}}{c} = \frac{1}{\hat{\theta}}(g - \frac{\dot{\lambda}}{\lambda}) = \frac{1}{\hat{\theta}}(f'(\tilde{k}) - \delta - \hat{\rho}). \tag{10.48}$$

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an upper bound on c, namely $c \leq Tf(\tilde{k})$, saying that the per capita consumption flow cannot exceed the per capita output flow. The role of such constraints is discussed in Feichtinger and Hartl (1986).

[24] See Appendix E.

[25] Decreasing c_t by one unit, increases $\tilde{k}_t$ by $1/T_t$ units, each of which are worth λ_t utility units to the social planner.

This is the social planner's Keynes-Ramsey rule. If the rate of time preference, $\hat{\rho}$, is lower than the net marginal productivity of capital, $f'(\tilde{k})-\delta$, the social planner will let per capita consumption be relatively low in the beginning in order to attain greater per capita consumption later. The lower the impatience relative to the return to capital, the more favorable it becomes to defer consumption.

Because $\tilde{c} \equiv c/T$, we get from (10.48) qualitatively the same differential equation for $\tilde{c}$ as we obtained in the decentralized market economy. And the dynamic resource constraint (10.43) is of course identical to that of the decentralized market economy. Thus, the dynamics are in principle unaltered and the phase diagram in Fig. 10.2 is still valid. The solution of the social planner implies that the economy will move along the saddle path toward the steady state. This trajectory, path I in the diagram, satisfies both the first-order conditions and the transversality condition. However, paths such as III in the figure do not satisfy the transversality condition of the social planner but imply permanent over-saving. And paths such as II in the figure will experience a sudden end when all the capital has been used up. Intuitively, they cannot be optimal. A rigorous argument is given in Appendix E, based on the fact that the Hamiltonian is *strictly concave* in $(\tilde{k}, \tilde{c})$. Thence, not only is the saddle path an optimal solution, it is the *unique* optimal solution.

Comparing with the market solution of the previous section, we have established:

PROPOSITION 2 *(equivalence theorem)* Consider an economy with neoclassical CRS technology and a representative infinitely-lived household with preferences as in (10.3) with $u(c) = c^{1-\theta}/(1-\theta)$. Assume (A1) and (A2). Let there be a given $\tilde{k}_0 > 0$. Then a perfectly competitive market economy brings about the same resource allocation as that brought about by a social planner with the same criterion function as the representative household, i.e., with $\hat{\theta} = \theta$ and $\hat{\rho} = \rho$.

This is a continuous time analogue to the discrete time equivalence theorem of Chapter 8.

The effective capital-labor ratio $\tilde{k}$ in the social planner's solution will not converge toward the golden rule level, $\tilde{k}_{GR}$, but toward a level whose distance to the golden rule level depends on how much $\hat{\rho} + \hat{\theta}g$ exceeds the natural growth rate, $g+n$. Even if society would be able to consume more in the long term if it aimed for the golden rule level, this would not compensate for the reduction in current consumption which would be necessary to achieve it. This consumption is relatively more valuable, the greater is the social planner's effective rate of time preference, $\hat{\rho} - n$. In line with the market economy, the social planner's solution ends up in a *modified golden rule.* In the long term, net marginal productivity of capital is determined by preference parameters and productivity growth and equals $\hat{\rho}+\hat{\theta}g > g+n$. Hereafter, given the net marginal productivity of capital, the effective capital-labor ratio and the level of the consumption path is determined by the production function.

Varieties of generational discounting* In the above analysis the social planner maximizes the sum of discounted per capita utilities *weighted* by generation size. This implies *utilitaristic discounting.* The *effective* utility discount rate, $\rho - n$, varies negatively (one to one) with the population growth rate. Since this corresponds to how the per capita rate of return on saving, $r - n$, is "diluted" by population growth, the net marginal productivity of capital in steady state becomes independent of n, namely equal to $\hat{\rho} + \hat{\theta}g$.

Some textbooks, Blanchard and Fischer (1989) for instance, let the social planner maximize the sum of discounted per capita utilities *without* weighting by generation size. Then the effective utility discount rate is independent of the population growth rate, n. With $\hat{\rho}$ still denoting the pure rate of time preference, the criterion function becomes

$$W_0 = \int_0^\infty \frac{{c_t}^{1-\hat{\theta}}}{1-\hat{\theta}} e^{-\hat{\rho}t} dt.$$

The social planner's solution then converges toward a steady state with net marginal productivity of capital equal to

$$f'(\tilde{k}^*) - \delta = \hat{\rho} + n + \hat{\theta}g. \tag{10.49}$$

Here, an increase in n will imply higher long-run net marginal productivity of capital and lower effective capital-labor ratio, everything else equal.

The representative household in the market economy described by a Ramsey model may of course also have a criterion function in line with this, that is, $U_0 = \int_0^\infty u(c_t)e^{-\rho t}dt$. Then, the interest rate in the economy will in the long run be $r^* = \rho + n + \theta g$ and so an increase in n will increase r^* and decrease $\tilde{k}^*$.

The more common approach is the utilitaristic accounting, which may be based on the argument: "if more people benefit, so much the better".

10.5.2 Ramsey's original zero discount rate and the overtaking criterion*

It was mostly the perspective of a social planner, rather than the market mechanism, which was at the center of Ramsey's original analysis (Ramsey, 1928). The case considered by Ramsey has $g = n = 0$. Ramsey maintained that the social planner should "not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of the imagination" (Ramsey 1928). So Ramsey has $\rho - n = \rho = 0$. Given the instantaneous utility function, u, where $u' > 0, u'' < 0$, and given $\rho = 0$, Ramsey's original problem was: choose $(c_t)_{t=0}^{\infty}$ so as to optimize (in some sense, see below)

$$\begin{aligned} W_0 &= \int_0^\infty u(c_t)dt \qquad \text{s.t.} \\ c_t &\geq 0, \\ \dot{k}_t &= f(k_t) - c_t - \delta k_t, \\ k_t &\geq 0 \qquad \text{for all } t \geq 0. \end{aligned}$$

A condition corresponding to our assumption (A1) above does not apply. So the improper integral W_0 will generally not be bounded[26] and Ramsey can not use maximization of W_0 as an optimality criterion. Instead he considers a criterion akin to the overtaking criterion we considered in a discrete time context in Chapter 8. We only have to reformulate this criterion for a continuous time setting.

Let $(c_t)_{t=0}^{\infty}$ be the consumption path associated with an arbitrary technically feasible path and let $(\hat{c}_t)$ be the consumption path associated with our candidate as an optimal path, that is, the path we wish to test for optimality. Define

$$D_T \equiv \int_0^T u(\hat{c}_t)dt - \int_0^T u(c_t)dt. \qquad (10.50)$$

Then the feasible path $(\hat{c}_t)_{t=0}^{\infty}$ is *overtaking optimal*, if for any feasible path, $(c_t)_{t=0}^{\infty}$, there exists a number $T' \geq 0$ such that $D_T \geq 0$ for all $T \geq T'$. That is, if for every alternative feasible path, the candidate path has from some date on, cumulative utility up to *all* later dates at least as great as that of the alternative feasible path, then the candidate path is overtaking optimal.

We say that the candidate path is *weakly preferred* in case we just know that $D_T \geq 0$ for all $T \geq T'$. If $D_T \geq 0$ can be replaced by $D_T > 0$, we say it is *strictly preferred*.[27]

Optimal control theory is also applicable for this criterion. The current-value Hamiltonian is

$$H(k, c, \lambda, t) = u(c) + \lambda [f(k) - c - \delta k].$$

The Maximum Principle states that an interior overtaking-optimal path will satisfy that there exists an adjoint variable λ such that for all $t \geq 0$ it holds along

[26] Suppose for instance that $c_t \to \bar{c}$ for $t \to \infty$. Then $\int_0^\infty u(c_t)dt = \pm\infty$ for $u(\bar{c}) \gtrless 0$, respectively.

[27] A slightly more generally applicable optimality criterion is the *catching-up* criterion. The meaning of this criterion in continuous time is analogue to its meaning in discrete time, cf. Chapter 8.3. The overtaking as well as the catching-up criterion entail generally only a *partial* ordering of alternative technically feasible paths.

this path that

$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0, \text{ and} \tag{10.51}$$

$$\frac{\partial H}{\partial k} = \lambda(f'(k) - \delta) = -\dot{\lambda}. \tag{10.52}$$

Since $\rho = 0$, the Keynes-Ramsey rule reduces to

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta(c_t)}(f'(k_t) - \delta), \qquad \text{where } \theta(c) \equiv -\frac{c}{u'(c)}u''(c).$$

One might conjecture that also the transversality condition,

$$\lim_{t\to\infty} k_t\lambda_t = 0, \tag{10.53}$$

is necessary for optimality but, as we will see below, this turns out to be wrong in this case with no discounting.

Our assumption (A2') here reduces to $\lim_{k\to 0} f'(k) > \delta > \lim_{k\to\infty} f'(k)$ (which requires $\delta > 0$). Apart from this, the phase diagram is fully analogue to that in Fig. 10.2, except that the steady state, E, is now at the top of the $\dot{k} = 0$ curve. This is because in steady state, $f'(k^*) - \delta = 0$. This equation also defines k_{GR} in this case. It can be shown that the saddle path is again the unique solution to the optimization problem (the method is essentially the same as in the discrete time case of Chapter 8). The intuitive background is that failing to approach the golden rule would imply a forgone "opportunity of infinite gain".

A noteworthy feature is that in this case the Ramsey model constitutes a counterexample to the widespread presumption that an optimal plan with infinite horizon *must* satisfy a transversality condition like (10.53). Indeed, by (10.51), $\lambda_t = u'(c_t) \to u'(c^*)$ for $t \to \infty$ along the overtaking-optimal path (the saddle path). Thus, instead of (10.53), we get

$$\lim_{t\to\infty} k_t\lambda_t = k^*u'(c^*) > 0.$$

With CRRA utility it is straightforward to generalize these results to the case $g \geq 0, n \geq 0$ and $\hat{\rho} - n = (1 - \hat{\theta})g$. The social planner's overtaking-optimal solution is still the saddle path approaching the golden rule steady state. And this solution violates the seemingly "natural" transversality condition, (10.47), which *is* necessary for optimality when $\hat{\rho} - n > (1 - \hat{\theta})g$, as in Section 10.5.1.

Note also that with zero effective utility discounting, there can not be equilibrium in the *market* economy version of this story. The real interest rate would in the long run be zero and thus the human wealth of the infinitely-lived household would be infinite. But then the demand for consumption goods would be unbounded and equilibrium thus be impossible.

10.6 Concluding remarks

The Ramsey model has played an important role as a way of structuring economists' thoughts about an array of macrodynamic phenomena. The popularity of the model probably derives from the fact that is allows taking microeconomic principles into account without worrying about the usual aggregation problems when going from micro to macro.

As illustrated in Fig. 10.3, the Ramsey model can be seen as situated at one end of a line segment where the Diamond OLG model is situated at the opposite end. Both models build on idealized assumptions. The Diamond model ignores any bequest motive and emphasizes life-cycle behavior and heterogeneity in the population. The Ramsey model implicitly assumes an altruistic bequest motive which is always operative and which turns households into homogeneous, infinitely-lived agents. In this way the Ramsey model ends up as an easy-to-apply framework, suggesting *inter alia* a clear-cut theory of the level of the real interest rate in the long run − the *modified golden rule*. Although this theory finds little empirical support (Hamilton et al., 2016), it facilitates general equilibrium analysis of an array of dynamic problems. The next chapter discusses some examples: effects of unanticipated and anticipated changes in taxation and endogenous growth theory.

The assumption of a representative household is a main limitation. The lack of heterogeneity in the model's population of households implies a danger that important interdependencies between different classes of agents are unduly neglected. For some problems these interdependencies may be of only secondary importance, but they are crucial for others (for instance, issues concerning public debt or interaction between private debtors and creditors). On the other hand, as Caselli and Ventura (2000) have shown, it is possible to extend the Ramsey model so as to allow heterogeneity in the population with respect to initial financial wealth and labor productivity. But regarding preferences only very limited heterogeneity can be embraced by the Ramsey model.

Another disputed feature of the model is that it endows the households with an extreme amount of information about the future. Solow (1990, p. 221) warns against overly reliance on saddle-point stability in the analysis of a market economy:

> "The problem is not just that perfect foresight into the indefinite future is so implausible away from steady states. The deeper problem is that in practice − if there is any practice − miscalculations about the equilibrium path may not reveal themselves for a long time. The mistaken path gives no signal that it will be "ultimately" infeasible. It is natural to comfort oneself: whenever the error is perceived there will be a jump to a better

> approximation to the converging arm. But a large jump may be required. In a decentralized economy it will not be clear who knows what, or where the true converging arm is, or, for that matter, exactly where we are now, given that some agents (speculators) will already have perceived the need for a mid-course correction while others have not. This thought makes it hard even to imagine what a long-run path would look like. It strikes me as more or less devastating for the interpretation of quarterly data as the solution of an infinite time optimization problem."

As we saw in Section 10.5.2, Ramsey's original analysis (Ramsey 1928) dealt with a social planner's infinite horizon optimal control problem. In that optimization problem there are well-defined shadow prices, as implied by an explicit social welfare function. In a decentralized market economy, however, there are a multitude of both agents and prices and no god-like auctioneer to ensure that the long-term price expectations coincide with the long-term shadow prices in the social planner's optimal control problem.

Fig. 10.3 about here (not yet available)

While the Ramsey and the Diamond model are polar cases along the line segment in Fig. 10.3, less abstract macro models are scattered between these poles, some being closer to one pole than to the other. Sometimes a given model open up for alternative *regimes*, one close to Ramsey's pole, another close to Diamond's. An example is Robert Barro's model with parental altruism discussed in Chapter 7. When the bequest motive in the Barro model is operative, the model coincides with a Ramsey model (in discrete time) as was shown in Chapter 8. But when the bequest motive is not operative, the Barro model coincides with a Diamond OLG model. This conditionality "places" the Barro model in the interior of the line segment, but in practice closer to Ramsey's pole than to Diamond's. model

Blanchard's OLG model in continuous time (to be analyzed and used in chapters 12, 13, and 15) also belongs to the interior of the line segment, but closer to Diamond's pole than to Ramsey's.

10.7 Literature notes

1. Frank Ramsey (1903-1930) died at the age of 26 but he managed to publish several path-breaking articles in economics. Ramsey discussed economic issues

with, among others, John Maynard Keynes. In an obituary published in the Economic Journal (March 1932) after Ramsey's death, Keynes described Ramsey's article about the optimal savings as "one of the most remarkable contributions to mathematical economics ever made, both in respect of the intrinsic importance and difficulty of its subject, the power and elegance of the technical methods employed, and the clear purity of illumination with which the writer's mind is felt by the reader to play about its subject".

2. The version of the Ramsey model we have considered is in accordance with the general tenet of neoclassical preference theory: saving is motivated only by higher consumption in the future. Extended versions assume that accumulation of wealth is to some extend an end in itself or perhaps motivated by a desire for social prestige and economic and political power rather than consumption. In Kurz (1968b) an extended Ramsey model is studied where wealth is an independent argument in the instantaneous utility function.

Also Tournemaine and Tsoukis (2008) and Long and Shimomura (2004).

3. The equivalence in the Ramsey model between the decentralized market equilibrium and the social planner's solution can be seen as an extension of the first welfare theorem as it is known from elementary textbooks, to the case where the market structure stretches infinitely far out in time, and the finite number of economic agents (family dynasties) face an infinite time horizon: in the absence of externalities etc., the allocation of resources under perfect competition will lead to a Pareto optimal allocation. The Ramsey model is indeed a special case in that all households are identical. But the result can be shown in a far more general setup, cf. Debreu (1954). The result, however, does not hold in overlapping generations models where an unbounded sequence of new generations enter and the "interests" of the new households have not been accounted for in advance.

4. The simple counter-example to the "standard" necessary transversality condition for an infinite horizon optimal control problem given in Section 10.5.2 was for a problem where the utility integral was not bounded from above. The optimality criterion was therefore not maximization but overtaking. But Sydsæter et al. (2008) contains a counter-example for a problem where maximization *is* the optimality criterion.

10.8 Appendix

A. Algebraic analysis of the dynamics around the steady state

To supplement the graphical approach of Section 10.3 with an exact analysis of the adjustment dynamics of the model, we compute the Jacobian matrix for the system of differential equations (10.28) - (10.29):

$$J(\tilde{k},\tilde{c}) = \begin{bmatrix} \partial\dot{\tilde{k}}/\partial\tilde{k} & \partial\dot{\tilde{k}}/\partial\tilde{c} \\ \partial\dot{\tilde{c}}/\partial\tilde{k} & \partial\dot{\tilde{c}}/\partial\tilde{c} \end{bmatrix} = \begin{bmatrix} f'(\tilde{k}) - (\delta + g + n) & -1 \\ \frac{1}{\theta}f''(\tilde{k})\tilde{c} & \frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho + \theta g) \end{bmatrix}.$$

Evaluated in the steady state this reduces to

$$J(\tilde{k}^*,\tilde{c}^*) = \begin{bmatrix} \rho - n - (1-\theta)g & -1 \\ \frac{1}{\theta}f''(\tilde{k}^*)\tilde{c}^* & 0 \end{bmatrix}$$

This matrix has the determinant

$$\frac{1}{\theta}f''(\tilde{k}^*)\tilde{c}^* < 0.$$

Since the product of the eigenvalues of the matrix equals the determinant, the eigenvalues are real and opposite in sign.

In standard math terminology a steady-state point in a two dimensional continuous-time dynamic system is called a *saddle point* if the associated eigenvalues are opposite in sign.[28] For the present case we conclude that the steady state is a saddle point. This mathematical definition of a saddle point is equivalent to that given in the text of Section 10.3. Indeed, with two eigenvalues of opposite sign, there exists, in a small neighborhood of the steady state, a stable arm consisting of two saddle paths which point in opposite directions. From the phase diagram in Fig. 10.2 we know that the stable arm has a positive slope. At least for $\tilde{k}_0$ sufficiently close to $\tilde{k}^*$ it is thus possible to start out on a saddle path. Consequently, there is a (unique) value of $\tilde{c}_0$ such that $(\tilde{k}_t, \tilde{c}_t) \to (\tilde{k}^*, \tilde{c}^*)$ for $t \to \infty$. Finally, the dynamic system has exactly one jump variable, $\tilde{c}$, and one predetermined variable, $\tilde{k}$. It follows that the steady state is (locally) *saddle-point stable.*

We claim that for the present model this can be strengthened to *global* saddle-point stability. Indeed, for *any* $\tilde{k}_0 > 0$, it is possible to start out on the saddle path. For $0 < \tilde{k}_0 \leq \tilde{k}^*$, this is obvious in that the extension of the saddle path toward the left reaches the y-axis at a non-negative value of $\tilde{c}^*$. That is to say that the extension of the saddle path cannot, according to the uniqueness theorem for differential equations, intersect the $\tilde{k}$-axis for $\tilde{k} > 0$ in that the positive part of the $\tilde{k}$-axis is a solution of (10.28) - (10.29).[29]

[28] Note the difference compared to a discrete time system, cf. Appendix D of Chapter 8. In the discrete time system we have next period's $\tilde{k}$ and $\tilde{c}$ on the left-hand side of the dynamic equations, not the increase in $\tilde{k}$ and $\tilde{c}$, respectively. Therefore, the criterion for a saddle point looks different in discrete time.

[29] Because the extension of the saddle path towards the left in Fig. 10.1 can not intersect the $\tilde{c}$-axis at a value of $\tilde{c} > f(0)$, it follows that if $f(0) = 0$, the extension of the saddle path ends up in the origin.

For $\tilde{k}_0 > \tilde{k}^*$, our claim can be verified in the following way: suppose, contrary to our claim, that there exists a $\tilde{k}_1 > \tilde{k}^*$ such that the saddle path does not intersect that region of the positive quadrant where $\tilde{k} \geq \tilde{k}_1$. Let $\tilde{k}_1$ be chosen as the smallest possible value with this property. The slope, $d\tilde{c}/d\tilde{k}$, of the saddle path will then have no upper bound when $\tilde{k}$ approaches $\tilde{k}_1$ from the left. Instead $\tilde{c}$ will approach ∞ along the saddle path. But then $\ln \tilde{c}$ will also approach ∞ along the saddle path for $\tilde{k} \to \tilde{k}_1$ $(\tilde{k} < \tilde{k}_1)$. It follows that $d \ln \tilde{c}/d\tilde{k} = (d\tilde{c}/d\tilde{k})/\tilde{c}$, computed along the saddle path, will have no upper bound. Nevertheless, we have

$$\frac{d \ln \tilde{c}}{d\tilde{k}} = \frac{d \ln \tilde{c}/dt}{d\tilde{k}/dt} = \frac{\dot{\tilde{c}}/\tilde{c}}{\dot{\tilde{k}}} = \frac{\frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho - \theta g)}{f(\tilde{k}) - \tilde{c} - (\delta + g + n)\tilde{k}}.$$

When $\tilde{k} \to \tilde{k}_1$ and $\tilde{c} \to \infty$, the numerator in this expression is bounded, while the denominator will approach $-\infty$. Consequently, $d \ln \tilde{c}/d\tilde{k}$ will approach zero from above, as $\tilde{k} \to \tilde{k}_1$. But this contradicts that $d \ln \tilde{c}/d\tilde{k}$ has no upper bound, when $\tilde{k} \to \tilde{k}_1$. Thus, the assumption that such a $\tilde{k}_1$ exists is false and our original hypothesis holds true.

B. Boundedness of the utility integral

We claimed in Section 10.3 that if the parameter restriction

$$\rho - n > (1 - \theta)g \tag{A1}$$

holds, then the utility integral, $U_0 = \int_0^\infty \frac{c^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt$, is bounded, from above as well as from below, along the steady-state path, $c_t = \tilde{c}^* T_t$. The proof is as follows. Recall that $\theta > 0$ and $g \geq 0$. For $\theta \neq 1$,

$$\begin{aligned}(1-\theta)U_0 &= \int_0^\infty c_t^{1-\theta} e^{-(\rho-n)t} dt = \int_0^\infty (c_0 e^{gt})^{1-\theta} e^{-(\rho-n)t} dt \\ &= c_0 \int_0^\infty e^{[(1-\theta)g - (\rho-n)]t} dt = \frac{c_0}{\rho - n - (1-\theta)g},\end{aligned} \tag{10.54}$$

which by (A1) is finite and positive since $c_0 > 0$. If $\theta = 1$, so that $u(c) = \ln c$, we get

$$U_0 = \int_0^\infty (\ln c_0 + gt) e^{-(\rho-n)t} dt, \tag{10.55}$$

which is also finite, in view of (A1) implying $\rho - n > 0$ in *this* case (the exponential term, $e^{-(\rho-n)t}$, declines faster than the linear term gt increases). It follows that also any path converging to the steady state will entail bounded utility, when (A1) holds.

On the other hand, suppose that (A1) does *not* hold, i.e., $\rho - n \leq (1-\theta)g$. Then by the third equality in (10.54) and $c_0 > 0$ follows that $(1-\theta)U_0 = \infty$ if $\theta \neq 0$. If instead $\theta = 1$, (10.55) implies $U_0 = \infty$.

C. The diverging paths

In Section 10.3 we stated that paths of types *II* and *III* in the phase diagram in Fig. 10.2 can not be equilibria with perfect foresight. Given the expectation corresponding to any of these paths, every single household will choose to *deviate* from the expected path (i.e., deviate from the expected "average behavior" in the economy). We will now show this formally.

We first consider a path of type *III*. A path of this type will not be able to *reach* the horizontal axis in Fig. 10.2. It will only *converge* toward the point $(\bar{\tilde{k}}, 0)$ for $t \to \infty$. This claim follows from the uniqueness theorem for differential equations with continuously differentiable right-hand sides. The uniqueness implies that two solution curves cannot intersect. And we see from (10.29) that the positive part of the x-axis is from a mathematical point of view a solution curve (and the point $(\bar{\tilde{k}}, 0)$ is a trivial steady state). This rules out another solution curve hitting the x-axis.

The convergence of $\tilde{k}$ toward $\bar{\tilde{k}}$ implies $\lim_{t\to\infty} r_t = f'(\bar{\tilde{k}}) - \delta < g + n$, where the inequality follows from $\bar{\tilde{k}} > \tilde{k}_{GR}$. So,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = \lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (r_s - g - n)ds} = \lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s) - \delta - g - n)ds} = \bar{\tilde{k}} e^{\infty} > 0. \tag{10.56}$$

Hence the transversality condition of the households is violated. Consequently, the household will choose higher consumption than along this path and can do so without violating the NPG condition.

Consider now instead a path of type *II*. We shall first show that if the economy follows such a path, then depletion of all capital occurs in finite time. Indeed, in the text it was shown that any path of type *II* will pass the $\dot{\tilde{k}} = 0$ locus in Fig. 10.2. Let t_0 be the point in time where this occurs. If path *II* lies above the $\dot{\tilde{k}}$ $= 0$ locus for all $t \geq 0$, then we set $t_0 = 0$. For $t > t_0$, we have

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t < 0.$$

By differentiation w.r.t. t we get

$$\ddot{\tilde{k}}_t = f'(\tilde{k}_t)\dot{\tilde{k}}_t - \dot{c}_t - (\delta + g + n)\dot{\tilde{k}}_t = [f'(\tilde{k}_t) - \delta - g - n]\dot{\tilde{k}}_t - \dot{c}_t < 0,$$

where the inequality comes from $\dot{\tilde{k}}_t < 0$ combined with the fact that $\tilde{k}_t < \tilde{k}_{GR}$ implies $f'(\tilde{k}_t) - \delta > f'(\tilde{k}_{GR}) - \delta = g + n$. Therefore, there exists a $t_1 > t_0 \geq 0$ such that

$$\tilde{k}_{t_1} = \tilde{k}_{t_0} + \int_{t_0}^{t_1} \dot{\tilde{k}}_t dt = 0,$$

as was to be shown. At time t_1, $\tilde{k}$ cannot fall any further and $\tilde{c}_t$ immediately drops to $f(0)$ and stay there hereafter.

Yet, this result does not in itself explain why the individual household will deviate from such a path. The individual household has a negligible impact on the movement of $\tilde{k}_t$ in society and correctly perceives r_t and w_t as essentially independent of its own consumption behavior. Indeed, the economy-wide $\tilde{k}$ is not the household's concern. What the household cares about is its own financial wealth and budget constraint. In the perspective of the household nothing prevents it from planning a negative financial wealth, a, and possibly a continuously declining financial wealth, if only the NPG condition,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0,$$

is satisfied.

But we can show that paths of type *II* will *violate* the NPG condition. The reasoning is as follows. The household plans to follow the Keynes-Ramsey rule. Given an expected evolution of r_t and w_t corresponding to path *II*, this will imply a planned gradual transition from positive financial wealth to debt. The transition to positive net debt, $\tilde{d}_t \equiv -\tilde{a}_t \equiv -a_t/T_t > 0$, takes place at time t_1 defined above.

The continued growth in the debt will meanwhile be so fast that the NPG condition is violated. To see this, note that the NPG condition implies the requirement

$$\lim_{t\to\infty} \tilde{d}_t e^{-\int_0^t (r_s - g - n)ds} \leq 0, \tag{NPG}$$

that is, the productivity-corrected debt, $\tilde{d}_t$, is allowed to grow in the long run only at a rate *less* than the growth-corrected real interest rate. For $t > t_1$ we get from the accounting equation $\dot{a}_t = (r_t - n)a_t + w_t - c_t$ that

$$\dot{\tilde{d}}_t = (r_t - g - n)\tilde{d}_t + \tilde{c}_t - \tilde{w}_t > 0,$$

where $\tilde{d}_t > 0$, $r_t > \rho + \theta g > g + n$, and where $\tilde{c}_t$ grows exponentially according to the Keynes-Ramsey rule, while $\tilde{w}_t$ is non-increasing in that $\tilde{k}_t$ does not grow. This implies

$$\lim_{t\to\infty} \frac{\dot{\tilde{d}}_t}{\tilde{d}_t} \geq \lim_{t\to\infty} (r_t - g - n),$$

which is in conflict with (NPG).

Consequently, the household will choose a lower consumption path and thus *deviate* from the reference path considered. Every household will do this and the evolution of r_t and w_t corresponding to path *II* is thus *not* an equilibrium with perfect foresight.

The conclusion is that all individual households understand that the only evolution which can be expected rationally is the one corresponding to the saddle path.

D. Constant saving-income ratio as a special case

As we noted in Section 10.4, Solow's growth model can be seen as a special case of the Ramsey model. Indeed, a constant saving-income ratio may, under certain conditions, emerge as an endogenous result in the Ramsey model.

Let the rate of saving, $(Y_t - C_t)/Y_t$, be s_t. We have generally

$$\tilde{c}_t = (1 - s_t)f(\tilde{k}_t), \qquad \text{and so} \tag{10.57}$$

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t = s_t f(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t. \tag{10.58}$$

In the Solow model the rate of saving is a constant, s, and we then get, by differentiating with respect to t in (10.57) and using (10.58),

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = f'(\tilde{k}_t)[s - \frac{(\delta + g + n)\tilde{k}_t}{f(\tilde{k}_t)}]. \tag{10.59}$$

By maximization of discounted utility in the Ramsey model, given a rate of time preference ρ and an elasticity of marginal utility θ, we get in equilibrium

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{1}{\theta}(f'(\tilde{k}_t) - \delta - \rho - \theta g). \tag{10.60}$$

There will not generally exist a constant, s, such that the right-hand sides of (10.59) and (10.60), respectively, are the same for varying $\tilde{k}$ (that is, outside steady state). But Kurz (1968a) showed the following:

CLAIM Let δ, g, n, α, and θ be given. If the elasticity of marginal utility θ is greater than 1 and the production function is $\tilde{y} = A\tilde{k}^{\alpha}$ with $\alpha \in (1/\theta, 1)$, then a Ramsey model with $\rho = \theta\alpha(\delta + g + n) - \delta - \theta g$ will generate a constant saving-income ratio $s = 1/\theta$. Thereby the same resource allocation and transitional dynamics arise as in the corresponding Solow model with $s = 1/\theta$.

Proof. Let $1/\theta < \alpha < 1$ and $f(\tilde{k}) = A\tilde{k}^{\alpha}$. Then $f'(\tilde{k}) = A\alpha\tilde{k}^{\alpha-1}$. The right-hand-side of the Solow equation, (10.59), becomes

$$A\alpha\tilde{k}^{\alpha-1}[s - \frac{(\delta+g+n)\tilde{k}_t}{A\tilde{k}^{\alpha}}] = sA\alpha\tilde{k}^{\alpha-1} - \alpha(\delta+g+n). \tag{10.61}$$

The right-hand-side of the Ramsey equation, (10.60), becomes

$$\frac{1}{\theta}A\alpha\tilde{k}^{\alpha-1} - \frac{\delta+\rho+\theta g}{\theta}.$$

By inserting $\rho = \theta\alpha(\delta+g+n) - \delta - \theta g$, this becomes

$$\begin{aligned} &\frac{1}{\theta}A\alpha\tilde{k}^{\alpha-1} - \frac{\delta+\theta\alpha(\delta+g+n)-\delta-\theta g+\theta g}{\theta} \\ = &\frac{1}{\theta}A\alpha\tilde{k}^{\alpha-1} - \alpha(\delta+g+n). \end{aligned} \tag{10.62}$$

For the chosen ρ we have $\rho = \theta\alpha(\delta+g+n) - \delta - \theta g > n + (1-\theta)g$, because $\theta\alpha > 1$ and $\delta+g+n > 0$. Thus, $\rho - n > (1-\theta)g$ and existence of equilibrium in the Ramsey model with this ρ is ensured. We can now make (10.61) and (10.62) the same by inserting $s = 1/\theta$. This also ensures that the two models require the same $\tilde{k}^*$ to obtain a *constant* $\tilde{c} > 0$. With this $\tilde{k}^*$, the requirement $\dot{\tilde{k}}_t = 0$ gives the same steady-state value of $\tilde{c}$ in both models, in view of (10.58). It follows that $(\tilde{k}_t, \tilde{c}_t)$ is the same in the two models for all $t \geq 0$. □

On the other hand, maintaining $\tilde{y} = A\tilde{k}^{\alpha}$, but allowing $\rho \neq \theta\alpha(\delta+g+n) -\delta - \theta g$, so that $\theta \neq 1/s^*$, then $s'(\tilde{k}) \neq 0$, i.e., the Ramsey model does not generate a constant saving-income ratio except in steady state. Defining s^* as in (10.40) and $\bar{\theta} \equiv (\delta+\rho)/[\alpha(\delta+g+n)-g]$, we have: When $\alpha(\delta+g+n) > g$ (which seems likely empirically), it holds that if $\theta \lesseqgtr 1/s^*$ (i.e., if $\theta \lesseqgtr \bar{\theta}$), then $s'(\tilde{k}) \lesseqgtr 0$, respectively; if instead $\alpha(\delta+g+n) \leq g$, then $\theta < 1/s^*$ and $s'(\tilde{k}) < 0$, unconditionally. These results follow by considering the slope of the saddle path in a phase diagram in the $(\tilde{k}, \tilde{c}/f(\tilde{k}))$ plane and using that $s(\tilde{k}) = 1 - \tilde{c}/f(\tilde{k})$, cf. Exercise 10.?? The intuition is that when $\tilde{k}$ is rising over time (i.e., society is becoming wealthier), then, when the desire for consumption smoothing is "high" (θ "high"), the prospect of high consumption in the future is partly taken out as high consumption already today, implying that saving is initially low, but rising over time until it eventually settles down in the steady state. But if the desire for consumption smoothing is "low" (θ "low"), saving will initially be high and then gradually fall in the process toward the steady state. The case where $\tilde{k}$ is falling over time gives symmetric results.

E. The social planner's solution

In the text of Section 10.5 we postponed some of the technical details. First, by (A2), the existence of the steady state, E, and the saddle path in Fig. 10.2 is ensured. Solving the linear differential equation (10.46) gives $\lambda_t = \lambda_0 e^{-\int_0^t (f'(\tilde{k}_s)-\delta-\hat{\rho}-g)ds}$. Substituting this into the transversality condition (10.47) gives

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s)-\delta-g-n)ds} = 0, \tag{10.63}$$

where we have eliminated the unimportant positive factor $\lambda_0 = c_0^{-\hat{\theta}} T_0 > 0$.

The condition (10.63) is essentially the same as the transversality condition (10.30) for the market economy and holds in the steady state, given the parameter restriction $\hat{\rho} - n > (1-\hat{\theta})g$, which is analogue to (A1). Thus, (10.63) also holds along the saddle path. Since we must have $\tilde{k} \geq 0$ for all $t \geq 0$, (10.63) has the form required by Mangasarian's sufficiency theorem. If we can show that the Hamiltonian is jointly concave in $(\tilde{k}, c)$ for all $t \geq 0$, then the saddle path *is* a solution to the social planner's problem. And if we can show strict concavity, the saddle path is the *unique* solution. We have:

$$\begin{aligned}
\frac{\partial H}{\partial \tilde{k}} &= \lambda(f'(\tilde{k}) - (\delta + g + n)), \quad \frac{\partial H}{\partial c} = c^{-\hat{\theta}} - \frac{\lambda}{T}, \\
\frac{\partial^2 H}{\partial \tilde{k}^2} &= \lambda f''(\tilde{k}) < 0 \quad (\text{by } \lambda = c^{-\hat{\theta}} T > 0), \quad \frac{\partial^2 H}{\partial c^2} = -\hat{\theta} c^{-\hat{\theta}-1} < 0, \\
\frac{\partial^2 H}{\partial \tilde{k} \partial c} &= 0.
\end{aligned}$$

So the leading principal minors of the Hessian matrix of H are

$$D_1 = -\frac{\partial^2 H}{\partial \tilde{k}^2} > 0, \quad D_2 = \frac{\partial^2 H}{\partial \tilde{k}^2}\frac{\partial^2 H}{\partial c^2} - \left(\frac{\partial^2 H}{\partial \tilde{k}\partial c}\right)^2 > 0.$$

Hence, H is strictly concave in $(\tilde{k}, c)$ and the saddle path is the *unique* optimal solution.

It also follows that the transversality condition (10.47) *is* a necessary optimality condition when the parameter restriction $\hat{\rho} - n > (1-\hat{\theta})g$ holds. Note that we have had to derive this conclusion in a different way than when solving the household's consumption/saving problem in Section 10.2. There we could appeal to a link between the No-Ponzi-Game condition (with strict equality) and the transversality condition to verify necessity of the transversality condition. But that proposition does not cover the social planner's problem where there is no NPG condition.

As to the diverging paths in Fig. 10.2, note that paths of type II (those paths which, as shown in Appendix C, in finite time deplete all capital) can not be optimal, in spite of the temporarily high consumption level. This follows from the fact that the saddle path is the unique solution. Finally, paths of type III in Fig. 10.2 behave as in (10.56) and thus violate the transversality condition (10.47), as claimed in the text.

10.9 Exercises

Makroøkonomi 2 Speciel Note 1

30.9.2016. Christian Groth

A glimpse of theory of the "level of interest rates"

This short note provides a brief sketch of what macroeconomics says about the general level around which rates of return fluctuate. We also give a "broad" summary of different circumstances that give rise to differences in rates of return on different assets.

In non-monetary models without uncertainty there is in equilibrium only one rate of return, r. If in addition there is a) perfect competition in all markets, b) the consumption good is physically indistinguishable from the capital good, and c) there are no capital adjustment costs, as in simple neoclassical models (like the Diamond OLG model and the Ramsey model), then the equilibrium real interest rate is at any time equal to the current net marginal productivity of capital evaluated at full employment ($r = \partial Y/\partial K - \delta$ in standard notation). Moreover, under conditions ensuring "well-behavedness" of these models, they predict that in the absence of disturbances, the technology-corrected capital-labor ratio, and thereby the marginal productivity of capital, adjusts over time to some long-run level (on which more below).

Different rates of return In simple neoclassical models with perfect competition and no uncertainty, the equilibrium short-term real interest rate is at any time equal to the net marginal productivity of capital ($r = \partial Y/\partial K - \delta$). In turn the marginal productivity of capital adjusts over time, via changes in the capital intensity, to some long-run level (on this more below). As we saw in Chapter 14, existence of convex *capital installation costs* loosens the link between r and $\partial Y/\partial K$. The convex adjustment costs create a wedge between the price of investment goods and the market value of the marginal unit of installed capital.

When *imperfect competition* in the output markets rules, prices are typically set as a mark-up on marginal cost. This implies a wedge between the net marginal productivity of capital and capital costs. And when *uncertainty* and limited opportunities for risk

	Arithmetic average	Standard deviation	Geometric average
	----------------	Percent	----------------
Nominal values			
Small Company Stocks	17,3	33,2	12,5
Large Company Stocks	12,7	20,2	10,7
Long-Term Corporate Bonds	6,1	8,6	5,8
Long-Term Government Bonds	5,7	9,4	5,3
Intermediate-Term Government Bonds	5,5	5,7	5,3
U.S. Treasury Bills	3,9	3,2	3,8
Cash	0,0	0,0	0,0
Inflation rate	3,1	4,4	3,1
Real values			
Small Company Stocks	13,8	32,6	9,2
Large Company Stocks	9,4	20,4	7,4
Long-Term Corporate Bonds	3,1	9,9	2,6
Long-Term Government Bonds	2,7	10,6	2,2
Intermediate-Term Government Bonds	2,5	7,0	2,2
U.S. Treasury Bills	0,8	4,1	0,7
Cash	-2,9	4,2	-3,0

Table 1: Average annual rates of return on a range of U.S. asset portfolios, 1926-2001. Source: Stocks, Bonds, Bills, and Inflation: Yearbook 2002, Valuation Edition. Ibbotson Associates, Inc.

diversification are added to the model, a wide spectrum of expected rates of return on different financial assets and expected marginal productivities of capital in different production sectors arise, depending on the risk profiles of the different assets and production sectors. On top of this comes the presence of taxation which may complicate the picture because of different tax rates on different asset returns.

Nominal and real average annual rates of return on a range of U.S. asset portfolios for the period 1926–2001 are reported in Table 1. By a *portfolio* of n assets, $i = 1, 2, \dots, n$ is meant a "basket", $(v_1, v_2, \dots, v_n)$, of the n assets in value terms, that is, $v_i = p_i x_i$ is the value of the investment in asset i, the price of which is denoted p_i and the quantity of which is denoted x_i. The total investment in the basket is $V = \sum_{i=1}^{n} v_i$. If R_i denotes the gross rate of return on asset i, the overall gross rate of return on the portfolio is

$$R = \frac{\sum_i^n v_i R_i}{V} = \sum_{i=1}^{n} w_i R_i,$$

where $w_i \equiv v_i / V$ is the *weight* or *fraction* of asset i in the portfolio. Defining $R_i \equiv 1 + r_i$, where r_i is the net rate of return on asset i, the net rate of return on the portfolio can be written

$$r = R - 1 = \sum_{i=1}^{n} w_i(1 + r_i) - 1 = \sum_{i=1}^{n} w_i + \sum_{i=1}^{n} w_i r_i - 1 = \sum_{i=1}^{n} w_i r_i.$$

The net rate of return is often just called "the rate of return".

In Table 1 we see that the portfolio consisting of small company stocks throughout the period 1926-2001 had an average annual real rate of return of 13.8 per cent (the arithmetic average) or 9.2 per cent (the geometric average, assuming annual compounding). This is more than the annual rate of return of any of the other considered portfolios. Small company stocks are also seen to be the most volatile. The standard deviation of the annual real rate of return of the portfolio of small company stocks is almost eight times higher than that of the portfolio of U.S. Treasury bills (government zero coupon bonds with 30 days to maturity), with an average annual real return of only 0.8 per cent (arithmetic average) or 0.7 per cent (geometric average) throughout the period. The displayed positive relation between high returns and high volatility is not without exceptions, however. The portfolio of long-term corporate bonds has performed better than the portfolio of long-term government bonds, although they have been slightly less volatile as here measured. The data is historical and expectations are not always met. Moreover, risk depends significantly on the *covariance* of asset returns within the total set of assets and specifically on the correlation of asset returns with the business cycle, a feature that can not be read off from Table 1. Share prices, for instance, are very sensitive to business cycle fluctuations.

The need for means of payment − money − is a further complicating factor. That is, besides dissimilarities in risk and expected return across different assets, also dissimilarities in their degree of liquidity are important, not least in times of financial crisis. The expected real rate of return on cash holding is minus the expected rate of inflation and is therefore negative in an economy with inflation, cf. the last row in Table 1. When agents nevertheless hold cash in their portfolios, it is because the low rate of return is compensated by the *liquidity* services of money. In the Sidrauski model of Chapter 17 this is modeled in a simple way, albeit ad hoc, by including real money holdings directly as an argument in the utility function. Another dimension along which the presence of money interferes with returns is through inflation. Real assets, like physical capital, land, houses, etc. are better protected against fluctuating inflation than are nominally denominated bonds (and money of course).

Without claiming too much we can say that investors facing such a spectrum of rates of return choose a composition of assets so as to balance the need for liquidity, the wish for a high expected return, and the wish for low risk. Finance theory teaches us that adjusted for differences in risk and liquidity, asset returns tend to be the same. This raises the question: at what level? This is where macroeconomics – as an empirically oriented theory about the economy as a whole – comes in.

Macroeconomic theory of the "average rate of return" The point of departure is that market forces by and large may be thought of as anchoring the rate of return of an average portfolio of interest-bearing assets to the net marginal productivity of capital in an aggregate production function, assuming a closed economy. Some popular phrases are:

- the net marginal productivity of capital acts as a centre of gravitation for the spectrum of asset returns; and
- movements of the rates of return are in the long run held in check by the net marginal productivity of capital.

Though such phrases seem to convey the right flavour, in themselves they are not very informative. The net marginal productivity of capital is not a given, but an endogenous variable which, via changes in the capital intensity, adjusts through time to more fundamental factors in the economy.

The different macroeconomic models we have encountered in previous chapters bring to mind different presumptions about what these fundamental factors are.

1. Solow's growth model The Solow growth model leads to the fundamental differential equation (standard notation)

$$\dot{\tilde{k}}_t = sf(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t,$$

where s is an exogenous and constant aggregate saving-income ratio, $0 < s < 1$. In steady state

$$r^* = f'(\tilde{k}^*) - \delta, \tag{1}$$

where $\tilde{k}^*$ is the unique steady state value of the (effective) capital intensity, $\tilde{k}$, satisfying

$$sf(\tilde{k}^*) = (\delta + g + n)\tilde{k}^*. \tag{2}$$

In society there is a debate and a concern that changed demography and less growth in the source of new technical ideas, i.e., the stock of educated human beings, will in the future result in lower n and lower g, respectively, making financing social security more difficult. On the basis of the Solow model we find by implicit differentiation in (2) $\partial \tilde{k}^*/\partial n = \partial \tilde{k}^*/\partial g = -\tilde{k}^* \left[\delta + g + n - sf'(\tilde{k}^*)\right]^{-1}$, which is negative since $sf'(\tilde{k}^*) < sf(\tilde{k}^*)/\tilde{k}^* = \delta + g + n$. Hence, by (1),

$$\frac{\partial r^*}{\partial n} = \frac{\partial r^*}{\partial g} = \frac{\partial r^*}{\partial \tilde{k}^*}\frac{\partial \tilde{k}^*}{\partial n} = f''(\tilde{k}^*)\frac{-\tilde{k}^*}{\delta + g + n - sf'(\tilde{k}^*)} > 0,$$

since $f''(\tilde{k}^*) < 0$. It follows that

$$n \downarrow \text{ or } g \downarrow \Rightarrow r^* \downarrow . \tag{3}$$

A limitation of this theory is of course the exogeneity of the saving-income ratio, which is a key co-determinant of $\tilde{k}^*$, hence of r^*. The next models are examples of different ways of integrating a theory of saving into the story about the long-run rate of return.

2. The Diamond OLG model In the Diamond OLG model, based on a life-cycle theory of saving, we again arrive at the formula $r^* = f'(\tilde{k}^*) - \delta$. Like in the Solow model, the long-run rate of return thus depends on the aggregate production function and on $\tilde{k}^*$. But now there is a logically complete theory about how $\tilde{k}^*$ is determined. In the Diamond model $\tilde{k}^*$ depends in a complicated way on the lifetime utility function and the aggregate production function. The steady state of a well-behaved Diamond model will nevertheless have the same qualitative property as indicated in (3).

3. The Ramsey model Like the Solow and Diamond models, the Ramsey model implies that $r_t = f'(\tilde{k}_t) - \delta$ for all t. But unlike in the Solow and Diamond models, the net marginal productivity of capital now converges in the long run to a specific value given by the *modified golden rule* formula. In a continuous time framework this formula says:

$$r^* = \rho + \theta g, \tag{4}$$

where the new parameter, θ, is the (absolute) elasticity of marginal utility of consumption. Because the Ramsey model is a representative agent model, the Keynes-Ramsey rule holds not only at the individual level, but also at the aggregate level. This is what gives rise to this simple formula for r^*.

Here there is no role for n, only for g. On the other hand, there is an alternative specification of the Ramsey model, namely the "average utilitarianism" specification. In this version of the model, we get $r^* = f'(\tilde{k}^*) - \delta = \rho + n + \theta g$, so that not only a lower g, but also a lower n implies lower r^*.

Also the Sidrauski model, i.e., the monetary Ramsey model of Chapter 17, results in the *modified golden rule* formula (4).

4. Blanchard's OLG model A continuous time OLG model, and thereby a model emphasizing life-cycle aspects of economic behavior, is developed in Blanchard (1985). In that model the net marginal productivity of capital adjusts to a value where, in addition to the production function, technology growth, and preference parameters, also demographic parameters, like birth rate, death rate, and retirement rate, play a role. One of the results is that when $\theta = 1$,

$$\rho + g - \lambda < r^* < \rho + g + b,$$

where λ is the retirement rate (reflecting how early in life the "average" person retire from the labor market) and b is the (crude) birth rate. The population growth rate is the difference between the birth rate, b, and the (crude) mortality rate, m, so that $n = b - m$. The qualitative property indicated in (3) becomes conditional. It still holds if the fall in n reflects a lower b, but not necessarily if it reflects a higher m.

5. What if technological change is embodied? The models in the list above assume a neoclassical aggregate production function with CRS and *disembodied* Harrod-neutral technological progress, that is,

$$Y_t = F(K_t, T_t L_t) \equiv T_t L_t f(\tilde{k}_t), \qquad f' > 0, f'' < 0. \tag{5}$$

This amounts to assuming that new technical knowledge advances the combined productivity of capital and labor *independently* of whether the workers operate old or new machines.

In contrast, we say that technological change is *embodied* if taking advantage of new technical knowledge requires construction of new investment goods. The newest technology is incorporated in the design of newly produced equipment; and this equipment will not participate in subsequent technological progress. Both intuition and empirics suggest that most technological progress is of this form. Indeed, Greenwood et al. (1997) estimate for the U.S. 1950-1990 that embodied technological change explains 60% of the growth in output per man hour.

So a theory of the rate of return should take this into account. Fortunately, this can be done with only minor modifications. We assume that the link between investment and capital accumulation takes the form

$$\dot{K}_t = Q_t I_t - \delta K_t, \tag{6}$$

where I_t is gross investment ($I = Y - C$) and Q_t measures the "quality" (efficiency) of newly produced investment goods. Suppose for instance that

$$Q_t = Q_0 e^{\gamma t}, \qquad \gamma > 0.$$

Then, even if no technological change directly appears in the production function, that is, even if (5) is replaced by

$$Y_t = F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}, \qquad 0 < \alpha < 1,$$

the economy will still experience a rising standard of living.[1] A given level of gross investment will give rise to greater and greater additions to the capital stock K, measured in efficiency units. Since at time t, Q_t capital goods can be produced at the same cost as one consumption good, the price, p_t, of capital goods in terms of the consumption good must in competitive equilibrium equal the inverse of Q_t, that is, $p_t = 1/Q_t$. In this way embodied technological progress results in a steady decline in the relative price of capital equipment.

This prediction is confirmed by the data. Greenwood et al. (1997) find for the U.S. that the relative price of capital equipment has been declining at an average rate of 0.03 per year in the period 1950-1990, a trend that has seemingly been fortified in the wake of the computer revolution.

Along a balanced growth path the constant growth rate of K will now exceed that of Y, and Y/K thus be falling. The output-capital ratio in value terms, $Y/(pK)$, will be constant, however. Embedding these features in a Ramsey-style framework, we find the long-run rate of return to be[2]

$$r^* = \rho + \theta \frac{\alpha\gamma}{1-\alpha}.$$

This is of the same form as (4) since growth in output per unit of labor in steady state is exactly $g = \alpha\gamma/(1-\alpha)$.

[1] We specify F to be Cobb-Douglas, because otherwise a model with embodied technical progress in the form (6) will not be able to generate balanced growth and comply with Kaldor's stylized facts.

[2] See Exercise 18.??

6. Adding uncertainty and risk of bankruptcy Although absent from many simple macroeconomic models, uncertainty and risk of bankruptcy are significant features of reality. Bankruptcy risk may lead to a conflict of interest between share owners and managers. Managers may want less debt and more equity than the share owners because bankruptcy can be very costly to managers who loose a well-paid job and a promising carrier. So managers are unwilling to finance all new capital investment by new debt in spite of the associated lower capital cost (there is generally a lower rate of return on debt than on equity). In this way the excess of the rate of return on equity over that on debt, the equity premium, is sustained.

A rough behavioral theory of the equity premium goes as follows.[3] Firm managers prefer a payout structure with a fraction, s_f, going to equity and the remaining fraction, $1 - s_f$, to debt (corporate bonds). That is, out of each unit of expected operating profit, managers are unwilling to commit more than $1 - s_f$ to bond owners. This is to reduce the risk of a failing payment ability in case of a bad market outcome. And those who finance firms by loans definitely also want debtor firms to have some equity at stake.

We let households' preferred portfolio consist of a fraction s_h in equities and the remainder, $1 - s_h$, in bonds. In view of households' risk aversion and memory of historical stock market crashes, it is plausible to assume that $s_h < s_f$.

As a crude adaptation of for instance the Blanchard OLG model to these features, we interpret the model's r^* as an average rate of return across firms. Let time be discrete and let aggregate financial wealth be $A = pK$, where p is the price of capital equipment in terms of consumption goods. In the frameworks 1 to 4 above we have $p \equiv 1$, but in framework 5 the relative price p equals $1/Q$ and is falling over time. Anyway, given A at time t, the aggregate gross return or payout is $(1 + r^*)A$. Out of this, $(1 + r^*)As_f$ constitutes the gross return to the equity owners and $(1 + r^*)A(1 - s_f)$ the gross return to the bond owners. Let r_e denote the rate of return on equity and r_b the rate of return on bonds.

To find r_e and r_b we have

$$\begin{aligned} (1 + r_e)As_h &= (1 + r^*)As_f, \\ (1 + r_b)A(1 - s_h) &= (1 + r^*)A(1 - s_f). \end{aligned}$$

[3] The following is inspired by Baker, DeLong, and Krugman (2005). These authors discuss the implied predictions for U.S. rates of return in the future and draw implications of relevance for the debate on social security reform.

Thus,

$$\begin{aligned} 1 + r_e &= (1 + r^*)\frac{s_f}{s_h} > 1 + r^*, \\ 1 + r_b &= (1 + r^*)\frac{1 - s_f}{1 - s_h} < 1 + r^*. \end{aligned}$$

We may define the *equity premium*, π, by $1 + \pi \equiv (1 + r_e)/(1 + r_b)$. Then

$$\pi = \frac{s_f(1 - s_h)}{s_h(1 - s_f)} - 1 > 0.$$

Of course these formulas have their limitations. The key variables s_f and s_h will depend on a lot of economic circumstances and should be endogenous in an elaborate model. Yet, the formulas establish a way of organizing one's thoughts about rates of return in a world with asymmetric information and risk of bankruptcy.

There is evidence that in the last decades of the twentieth century the equity premium had become lower than in the long aftermath of the Great Depression in the 1930s.[4] A likely explanation is that s_h had gone up, along with rising confidence. The computer and the World Wide Web have made it much easier for individuals to invests in stocks of shares. On the other hand, the recent financial and economic crisis, known as the Great Recession 2008- , and the associated rise in mistrust seems to have halted and possibly reversed this tendency for some time (source ??).

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[4] Blanchard (2003, p. 333).

Chapter 11

The Ramsey model in use

The Ramsey representative agent framework has, rightly or wrongly, been a workhorse for the study of many macroeconomic issues. Among these are public finance themes and themes relating to endogenous productivity growth. In this chapter we consider issues within these two areas. Section 11.1 deals with a market economy with a public sector. We consider general equilibrium effects of government spending and taxation. The focus is on effects of shifts in fiscal policy and how these effects depend on whether the shift is unanticipated or anticipated. In Section 11.2 we set up and analyze a model of technology growth based on learning by investing. The analysis leads to a characterization of "first-best policy".

11.1 Fiscal policy and announcement effects

In this section we extend the Ramsey model of a competitive market economy by adding a government sector that spends on goods and services, makes transfers to the private sector, and levies taxes.

Subsection 11.1.1 addresses the effect of government spending on goods and services, assuming a balanced budget where all taxes are lump sum. One issue is what is meant by a one-off policy shock in a context of perfect foresight – sounds like a contradiction. A related issue is how to model the effects of such shocks. In subsections 11.1.2 and 11.1.3 we consider income taxation and how the economy responds to the arrival of new information about future fiscal policy. Finally, Subsection 11.1.4 introduces financing by temporary budget deficits. In view of the representative agent character of the Ramsey model, it is not surprising that Ricardian equivalence will hold in the model.

11.1.1 Public consumption financed by lump-sum taxes

The representative household (family dynasty) has $L_t = L_0 e^{nt}$ members each of which supplies one unit of labor inelastically per time unit, $n \geq 0$. The household's preferences can be represented by a time separable utility function

$$\int_0^\infty \tilde{u}(c_t, G_t) L_t e^{-\rho t} dt,$$

where $c_t \equiv C_t/L_t$ is consumption per family member and G_t is public consumption in the form of a service delivered by the government, while ρ is the rate of time preference. We assume, for simplicity, that the instantaneous utility function is additive: $\tilde{u}(c, G) = u(c) + v(G)$, where $u' > 0, u'' < 0$, i.e., there is positive but diminishing marginal utility of private consumption; the properties of the utility function v are immaterial for the questions to be studied (but hopefully $v' > 0$). The public service consists in making a non-rival good, say "law and order" or TV-transmitted theatre, available for the households free of charge. That the argument of the function v is total G_t, not per capita G_t, is due to the non-rival character of the public service.

Until further notice, the government budget is always balanced. In the present subsection the government spending, G_t, is financed by a per capita lump-sum tax, τ_t, so that

$$\tau_t L_t = G_t. \tag{11.1}$$

To allow for balanced growth under technological progress we assume that u is a CRRA function. Thus, the criterion function of the representative household can be written

$$U_0 = \int_0^\infty \left(\frac{c_t^{1-\theta}}{1-\theta} + v(G_t) \right) e^{-(\rho-n)t} dt, \tag{11.2}$$

where $\theta > 0$ is the constant (absolute) elasticity of marginal utility of private consumption. For convenience, we assume $\rho > 0$ throughout.

Let the real interest rate and the real wage be denoted r_t and w_t, respectively. The household's dynamic book-keeping equation reads

$$\dot{a}_t = (r_t - n)a_t + w_t - \tau_t - c_t, \qquad a_0 \text{ given}, \tag{11.3}$$

where a_t is per capita financial wealth. The financial wealth is held in claims of a form similar to a variable-rate deposit in a bank. Hence, at any point in time a_t is historically determined and independent of the current and future interest rates. The No-Ponzi-Game condition (solvency condition) is

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n) ds} \geq 0. \tag{NPG}$$

We see from (11.2) that leisure does not enter the instantaneous utility function. So per capita labor supply is exogenous. We fix it to be one unit of labor per time unit, as is indicated by (11.3).

In view of the additive instantaneous utility function in (11.2), marginal utility of private consumption is not affected by G_t. The Keynes-Ramsey rule resulting from the household's optimization will therefore be as if there were no government sector:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho).$$

The transversality condition of the household is that (NPG) holds with strict equality, i.e.,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = 0.$$

GDP is produced via an aggregate neoclassical production function with CRS:

$$Y_t = F(K_t^d, \mathcal{T}_t L_t^d),$$

where K_t^d and L_t^d are inputs of capital and labor, respectively, and $\mathcal{T}_t$ is the technology level, assumed to grow at an exogenous and constant rate $g \geq 0$. For simplicity we assume that F satisfies the Inada conditions. It is further assumed that in the production of G_t, the same technology (production function) is applied as in the production of the other components of GDP. So the same unit production costs are involved. A possible role of G_t for productivity is ignored (so we should not interpret G_t as related to such things as infrastructure, health, education, or research).

All capital in the economy is assumed to belong to the private sector. The economy is closed. In accordance with the standard Ramsey model, there is perfect competition in all markets. Hence there is market clearing so that $K_t^d = K_t$ and $L_t^d = L_t$ for all t.

General equilibrium and dynamics

The increase in the capital stock, K, per time unit equals aggregate gross saving:

$$\dot{K}_t = Y_t - C_t - G_t - \delta K_t = F(K_t, \mathcal{T}_t L_t) - c_t L_t - G_t - \delta K_t, \qquad K_0 > 0 \text{ given.} \quad (11.4)$$

We assume G_t is proportional to the work force measured in efficiency units, that is

$$G_t = \tilde{\gamma}\mathcal{T}_t L_t, \quad (11.5)$$

where the size of $\tilde{\gamma} \geq 0$ is decided by the government. The balanced budget (11.1) now implies that the per capita lump-sum tax grows at the same rate as technology:

$$\tau_t = G_t/L_t = \tilde{\gamma}\mathcal{T}_t = \tilde{\gamma}\mathcal{T}_0 e^{gt} = \tau_0 e^{gt}. \quad (11.6)$$

Defining $\tilde{k}_t \equiv K_t/(\mathcal{T}_t L_t) \equiv k_t/\mathcal{T}_t$ and $\tilde{c}_t \equiv C_t/(\mathcal{T}_t L_t) \equiv c_t/\mathcal{T}_t$, the dynamic aggregate resource constraint (11.4) can be written

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - \tilde{\gamma} - (\delta + g + n)\tilde{k}_t, \qquad \tilde{k}_0 > 0 \text{ given}, \tag{11.7}$$

where f is the production function in intensive form, $f' > 0, f'' < 0$. As F satisfies the Inada conditions, f satisfies

$$f(0) = 0, \qquad \lim_{\tilde{k}\to 0} f'(\tilde{k}) = \infty, \qquad \lim_{\tilde{k}\to\infty} f'(\tilde{k}) = 0.$$

As usual, by the golden-rule capital intensity, $\tilde{k}_{GR}$, we mean that capital intensity which maximizes sustainable consumption per unit of effective labor, $\tilde{c} + \tilde{\gamma}$. By setting the left-hand side of (11.7) to zero, eliminating the time indices on the right-hand side, and rearranging, we get $\tilde{c} + \tilde{\gamma} = f(\tilde{k}) - (\delta + g + n)\tilde{k} \equiv c(\tilde{k})$. In view of the Inada conditions, the problem $\max_{\tilde{k}} c(\tilde{k})$ has a unique solution, $\tilde{k} > 0$, characterized by the condition $f'(\tilde{k}) = \delta + g + n$. This $\tilde{k}$ is, by definition, $\tilde{k}_{GR}$.

In general equilibrium the real interest rate, r_t, equals $f'(\tilde{k}_t) - \delta$. Expressed in terms of $\tilde{c}$, the Keynes-Ramsey rule thus becomes

$$\dot{\tilde{c}}_t = \frac{1}{\theta}\left[f'(\tilde{k}_t) - \delta - \rho - \theta g\right]\tilde{c}_t. \tag{11.8}$$

Moreover, we have $a_t = k_t \equiv \tilde{k}_t T_t = \tilde{k}_t T_0 e^{gt}$, and so the transversality condition of the representative household can be written

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t \left(f'(\tilde{k}_s) - \delta - n - g\right)ds} = 0. \tag{11.9}$$

The phase diagram of the dynamic system (11.7) - (11.8) is shown in Fig. 11.1 where the $\dot{\tilde{k}} = 0$ locus is represented by the stippled inverse-U curve. Apart from a vertical downward shift of the $\dot{\tilde{k}} = 0$ locus, when we have $\tilde{\gamma} > 0$ instead of $\tilde{\gamma} = 0$, the phase diagram is similar to that of the Ramsey model without government. Although the per capita lump-sum tax is not visible in the reduced form of the model consisting of (11.7), (11.8), and (11.9), it is indirectly present. This is because it ensures that for all $t \geq 0$, the $\tilde{c}_t$ and $\dot{\tilde{k}}_t$ appearing in (11.7) represent exactly the consumption demand and net saving coming from the households' choices given its intertemporal budget constraint which depends on the lump-sum tax, cf. (11.11) below.

We assume $\tilde{\gamma}$ is of "moderate size" compared to the productive capacity of the economy so as to not rule out the existence of a steady state. Moreover, to

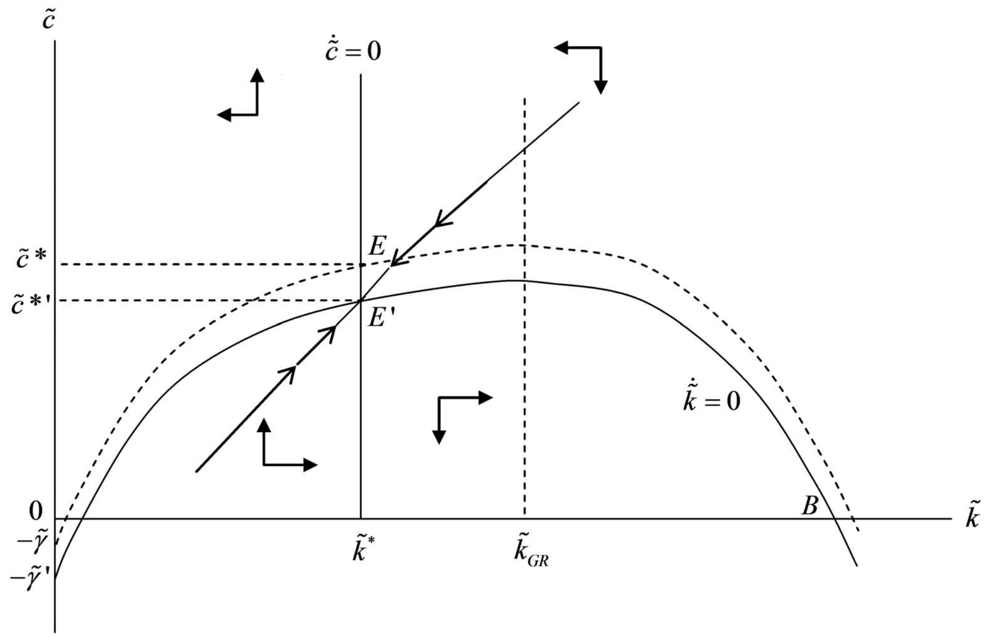

Figure 11.1: Phase portrait of an unanticipated permanent increase in government spending from $\tilde{\gamma}$ to $\tilde{\gamma}' > \tilde{\gamma}$.

guarantee bounded discounted utility and existence of general equilibrium, we impose the "sufficient impatience" restriction

$$\rho - n > (1 - \theta)g. \tag{A1}$$

How to model effects of unanticipated policy shifts

In a perfect foresight model, as the present one, agents' expectations and actions never incorporate that unanticipated events, "shocks", may arrive. That is, if a shock occurs in historical time, it must be treated as a complete surprise, a one-off shock not expected to be replicated in any sense.

Suppose that up until time $t_0 > 0$ government spending maintains the given ratio $G_t/(\mathcal{T}_t L_t) = \tilde{\gamma}$. Suppose further that before time t_0, the households expected this state of affairs to continue forever. But, unexpectedly, at time t_0 there is a shift to a higher constant spending ratio, $\tilde{\gamma}'$, which is maintained for a long time.

We assume that the upward shift in public spending goes hand in hand with higher lump-sum taxes so as to maintain a balanced budget. Thereby the after-tax human wealth of the household is at time t_0 immediately reduced. As the households are now less wealthy, cf. (11.11) below, private consumption immediately drops.

Mathematically, the time path of c_t will therefore have a discontinuity at $t = t_0$. To fix ideas, we will generally consider *control* variables, e.g., consumption, to be *right-continuous* functions of time in such cases. This means that $c_{t_0} = \lim_{t\to t_0^+} c_t$. Likewise, at such points of discontinuity of the control variable the "time derivative" of the *state* variable a in (11.3) is generally not well-defined without an amendment. In line with the right-continuity of the control variable, we define the time derivative of a state variable at a point of discontinuity of the control variable as the *right-hand time derivative*, i.e., $\dot{a}_{t_0} = \lim_{t\to t_0^+}(a_t - a_{t_0})/(t - t_0)$.[1] We say that the control variable has a *jump* at time t_0, we call the point where this jump occurs a *switch point*, and we say that the state variable, which remains a continuous function of t, has a *kink* at time t_0.

In line with this, control variables are called *jump variables* or *forward-looking variables.* The latter name comes from the notion that a decision variable can immediately shift to another value if new information arrives. In contrast, a state variable is said to be *pre-determined* because its value is an outcome of the past and it cannot jump.

An unanticipated permanent upward shift in government spending Returning to our specific example, suppose that the economy has been in steady state for $t < t_1$. Then, unexpectedly, the new spending policy $\tilde{\gamma}' > \tilde{\gamma}$ is introduced, followed by an increase in taxation so as to maintain a balanced budget. Let the households rightly expect this new policy to be maintained forever. As a consequence, the $\dot{\tilde{k}} = 0$ locus in Fig. 11.1 is shifted downwards while the $\dot{\tilde{c}} = 0$ locus remains where it is. It follows that $\tilde{k}$ stays unchanged at its old steady-state level, $\tilde{k}^*$, while $\tilde{c}$ jumps down to the new steady-state value, $\tilde{c}^{*\prime}$. There is immediate *crowding out* of private consumption to the exact extent of the rise in public consumption.[2]

To understand the mechanism, note that Per capita consumption of the household is

$$c_t = \beta_t(a_t + h_t), \tag{11.10}$$

where h_t is the after-tax human wealth per family member and is given by

$$h_t = \int_t^\infty (w_s - \tau_s)e^{-\int_t^s (r_z - n)dz} ds, \tag{11.11}$$

[1] While these conventions help to fix ideas, they are mathematically inconsequential. Indeed, the value of the consumption intensity at each isolated point of discontinuity will affect neither the utility integral of the household nor the value of the state variable, a.

[2] The conclusion is modified, of course, if G_t encompasses public investment and this has an impact on the productivity of the private sector.

and β_t is the propensity to consume out of wealth,

$$\beta_t = \frac{1}{\int_t^\infty e^{\int_t^s (\frac{(1-\theta)r_z - \rho}{\theta} + n)dz} ds}, \tag{11.12}$$

as derived in the previous chapter. The upward shift in public spending is accompanied by higher lump-sum taxes, $\tau_t' = \tilde{\gamma}' L_t$, forever, implying that h_t is reduced, which in turn reduces consumption.

Had the unanticipated shift in public spending been *downward*, say from $\tilde{\gamma}'$ to $\tilde{\gamma}$, the effect would be an *upward* jump in consumption but no change in $\tilde{k}$, that is, a jump E' to E in Fig. 11.1.

Many kinds of disturbances of a steady state will result in a *gradual* adjustment process, either to a new steady state or back to the original steady state. It is otherwise in this example where there is an *immediate jump* to a new steady state.

11.1.2 Income taxation

We now replace the assumed lump-sum taxation by income taxation of different kinds. In addition, we introduce lump-sum income transfers to the households. The path of spending on goods and services remains unchanged throughout, i.e., $G_t = \tilde{\gamma}\mathcal{T}_t L_t$ for all $t \geq 0$.

Taxation of labor income

Consider a tax on wage income at the constant rate τ_w, $0 < \tau_w < 1$. Since labor supply is exogenous, it is unaffected by the wage income tax. While (11.7) is still the dynamic resource constraint of the economy, the household's dynamic book-keeping equation now reads

$$\dot{a}_t = (r_t - n)a_t + (1 - \tau_w)w_t + x_t - c_t, \qquad a_0 \text{ given},$$

where x_t is the per capita lump-sum transfers at time t. Maintaining the assumption of a balanced budget, the tax revenue at every t exactly covers government expenditure, that is, spending on goods and services plus the lump-sum transfers to the private sector. This means that

$$\tau_w w_t L_t = G_t + x_t L_t \qquad \text{for all } t \geq 0. \tag{11.13}$$

As G_t and τ_w are given, the interpretation is that for all $t \geq 0$, transfers adjust so as to balance the budget. This requires that $x_t = \tau_w w_t - G_t/L_t = \tau_w w_t - \tilde{\gamma}\mathcal{T}_t$, for all $t \geq 0$; if x_t need be negative to satisfy this equation, so be it. Then $-x_t$ would act as a positive lump-sum tax. Disposable income at time t is

$$(1-\tau_w)w_t + x_t = w_t - \tilde{\gamma}\mathcal{T}_t,$$

and human wealth at time t per member of the representative household is thus

$$h_t = \int_t^\infty \left[(1-\tau_w)w_s + x_s\right] e^{-\int_t^s (r_z - n)dz} ds = \int_t^\infty (w_s - \tilde{\gamma}\mathcal{T}_s) e^{-\int_t^s (r_z-n)dz} ds. \tag{11.14}$$

Owing to the given $\tilde{\gamma}$, a shift in the value of τ_w is immediately compensated by an adjustment of the path of transfers in the same direction so as to maintain a balanced budget. Neither disposable income nor h_t is affected. So the shift in τ_w leaves the determinants of per capita consumption in this model unaffected. As also disposable income is unaffected, it follows that private saving is unaffected. This is why τ_w nowhere enters the model in its reduced form, consisting of (11.7), (11.8), and (11.9). The phase diagram for the economy with labor income taxation is completely identical to that in Fig. 11.1 where there is no tax on labor income. The evolution of the economy is independent of the size of τ_w (if the model were extended with endogenous labor supply, the result would generally be different). The intuitive explanation is that the three conditions: (a) inelastic labor supply, (b) a balanced budget,[3] and (c) a given path for G_t, imply that a labor income tax affects neither the marginal trade-offs (consumption versus saving and working versus enjoying leisure) nor the intertemporal budget constraint of the household.

Taxation of capital income

It is different when it comes to a tax on capital income because saving in the Ramsey model responds to incentives. Consider a constant capital income tax at the rate τ_r, $0 < \tau_r < 1$. The household's dynamic budget identity becomes

$$\dot{a}_t = \left[(1-\tau_r)r_t - n\right] a_t + w_t + x_t - c_t, \qquad a_0 \text{ given},$$

where, if $a_t < 0$, the tax acts as a rebate. As above, x_t is a per capita lump-sum transfer. In view of a balanced budget, we have at the aggregate level

$$G_t + x_t L_t = \tau_r r_t K_t. \tag{11.15}$$

[3] In fact, as we shall see in Section 11.1.4, the key point is not that, to fix ideas, we have assumed the budget is balanced for every t. It is enough that the government satisfies its intertemporal budget constraint.

As G_t and τ_r are given, the interpretation is that for all $t \geq 0$, transfers adjust so as to balance the budget. This requires that

$$x_t = \tau_r r_t k_t - G_t/L_t = \tau_r r_t k_t - \tilde{\gamma}\mathcal{T}_t. \tag{11.16}$$

We may rewrite the balanced budget condition (11.15) this way:

$$\tau_r r_t K_t - x_t L_t = G_t \qquad \text{for all } t \geq 0.$$

We see that as long as the path of G_t is given, so is that of "net taxes" on the representative household on the left-hand side. An immediate effect of a change in the tax rate τ_r will thus reflect the effect of this change *in isolation* from any change in the current net-tax payment as such because there is no such change. Within the model we study: (a) the pure effect on the consumption-saving split of a change in the tax rate τ_r, and (b) the resulting dynamic repercussions in the economy as a whole.

The No-Ponzi-Game condition of the representative household is

$$\lim_{t\to\infty} a_t e^{-\int_0^t [(1-\tau_r)r_s - n]ds} \geq 0,$$

and the Keynes-Ramsey rule takes the form

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[(1-\tau_r)r_t - \rho].$$

In general equilibrium we have

$$\dot{\tilde{c}}_t = \frac{1}{\theta}\left[(1-\tau_r)(f'(\tilde{k}_t) - \delta) - \rho - \theta g\right]\tilde{c}_t. \tag{11.17}$$

The differential equation for $\tilde{k}$ is still (11.7).

In a steady state $\tilde{k}^*$ satisfies $(f'(\tilde{k}^*) - \delta)(1-\tau_r) = \rho + \theta g$, that is,

$$f'(\tilde{k}^*) - \delta = \frac{\rho + \theta g}{1-\tau_r} > \rho + \theta g > g + n,$$

where the last inequality comes from the "sufficient impatience" assumption (A1). The higher is the tax rate τ_r, the lower is $\tilde{k}^*$. This is implied by $f'' < 0$. Consequently, in the long run consumption is lower as well.[4] The resulting resource allocation is not Pareto optimal. There exist an alternative technically feasible resource allocation that makes everyone in society better off. This is because the capital income tax implies a wedge between the marginal rate of transformation over time in production, $f'(\tilde{k}_t) - \delta$, and the marginal rate of transformation over time to which consumers adapt, $(1-\tau_r)(f'(\tilde{k}_t) - \delta)$.

[4] In the Diamond OLG model a capital income tax, which finances lump-sum transfers to the old generation, has an ambiguous effect on capital accumulation, depending on whether $\theta < 1$ or $\theta > 1$, cf. Exercise 5.?? in Chapter 5.

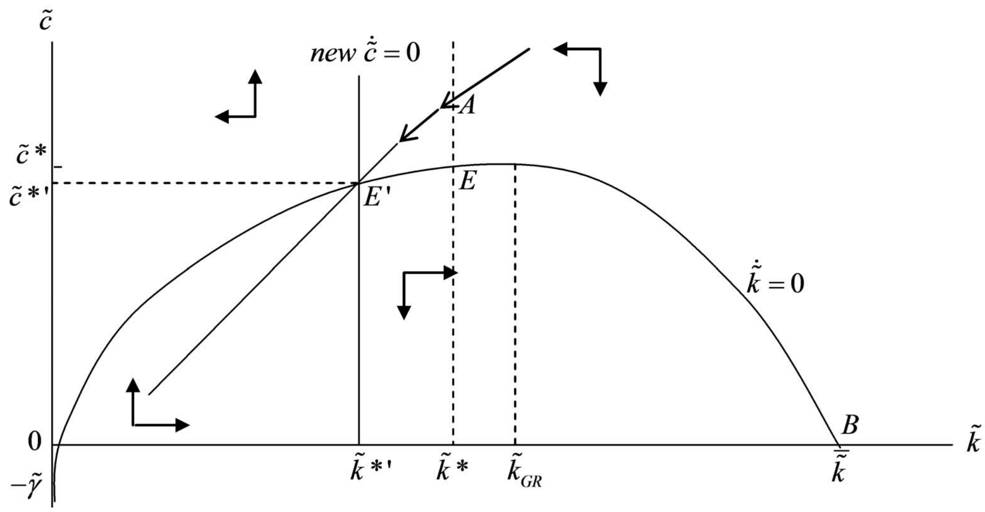

Figure 11.2: Phase portrait of an unanticipated permanent rise in τ_r.

11.1.3 Effects of shifts in the capital income tax rate

We shall study effects of a rise in the tax on capital income. The effects depend on whether the change is anticipated in advance or not and whether the change is permanent or only temporary. So there are four cases to consider. Throughout, the path of spending on goods and services remains unchanged, i.e., $G_t = \tilde{\gamma}\mathcal{T}_t L_t$ for all $t \geq 0$.

(i) Unanticipated permanent upward shift in τ_r

Until time t_1 the economy has been in steady state with a tax-transfer scheme based on some given constant tax rate, τ_r, on capital income. At time t_1, unexpectedly, the government introduces a new tax-transfer scheme, involving a higher constant tax rate, τ_r', on capital income, i.e., $0 < \tau_r < \tau_r' < 1$. Since the path of spending on goods and services is unchanged, to maintain a balanced budget, the lump-sum transfers, x_t, must be raised. We assume it is credibly announced that the new tax-transfer scheme will be adhered to forever. So households expect the real after-tax interest rate (rate of return on saving) to be $(1-\tau_r')r_t$ for all $t \geq t_1$.

For $t < t_1$ the dynamics are governed by (11.7) and (11.17) with $0 < \tau_r < 1$. The corresponding steady state, E, has $\tilde{k} = \tilde{k}^*$ and $\tilde{c} = \tilde{c}^*$ as indicated in the phase diagram in Fig. 11.2. The new tax-transfer scheme ruling after time t_1 shifts the steady state point to E' with $\tilde{k} = \tilde{k}^{*\prime}$ and $\tilde{c} = \tilde{c}^{*\prime}$. The new $\dot{\tilde{c}} = 0$ line and the new saddle path are to the left of the old, i.e., $\tilde{k}^{*\prime} < \tilde{k}^*$. Until time t_1 the economy is at the point E. Immediately after the shift in the tax on capital income, equilibrium requires that the economy is on the new saddle path. So there will be a jump from point E to point A in Fig. 11.2.

This upward jump in consumption is intuitively explained the following way. We know that individual consumption immediately after the policy shock satisfies

$$c_{t_1} = \beta_{t_1}(a_{t_1} + h_{t_1}), \tag{11.18}$$

where

$$h_{t_1} = \int_{t_1}^{\infty} (w_t + x_t) e^{-\int_{t_1}^{t}((1-\tau_r')r_z - n)dz} dt = \int_{t_1}^{\infty} (w_t + \tau_r' r_t k_t - \tilde{\gamma}\mathcal{T}_t) e^{-\int_{t_1}^{t}((1-\tau_r')r_z - n)dz} dt,$$

by (11.16), and

$$\beta_{t_1} = \frac{1}{\int_{t_1}^{\infty} e^{\int_{t_1}^{t}(\frac{(1-\theta)(1-\tau_r')r_z - \rho}{\theta} + n)dz} dt}.$$

Two effects are present. First, both the higher transfers and the lower after-tax rate of return after time t_1 contribute to a higher h_{t_1}. There is thereby a positive wealth effect on current consumption through a higher h_{t_1}. Second, the propensity to consume, β_{t_1}, will generally be affected. If $\theta < 1$, the reduction in the after-tax rate of return will have a positive effect on β_{t_1}. The positive effect on β_{t_1} when $\theta < 1$ reflects that the positive substitution effect on c_{t_1} of a lower after-tax rate of return dominates the negative income effect. If instead $\theta > 1$, the positive substitution effect on c_{t_1} is dominated by the negative income effect. Whatever happens to β_{t_1}, however, the phase diagram shows that in general equilibrium there will necessarily be an *upward* jump in c_{t_1}. The implication is lower saving. We get this result even if θ is much higher than 1. The explanation lies in the assumption that all the extra tax revenue obtained by the rise in τ_r is immediately transferred back to the households lump sum, thereby strengthening the positive wealth effect on current consumption through the lower discount rate implied by $(1 - \tau_r')r_z < (1 - \tau_r)r_z$.

In response to the rise in τ_r, we thus have $\tilde{c}_{t_1} > f(\tilde{k}_{t_1}) - (\delta + g + n)\tilde{k}_{t_1}$, implying that saving is too low to sustain $\tilde{k}$, which thus begins to fall. This results in lower real wages and higher before-tax interest rates, that is two *negative* feedbacks on human wealth. Could these feedbacks not fully offset the initial tendency for (after-tax) human wealth to rise? The answer is no, see Box 11.1.

As indicated by the arrows in Fig. 11.2, the economy moves along the new saddle path towards the new steady state E'. Because $\tilde{k}$ is lower in the new steady state than in the old, so is $\tilde{c}$. The evolution of the technology level, $\mathcal{T}$, is by assumption exogenous; thus, also actual per capita consumption, $c \equiv \tilde{c}^*\mathcal{T}$, is lower in the new steady state.

Box 11.1. A mitigating feedback can not instantaneously fully offset the force that activates it.

Can the story told by Fig. 11.2 be true? Can it be true that the net effect of the higher tax on capital income is an upward jump in consumption at time t_1 as indicated in Fig. 11.2? Such a jump means that $\tilde{c}_{t_1} > f(\tilde{k}_{t_1}) - (\delta + g + n)\tilde{k}_{t_1}$ and the resulting reduced saving will make the future k lower than otherwise and thereby make expected future real wages lower and expected future before-tax interest rates higher. Both feedbacks partly counteract the initial upward shift in human wealth due to higher transfers and a lower effective discount rate that were the direct result of the rise in τ_w. Could the two mentioned counteracting feedbacks fully offset the initial tendency for (after-tax) human wealth, and therefore current consumption, to rise?

The phase diagram says no. But what is the intuition? That the two feedbacks can not fully offset (or even reverse) the tendency for (after-tax) human wealth to rise at time t_1 is explained by the fact that if they could, then the two feedbacks would not be there in the first place. We cannot at the same time have both a rise in the human wealth that triggers higher consumption (and thereby lower saving and investment in the economy) and a neutralization, or a complete reversal, of this rise in the human wealth caused by the higher consumption. The two feedbacks can only partly offset the initial tendency for human wealth to rise.

Instead of all the extra tax revenue obtained being transferred back lump sum to the households, we may alternatively assume that a major part of it is used to finance a rise in government consumption to the level $G'_t = \tilde{\gamma}' T_t L_t$, where $\tilde{\gamma}' > \tilde{\gamma}$.[5] In addition to the leftward shift of the $\dot{\tilde{c}} = 0$ locus this will result in a downward shift of the $\dot{\tilde{k}} = 0$ locus. The phase diagram would look like a convex combination of Fig. 11.1 and Fig. 11.2. Then it is possible that the jump in consumption at time t_0 becomes downward instead of upward.

Returning to the case where the extra tax revenue is fully transferred, the next subsection splits the change in taxation policy into two events.

[5] It is understood that also $\tilde{\gamma}'$ is not larger than what allows a steady state to exist. Moreover, the government budget is still balanced for all t so that any temporary surplus or shortage of tax revenue, $\tau'_r r_t K_t - G'_t$, is immediately transferred or levied lump-sum, respectively.

(ii) Anticipated permanent upward shift in τ_r

Until time t_1 the economy has been in steady state with a tax-transfer scheme based on some given constant tax rate, τ_r, on capital income.

At time t_1, unexpectedly, the government credibly announces that a new fiscal policy with $\tau_r' > \tau_r$ is to be implemented at time $t_2 > t_1$, and that transfers will be adjusted so as to maintain a balanced budget, given no change in the path of G_t. We assume people believe in this announcement and that the new policy is actually implemented at time t_2 as announced. The shock to the economy is now not the event of a higher tax being implemented at time t_2. Already immediately after time t_1, this event is foreseen. It is at time t_1 that a "shock" occurs, namely in the form of an unexpected announcement.

The phase diagram in Fig. 11.3 illustrates the evolution of the economy for $t \geq t_1$. There are two time intervals to consider. For $t \in [t_2, \infty)$, the dynamics are governed by (11.7) and (11.17) with τ_r replaced by τ_r', starting at some point on the new saddle path, namely the point which has abscissas equal to the so far unknown value obtained by $\tilde{k}$ at time t_1.

In the time interval $[t_1, t_2)$, however, the "old dynamics", with the lower tax rate, τ_r, still hold. Yet the path the economy follows immediately after time t_1 is different from what it would have been without the information that capital income will be taxed heavily from time t_2, where also transfers will become higher. On the one hand, the expectation of a higher after-tax interest rate and higher transfers from time t_2 and onwards immediately raises the present value, as seen from time t_1, of future after-tax labor and transfer income. This implies that already at time t_1 do people feel more wealthy. Consequently, an upward jump in consumption occurs, say to a point like point C in Fig. 11.3.

On the other hand, since the actual shift to a higher tax rate does not occur until time t_2, the rise in the present value of expected future labor and transfer income is lower than in Case (i) above. This explains that the point C is below point A in Fig. 11.3 (point A itself is the same as point A in Fig. 11.2). How far below? The answer follows from the fact that there cannot be an *expected* discontinuity of marginal utility at time t_2, since that would contradict the preference for consumption smoothing over time implied by strict concavity of the instantaneous utility function. To put it differently: as soon as people become aware of the upcoming rise in both tax rate and transfers, they adjust their consumption level so as to be on their preferred smooth consumption path under the new circumstances. When the shift to a higher tax rate occurs at time t_2, it has been anticipated and triggers no jump, neither in consumption, c_{t_2}, nor in human wealth, h_{t_2}.[6] Indeed, if on the contrary there were a discontinuity in c_t

[6] Replace t_1 in the formula for human wealth in (11.18) by some $t \in (t_1, t_2)$, and consider h_t

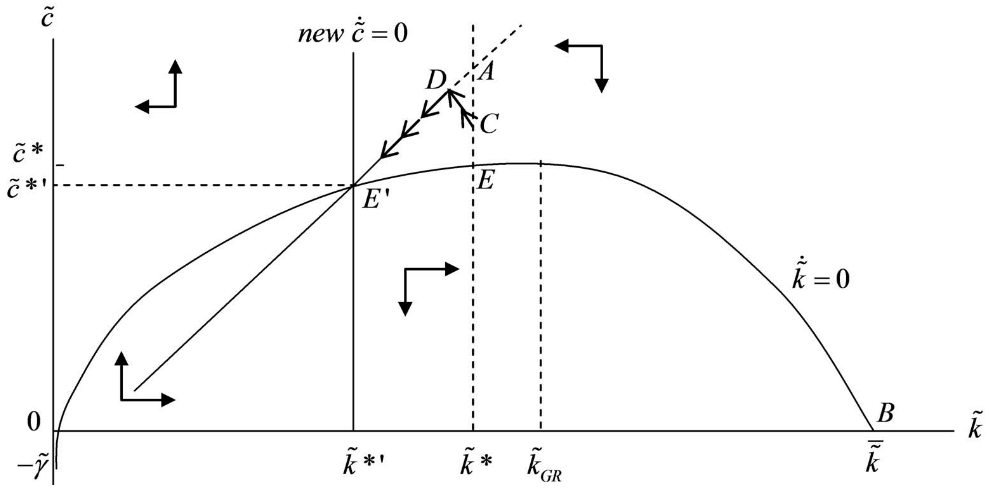

Figure 11.3: Phase portrait of an anticipated permanent rise in τ_r.

at time t_2, there would be gains to be obtained by removing this discontinuity. This is due to $u''(c) < 0$.

To avoid existence of an expected discontinuity in consumption, the point C on the vertical line $\tilde{k} = \tilde{k}^*$ in Fig. 11.3 must be such that, following the "old dynamics", it takes exactly $t_2 - t_0$ time units to reach the new saddle path. This dictates a unique position of the point C between E and A. If C were at a lower position, the journey to the saddle path would take longer than $t_2 - t_0$. And if C were at a higher position, the journey would not take as long as $t_2 - t_0$.

Immediately after time t_0, $\tilde{k}$ will be decreasing (because saving is smaller than what is required to sustain a constant $\tilde{k}$); and $\tilde{c}$ will be *increasing* in view of the Keynes-Ramsey rule, since the rate of return on saving is above $\rho + \theta g$ as long as $\tilde{k} < \tilde{k}^*$ and τ_r low. Precisely at time t_2 the economy reaches the new saddle path, the high taxation of capital income begins, and the after-tax rate of return becomes lower than $\rho + \theta g$. Hence, per-capita consumption begins to fall and the economy gradually approaches the new steady state E'.

This analysis illustrates that when economic agents' behavior depend on forward-looking expectations, a credible announcement of a future change in policy has an effect already before the new policy is implemented. Such effects are known as *announcement effects* or *anticipation effects.*

As a kind of parallel to our claim that there can be no *planned* jump in consumption, consider an asset price. In the asset market arbitrage rules out the possibility of a generally expected jump in the asset price at a given point in time

as the sum of two integrals, one from t to t_2 and one from t_2 to ∞. Then let t approach t_2 from below.

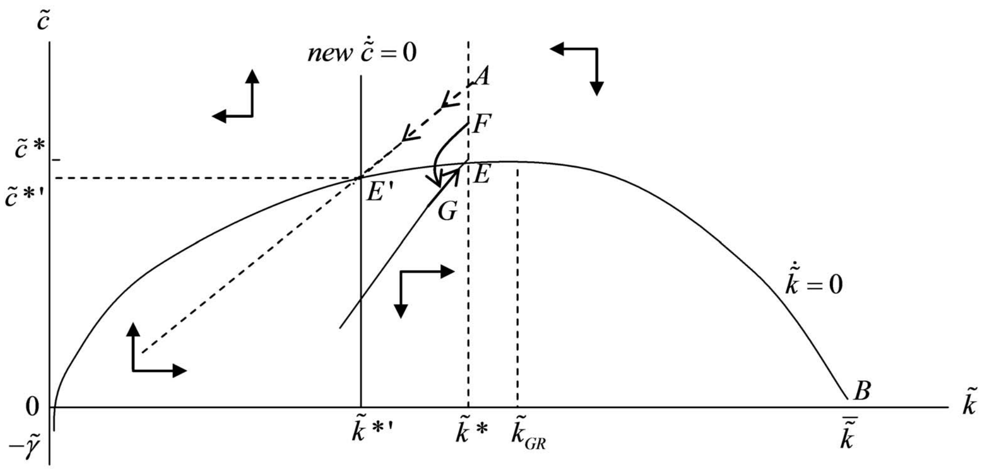

Figure 11.4: Phase portrait of an unanticipated temporary rise in τ_r.

in the future. If we imagine the expected jump is upward, an infinite positive rate of return *could be obtained* by buying the asset immediately before the jump. This would generate *excess demand* of the asset before time t_2 and drive up its price in advance thus *preventing* an expected upward jump to occur at time t_2. And if we on the other hand imagine the expected jump is downward, an infinite negative rate of return *could be avoided* by selling the asset immediately before the jump. This would generate *excess supply* of the asset before time t_2 and drive its price down in advance thus preventing an expected downward jump at t_2.

In the household's optimal control problem, cf. Chapter 10.2, the adjoint variable, λ, can be interpreted as a shadow price, and this has some resemblance to an asset price. Recalling the optimality condition $u'(c_{t_2}) = \lambda_{t_2}$, we could also say that due to $u''(c) < 0$, along an optimal path there can be no *expected* discontinuity in the shadow price of financial wealth, λ_{t_2}.

(iii) Unanticipated temporary upward shift in τ_r

Once again we change the scenario. The economy with low capital taxation has been in steady state up until time t_1. Then a new tax-transfer scheme is unexpectedly introduced. At the same time it is credibly announced that the high taxes on capital income and the corresponding transfers will cease at time $t_2 > t_1$. The path of spending on goods and services remains unchanged throughout, i.e., $G_t = \tilde{\gamma}\mathcal{T}_t L_t$ for all $t \geq 0$.

The phase diagram in Fig. 11.4 illustrates the evolution of the economy for $t \geq t_1$. For $t \geq t_2$, the dynamics are governed by (11.7) and (11.17), again with the old τ_r, starting from whatever value obtained by $\tilde{k}$ at time t_2.

In the time interval $[t_1, t_2)$ the "new, temporary dynamics" with the high τ_r'

and high transfers hold sway. Yet the path that the economy takes immediately after time t_1 is different from what it would have been without the information that the new tax-transfers scheme is only temporary. Indeed, the expectation of a shift to a higher after-tax rate of return and cease of high transfers as of time t_2 implies lower present value of expected future labor and transfer earnings than without this information. Hence, the upward jump in consumption at time t_1 is smaller than in Fig. 11.2. How much smaller? Again, the answer follows from the fact that there can not be an *expected* discontinuity of marginal utility at time t_2, since that would violate the principle of smoothing of planned consumption. Thus the point F on the vertical line $\tilde{k} = \tilde{k}^*$ in Fig. 11.4 must be such that, following the "new, temporary dynamics", it takes exactly $t_2 - t_1$ time units to reach the solid saddle path in Fig. 11.4 (which is in fact the same as the saddle path before time t_1). The implied position of the economy at time t_2 is indicated by the point G in the figure.

Immediately after time t_1, $\tilde{k}$ will be decreasing (because saving is smaller than what is required to sustain a constant $\tilde{k}$) and $\tilde{c}$ will be *decreasing* in view of the Keynes-Ramsey rule in a situation with an after-tax rate of return lower than $\rho + \theta g$. Precisely at time t_2, when the temporary tax-transfers scheme based on τ_r' is abolished (as announced and expected), the economy reaches the solid saddle path. From that time the return on saving is high both because of the abolition of the high capital income tax and because $\tilde{k}$ is relatively low. The general equilibrium effect of this is higher saving, and so the economy moves along the solid saddle path back to the original steady-state point E.

There is a last case to consider, namely an *anticipated temporary* in τ_r. We leave that for an exercise, see Exercise 11.??

11.1.4 Ricardian equivalence

We now drop the balanced budget assumption and allow public spending to be financed partly by issuing government bonds and partly by lump-sum taxation. Transfers and gross tax revenue as of time t are called X_t and $\tilde{T}_t$ respectively, while the real value of government net debt is called B_t. Taxes are lump sum. For simplicity, we assume all public debt is short-term. Ignoring any money-financing of the spending, the increase per time unit in government debt is identical to the government budget deficit:

$$\dot{B}_t = r_t B_t + G_t + X_t - \tilde{T}_t. \tag{11.19}$$

As we ignore uncertainty, on its debt the government has to pay the same interest rate, r_t, as other borrowers.

Because of the "sufficient impatience" assumption (A1), in the Ramsey model the long-run interest rate necessarily exceeds the long-run GDP growth rate. As we saw in Chapter 6, to remain solvent, the government must then, as a debtor, fulfil a solvency requirement analogous to that of the households in the Ramsey model:

$$\lim_{t\to\infty} B_t e^{-\int_0^t r_s ds} \leq 0. \tag{11.20}$$

This NPG condition says that the debt is in the long run allowed to grow at most at a rate less than the interest rate. As in discrete time, given the accounting relationship (11.19), the NPG condition is equivalent to the intertemporal budget constraint

$$\int_0^\infty (G_t + X_t)e^{-\int_0^t r_s ds}dt \leq \int_0^\infty \tilde{T}_t e^{-\int_0^t r_s ds}dt - B_0. \tag{GIBC}$$

This says that the present value of the credibly planned public expenditure cannot exceed government net wealth consisting of the present value of the expected future tax revenues minus initial government debt, i.e., assets minus liabilities.

Assuming that the government does not want to be a net creditor to the private sector in the long run, it will not collect more taxes than is necessary to satisfy (GIBC). Hence, we replace "$\leq$" by "=" and rearrange to obtain

$$\int_0^\infty \tilde{T}_t e^{-\int_0^t r_s ds}dt = \int_0^\infty (G_t + X_t)e^{-\int_0^t r_s ds}dt + B_0. \tag{11.21}$$

Thus, for a given path of G_t and X_t, the stream of the expected tax revenue must be such that its present value equals the present value of total liabilities on the right-hand-side of (11.21). A temporary budget deficit leads to more debt and therefore also higher taxes in the future. A budget deficit merely implies a deferment of tax payments. The condition (11.21) can be reformulated as

$$\int_0^\infty (\tilde{T}_t - G_t - X_t)e^{-\int_0^t r_s ds}dt = B_0.$$

This shows that *if net debt is positive today, then to satisfy its intertemporal budget constraint, the government has to run a positive primary budget surplus* (that is, $\tilde{T}_t - G_t - X_t > 0$) *in a sufficiently long time in the future.*

We will now show that when taxes are lump sum, then *Ricardian equivalence* holds in the Ramsey model with a public sector.[7] That is, a temporary tax cut will have no consequences for aggregate consumption. The time profile of lump-sum taxes does not matter.

Consider the intertemporal budget constraint of the representative household,

$$\int_0^\infty c_t L_t e^{-\int_0^t r_s ds}dt \leq A_0 + H_0 = K_0 + B_0 + H_0, \tag{11.22}$$

[7] It is enough that just those taxes that are varied in the thought experiment are lump-sum.

where H_0 is human wealth of the household. This says, that the present value of the planned consumption stream can not exceed the total wealth of the household. In the optimal plan of the household, we have strict equality in (11.22).

Let τ_t denote the lump-sum per capita *net* tax. Then, $\tilde{T}_t - X_t = \tau_t L_t$ and

$$\begin{aligned} H_0 &= h_0 L_0 = \int_0^\infty (w_t - \tau_t) L_t e^{-\int_0^t r_s ds} dt = \int_0^\infty (w_t L_t + X_t - \tilde{T}_t) e^{-\int_0^t r_s ds} dt \\ &= \int_0^\infty (w_t L_t - G_t) e^{-\int_0^t r_s ds} dt - B_0, \end{aligned} \tag{11.23}$$

where the last equality comes from rearranging (11.21). It follows that

$$B_0 + H_0 = \int_0^\infty (w_t L_t - G_t) e^{-\int_0^t r_s ds} dt.$$

We see that the time profiles of transfers and taxes have fallen out. What matters for total wealth of the forward-looking household is just the spending on goods and services, not the time profile of transfers and taxes. A higher initial debt has no effect on the *sum*, $B_0 + H_0$, because H_0, which incorporates transfers and taxes, becomes equally much lower. Total private wealth is thus unaffected by government debt. So is therefore also private consumption when net taxes are lump sum. A temporary tax cut will not make people feel wealthier and induce them to consume more. Instead they will increase their saving by the same amount as taxes have been reduced, thereby preparing for the higher taxes in the future.

This is the *Ricardian equivalence* result, which we encountered also in Barro's discrete time dynasty model in Chapter 7:

> In a representative agent model with full employment, rational expectations, and no credit market imperfections, if taxes are lump sum, then, for a given evolution of public expenditure, aggregate private consumption is independent of whether current public expenditure is financed by taxes or by issuing bonds. The latter method merely implies a deferment of tax payments. Given the government's intertemporal budget constraint, (11.21), a cut in current taxes has to be offset by a rise in future taxes of the same present value. Since, with lump-sum taxation, it is only the present value of the stream of taxes that matters, the "timing" is irrelevant.

Of key importance are the assumption of a representative agent and the assumption (A1), leading to a long-run interest rate in excess of the long-run GDP growth rate. As pointed out in Chapter 6, Ricardian equivalence breaks down in

OLG models without an operative Barro-style bequest motive. Such a bequest motive is implicit in the infinite horizon of the Ramsey household. In OLG models, where finite life time is emphasized, there is a turnover in the population of tax payers so that taxes levied at different times are levied on partly different sets of agents. In the future there are newcomers and they will bear part of the higher future tax burden. Therefore, a current tax cut makes current generations feel wealthier and this leads to an increase in current consumption, implying a decrease in national saving, as a result of the temporary deficit finance. The present generations benefit, but future generations bear the cost in the form of smaller national wealth than otherwise. We return to further reasons for absence of Ricardian equivalence in chapters 13 and 19.

11.2 Learning by investing and investment-enhancing policy

In *endogenous growth theory* the Ramsey framework has been applied extensively as a simplifying description of the household sector. In most endogenous growth theory the focus is on mechanisms that generate and shape technological change. Different hypotheses about the generation of new technologies are then often combined with a simplified picture of the household sector as in the Ramsey model. Since this results in a simple determination of the long-run interest rate (the modified golden rule), the analyst can in a first approach concentrate on the main issue, technological change, without being disturbed by aspects that are often secondary to this issue.

As an example, let us consider one of the basic endogenous growth models, the *learning-by-investing model,* sometimes called the *learning-by-doing model.* Learning from investment experience and diffusion across firms of the resulting new technical knowledge (positive externalities) play an important role.

There are two popular alternative versions of the model. The distinguishing feature is whether the learning parameter (see below) is less than one or equal to one. The first case corresponds to a model by Nobel laureate Kenneth Arrow (1962). The second case has been drawn attention to by Paul Romer (1986) who assumes that the learning parameter equals one. We first consider the common framework shared by these two models. Next we describe and analyze Arrow's model (in a simplified version) and finally we compare it to Romer's.

11.2.1 The common framework

We consider a closed economy with firms and households interacting under conditions of perfect competition. Later, a government attempting to internalize the positive investment externality is introduced.

Let there be N firms in the economy (N "large"). Suppose they all have the same neoclassical production function, F, with CRS. Firm no. i faces the technology

$$Y_{it} = F(K_{it}, \mathcal{T}_t L_{it}), \qquad i = 1, 2, ..., N, \tag{11.24}$$

where the economy-wide technology level $\mathcal{T}_t$ is an increasing function of society's previous experience, approximated by cumulative aggregate net investment:

$$\mathcal{T}_t = \left(\int_{-\infty}^{t} I_s^n ds \right)^{\lambda} = K_t^{\lambda}, \quad 0 < \lambda \leq 1, \tag{11.25}$$

where I_s^n is aggregate net investment and $K_t = \sum_i K_{it}$.[8]

The idea is that investment − the production of capital goods − as an unintended *by-product* results in *experience.* The firm and its employees learn from this experience. Producers recognize opportunities for process and quality improvements. In this way knowledge is achieved about how to produce the capital goods in a cost-efficient way and how to design them so that in combination with labor they are more productive and satisfy better the needs of the users. Moreover, as emphasized by Arrow,

> "each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning is taking place with continually new stimuli" (Arrow, 1962).[9]

The learning is assumed to benefit essentially all firms in the economy. There are knowledge spillovers across firms and these spillovers are reasonably fast relative to the time horizon relevant for growth theory. In our macroeconomic approach both F and $\mathcal{T}$ are in fact assumed to be exactly the same for all firms in the economy. That is, in this specification the firms producing consumption-goods benefit from the learning just as much as the firms producing capital-goods.

The parameter λ indicates the elasticity of the general technology level, $\mathcal{T}$, with respect to cumulative aggregate net investment and is named the "learning

[8] For arbitrary units of measurement for labor and output the hypothesis is $T_t = BK_t^{\lambda}$, $B > 0$. In (11.25) measurement units are chosen such that $B = 1$.

[9] Concerning empirical evidence of learning-by-doing and learning-by-investing, see Literature Notes. The citation of Arrow indicates that it was experience from cumulative *gross* investment he had in mind as the basis for learning. Yet, to simplify, we stick to the hypothesis in (11.25), where it is cumulative net investment that matters.

parameter". Whereas Arrow assumes $\lambda < 1$, Romer focuses on the case $\lambda = 1$. The case of $\lambda > 1$ is ruled out since it would lead to explosive growth (infinite output in finite time) and is therefore not plausible.

The individual firm

In the simple Ramsey model we assumed that households directly own the capital goods in the economy and rent them out to the firms. When discussing learning-by-investment, it fits the intuition better if we (realistically) assume that the firms generally own the capital goods they use. They then finance their capital investment by issuing shares and bonds. Households' financial wealth then consists of these shares and bonds.

Consider firm i. There is perfect competition in all markets. So the firm is a price taker. Its problem is to choose a production and investment plan which maximizes the present value, V_i, of expected future cash-flows. The firm thus chooses $(L_{it}, I_{it})_{t=0}^{\infty}$ to maximize

$$V_{i0} = \int_0^{\infty} \left[F(K_{it}, \mathcal{T}_t L_{it}) - w_t L_{it} - I_{it}\right] e^{-\int_0^t r_s ds} dt$$

subject to $\dot{K}_{it} = I_{it} - \delta K_{it}$. Here w_t and I_t are the real wage and gross investment, respectively, at time t, r_s is the real interest rate at time s, and $\delta \geq 0$ is the capital depreciation rate. Rising marginal capital installation costs and other kinds of adjustment costs are assumed minor and can be ignored. It can be shown, cf. Chapter 14, that in this case the firm's problem is equivalent to maximization of current pure profits in every short time interval. So, as hitherto, we can describe the firm as just solving a series of static profit maximization problems.

We suppress the time index when not needed for clarity. At any date firm i maximizes current pure profits, $\Pi_i = F(K_i, \mathcal{T} L_i) - (r + \delta)K_i - wL_i$, where $r + \delta$ is the imputed cost (opportunity cost) per unit of capital used by the firm itself. This leads to the first-order conditions for an interior solution:

$$\begin{aligned} \partial \Pi_i / \partial K_i &= F_1(K_i, \mathcal{T} L_i) - (r + \delta) = 0, \qquad (11.26) \\ \partial \Pi_i / \partial L_i &= F_2(K_i, \mathcal{T} L_i)\mathcal{T} - w = 0. \end{aligned}$$

Behind (11.26) is the presumption that each firm is small relative to the economy as a whole, so that each firm's investment has a negligible effect on the economy-wide technology level $\mathcal{T}_t$. Since F is homogeneous of degree one, by Euler's theorem,[10] the first-order partial derivatives, F_1 and F_2, are homogeneous of degree 0. Thus, we can write (11.26) as

$$F_1(k_i, \mathcal{T}) = r + \delta, \qquad (11.27)$$

[10] See Math tools.

where $k_i \equiv K_i/L_i$. Since F is neoclassical, $F_{11} < 0$. Therefore (11.27) determines k_i uniquely. From (11.27) follows that the chosen capital-labor ratio, k_i, will be the same for all firms, say $\bar{k}$.

The individual household

The household sector is described by our standard Ramsey framework with inelastic labor supply and a constant population growth rate $n \geq 0$. The households have CRRA instantaneous utility with parameter $\theta > 0$. The pure rate of time preference is a constant, ρ. The flow budget identity in per capita terms is

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t, \qquad a_0 \text{ given},$$

where a is per capita financial wealth. The NPG condition is

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0.$$

The resulting consumption-saving plan implies that per capita consumption follows the Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho),$$

and the transversality condition that the NPG condition is satisfied with strict equality. In general equilibrium of our closed economy with no role for natural resources and no government debt, a_t will equal K_t/L_t.

Equilibrium in factor markets

For every t we have in equilibrium that $\sum_i K_i = K$ and $\sum_i L_i = L$, where K and L are the available amounts of capital and labor, respectively (both predetermined). Since $K = \sum_i K_i = \sum_i k_i L_i = \sum_i \bar{k} L_i = \bar{k}L$, the chosen capital intensity, k_i, satisfies

$$k_i = \bar{k} = \frac{K}{L} \equiv k, \qquad i = 1, 2, ..., N. \tag{11.28}$$

As a consequence we can use (11.27) to *determine* the equilibrium interest rate:

$$r_t = F_1(k_t, \mathcal{T}_t) - \delta. \tag{11.29}$$

That is, whereas in the firm's first-order condition (11.27) causality goes from r_t to k_{it}, in (11.29) causality goes from k_t to r_t. Note also that in our closed economy with no natural resources and no government debt, a_t will equal k_t.

The implied aggregate production function is

$$\begin{aligned} Y &= \sum_i Y_i \equiv \sum_i y_i L_i = \sum_i F(k_i, \mathcal{T})L_i = \sum_i F(k, \mathcal{T})L_i \\ &= F(k, \mathcal{T})\sum_i L_i = F(k, \mathcal{T})L = F(K, \mathcal{T}L) = F(K, K^{\lambda}L), \quad (11.30) \end{aligned}$$

where we have used (11.24), (11.28), and (11.25) and the assumption that F is homogeneous of degree one.

11.2.2 The arrow case: $\lambda < 1$

The Arrow case is the robust case where the learning parameter satisfies $0 < \lambda < 1$. The method for analyzing the Arrow case is analogue to that used in the study of the Ramsey model with exogenous technical progress. In particular, aggregate capital per unit of effective labor, $\tilde{k} \equiv K/(\mathcal{T}L)$, is a key variable. Let $\tilde{y} \equiv Y/(\mathcal{T}L)$. Then

$$\tilde{y} = \frac{F(K, \mathcal{T}L)}{\mathcal{T}L} = F(\tilde{k}, 1) \equiv f(\tilde{k}), \qquad f' > 0, f'' < 0. \tag{11.31}$$

We can now write (11.29) as

$$r_t = f'(\tilde{k}_t) - \delta, \tag{11.32}$$

where $\tilde{k}_t$ is pre-determined.

Dynamics

From the definition $\tilde{k} \equiv K/(\mathcal{T}L)$ follows

$$\begin{aligned} \frac{\dot{\tilde{k}}}{\tilde{k}} &= \frac{\dot{K}}{K} - \frac{\dot{\mathcal{T}}}{\mathcal{T}} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - \lambda\frac{\dot{K}}{K} - n \quad \text{(by (11.25))} \\ &= (1-\lambda)\frac{Y - C - \delta K}{K} - n = (1-\lambda)\frac{\tilde{y} - \tilde{c} - \delta\tilde{k}}{\tilde{k}} - n, \quad \text{where } \tilde{c} \equiv \frac{C}{\mathcal{T}L} \equiv \frac{c}{\mathcal{T}}. \end{aligned}$$

Multiplying through by $\tilde{k}$ we have

$$\dot{\tilde{k}} = (1-\lambda)(f(\tilde{k}) - \tilde{c}) - [(1-\lambda)\delta + n]\,\tilde{k}. \tag{11.33}$$

In view of (11.32), the Keynes-Ramsey rule implies

$$g_c \equiv \frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho) = \frac{1}{\theta}\left(f'(\tilde{k}) - \delta - \rho\right). \tag{11.34}$$

Defining $\tilde{c} \equiv c/A$, now follows

$$\begin{aligned} \frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{\dot{c}}{c} - \frac{\dot{\mathcal{T}}}{\mathcal{T}} = \frac{\dot{c}}{c} - \lambda\frac{\dot{K}}{K} = \frac{\dot{c}}{c} - \lambda\frac{Y - cL - \delta K}{K} = \frac{\dot{c}}{c} - \frac{\lambda}{\tilde{k}}(\tilde{y} - \tilde{c} - \delta\tilde{k}) \\ &= \frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho) - \frac{\lambda}{\tilde{k}}(\tilde{y} - \tilde{c} - \delta\tilde{k}). \end{aligned}$$

Multiplying through by $\tilde{c}$ we have

$$\dot{\tilde{c}} = \left[\frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho) - \frac{\lambda}{\tilde{k}}(f(\tilde{k}) - \tilde{c} - \delta\tilde{k})\right]\tilde{c}. \tag{11.35}$$

The two coupled differential equations, (11.33) and (11.35), determine the evolution over time of the economy.

Phase diagram Fig. 11.5 depicts the phase diagram. The $\dot{\tilde{k}} = 0$ locus comes from (11.33), which gives

$$\dot{\tilde{k}} = 0 \text{ for } \tilde{c} = f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}, \tag{11.36}$$

where we realistically may assume that $\delta + n/(1-\lambda) > 0$. As to the $\dot{\tilde{c}} = 0$ locus, we have

$$\begin{aligned} \dot{\tilde{c}} &= 0 \text{ for } \tilde{c} = f(\tilde{k}) - \delta\tilde{k} - \frac{\tilde{k}}{\lambda\theta}(f'(\tilde{k}) - \delta - \rho) \\ &= f(\tilde{k}) - \delta\tilde{k} - \frac{\tilde{k}}{\lambda}g_c \equiv c(\tilde{k}) \quad \text{(from (11.34))}. \end{aligned} \tag{11.37}$$

Before determining the slope of the $\dot{\tilde{c}} = 0$ locus, it is convenient to consider the steady state, $(\tilde{k}^*, \tilde{c}^*)$.

Steady state In a steady state $\tilde{c}$ and $\tilde{k}$ are constant so that the growth rate of C as well as K equals $\dot{A}/A + n$, i.e.,

$$\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{\dot{\mathcal{T}}}{\mathcal{T}} + n = \lambda\frac{\dot{K}}{K} + n.$$

Solving gives

$$\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{n}{1-\lambda}.$$

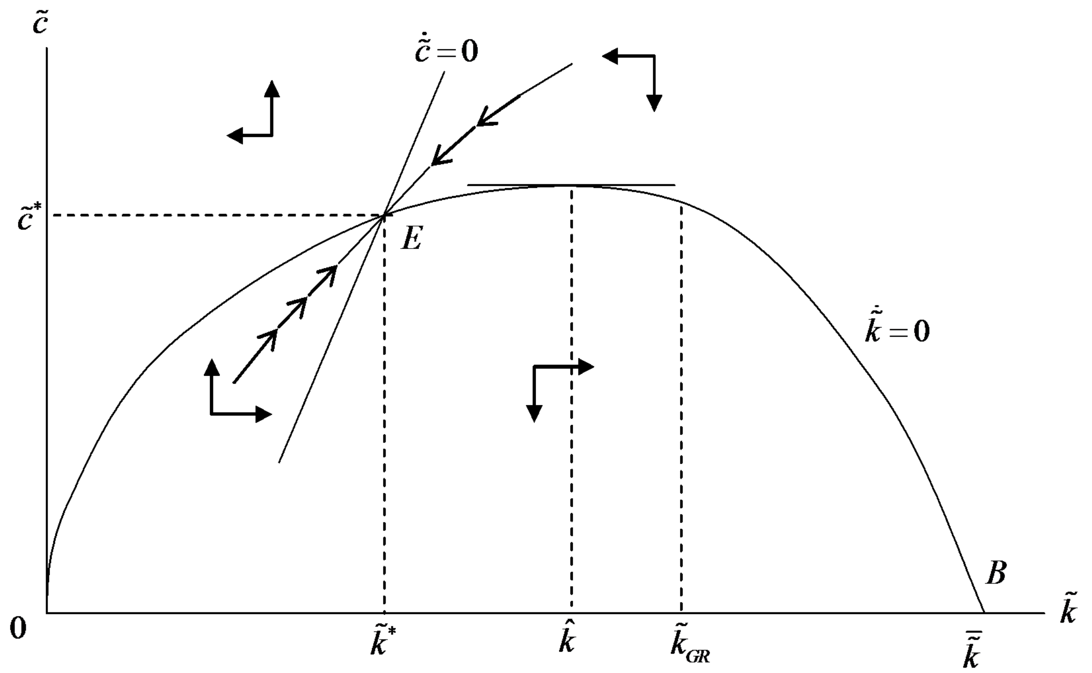

Figure 11.5: Phase diagram for the Arrow model.

Thence, in a steady state

$$g_c = \frac{\dot{C}}{C} - n = \frac{n}{1-\lambda} - n = \frac{\lambda n}{1-\lambda} \equiv g_c^*, \quad \text{and} \tag{11.38}$$

$$\frac{\dot{\mathcal{T}}}{\mathcal{T}} = \lambda \frac{\dot{K}}{K} = \frac{\lambda n}{1-\lambda} = g_c^*. \tag{11.39}$$

The steady-state values of r and $\tilde{k}$, respectively, will therefore satisfy, by (11.34),

$$r^* = f'(\tilde{k}^*) - \delta = \rho + \theta g_c^* = \rho + \theta \frac{\lambda n}{1-\lambda}. \tag{11.40}$$

To ensure existence of a steady state we assume that the private marginal productivity of capital is sufficiently sensitive to capital per unit of effective labor, from now called the "capital intensity":

$$\lim_{\tilde{k}\to 0} f'(\tilde{k}) > \delta + \rho + \theta \frac{\lambda n}{1-\lambda} > \lim_{\tilde{k}\to\infty} f'(\tilde{k}). \tag{A1}$$

The transversality condition of the representative household is that $\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n) ds} = 0$, where a_t is per capita financial wealth. In general equilibrium

$a_t = k_t \equiv \tilde{k}_t \mathcal{T}_t$, where $\mathcal{T}_t$ in steady state grows according to (11.39). Thus, in steady state the transversality condition can be written

$$\lim_{t\to\infty} \tilde{k}^* e^{(g_c^* - r^* + n)t} = 0. \tag{TVC}$$

For this to hold, we need

$$r^* > g_c^* + n = \frac{n}{1-\lambda}, \tag{11.41}$$

by (11.38). In view of (11.40), this is equivalent to

$$\rho - n > (1-\theta)\frac{\lambda n}{1-\lambda}, \tag{A2}$$

which we assume satisfied.

As to the slope of the $\dot{\tilde{c}} = 0$ locus we have from (11.37),

$$c'(\tilde{k}) = f'(\tilde{k}) - \delta - \frac{1}{\lambda}(\tilde{k}\frac{f''(\tilde{k})}{\theta} + g_c) > f'(\tilde{k}) - \delta - \frac{1}{\lambda}g_c, \tag{11.42}$$

since $f'' < 0$. At least in a small neighborhood of the steady state we can sign the right-hand side of this expression. Indeed,

$$f'(\tilde{k}^*) - \delta - \frac{1}{\lambda}g_c^* = \rho + \theta g_c^* - \frac{1}{\lambda}g_c^* = \rho + \theta\frac{\lambda n}{1-\lambda} - \frac{n}{1-\lambda} = \rho - n - (1-\theta)\frac{\lambda n}{1-\lambda} > 0, \tag{11.43}$$

by (11.38) and (A2). So, combining with (11.42), we conclude that $c'(\tilde{k}^*) > 0$. By continuity, in a small neighborhood of the steady state, $c'(\tilde{k}) \approx c'(\tilde{k}^*) > 0$. Therefore, close to the steady state, the $\dot{\tilde{c}} = 0$ locus is positively sloped, as indicated in Fig. 11.5.

Still, we have to check the following question: In a neighborhood of the steady state, which is steeper, the $\dot{\tilde{c}} = 0$ locus or the $\dot{\tilde{k}} = 0$ locus? The slope of the latter is $f'(\tilde{k}) - \delta - n/(1-\lambda)$, from (11.36). At the steady state this slope is

$$f'(\tilde{k}^*) - \delta - \frac{1}{\lambda}g_c^* \in (0, c'(\tilde{k}^*)),$$

in view of (11.43) and (11.42). The $\dot{\tilde{c}} = 0$ locus is thus steeper. So, the $\dot{\tilde{c}} = 0$ locus crosses the $\dot{\tilde{k}} = 0$ locus from below and can only cross once.

The assumption (A1) ensures existence of a $\tilde{k}^* > 0$ satisfying (11.40). As Fig. 11.5 is drawn, a little more is implicitly assumed namely that there exists a

$\hat{k} > 0$ such that the *private* net marginal productivity of capital equals the the steady-state growth rate of output, i.e.,

$$f'(\hat{k}) - \delta = (\frac{\dot{Y}}{Y})^* = (\frac{\dot{\mathcal{T}}}{\mathcal{T}})^* + \frac{\dot{L}}{L} = \frac{\lambda n}{1-\lambda} + n = \frac{n}{1-\lambda}, \tag{11.44}$$

where we have used (11.39). Thus, the tangent to the $\dot{\tilde{k}} = 0$ locus at $\tilde{k} = \hat{k}$ is horizontal and $\hat{k} > \tilde{k}^*$ as indicated in the figure.

Note, however, that $\hat{k}$ is not the golden-rule capital intensity. The latter is the capital intensity, $\tilde{k}_{GR}$, at which the *social* net marginal productivity of capital equals the steady-state growth rate of output (see Appendix). If $\tilde{k}_{GR}$ exists, it will be larger than $\hat{k}$ as indicated in Fig. 11.5. To see this, we now derive a convenient expression for the social marginal productivity of capital. From (11.30) we have

$$\begin{aligned}
\frac{\partial Y}{\partial K} &= F_1(\cdot) + F_2(\cdot)\lambda K^{\lambda-1}L = f'(\tilde{k}) + F_2(\cdot)K^{\lambda}L(\lambda K^{-1}) \quad \text{(by (11.31))} \\
&= f'(\tilde{k}) + (F(\cdot) - F_1(\cdot)K)\,\lambda K^{-1} \qquad \text{(by Euler's theorem)} \\
&= f'(\tilde{k}) + (f(\tilde{k})K^{\lambda}L - f'(\tilde{k})K)\lambda K^{-1} \qquad \text{(by (11.31) and (11.25))} \\
&= f'(\tilde{k}) + (f(\tilde{k})K^{\lambda-1}L - f'(\tilde{k}))\lambda = f'(\tilde{k}) + \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}} > f'(\tilde{k}).
\end{aligned}$$

in view of $\tilde{k} = K/(K^{\lambda}L) = K^{1-\lambda}L^{-1}$ and $f(\tilde{k})/\tilde{k} - f'(\tilde{k}) > 0$. As expected, the positive externality makes the social marginal productivity of capital larger than the private one. Since we can also write $\partial Y/\partial K = (1-\lambda)f'(\tilde{k}) + \lambda f(\tilde{k})/\tilde{k}$, we see that $\partial Y/\partial K$ is a decreasing function of $\tilde{k}$ (both $f'(\tilde{k})$ and $f(\tilde{k})/\tilde{k}$ are decreasing in $\tilde{k}$.

Now, the golden-rule capital intensity, $\tilde{k}_{GR}$, will be that capital intensity which satisfies

$$f'(\tilde{k}_{GR}) + \lambda\frac{f(\tilde{k}_{GR}) - \tilde{k}_{GR}f'(\tilde{k}_{GR})}{\tilde{k}_{GR}} - \delta = (\frac{\dot{Y}}{Y})^* = \frac{n}{1-\lambda}.$$

To ensure there exists such a $\tilde{k}_{GR}$, we strengthen the right-hand side inequality in (A1) by the assumption

$$\lim_{\tilde{k}\to\infty}\left(f'(\tilde{k}) + \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}}\right) < \delta + \frac{n}{1-\lambda}. \tag{A3}$$

This, together with (A1) and $f'' < 0$, implies existence of a unique $\tilde{k}_{GR}$, and in view of our additional assumption (A2), we have $0 < \tilde{k}^* < \hat{k} < \tilde{k}_{GR}$, as displayed in Fig. 11.5.

Stability The arrows in Fig. 11.5 indicate the direction of movement as determined by (11.33) and (11.35). We see that the steady state is a saddle point. The dynamic system has one pre-determined variable, $\tilde{k}$, and one jump variable, $\tilde{c}$. The saddle path is not parallel to the jump variable axis. We claim that for a given $\tilde{k}_0 > 0$, (i) the initial value of $\tilde{c}_0$ will be the ordinate to the point where the vertical line $\tilde{k} = \tilde{k}_0$ crosses the saddle path; (ii) over time the economy will move along the saddle path towards the steady state. Indeed, this time path is consistent with all conditions of general equilibrium, including the transversality condition (TVC). And the path is the *only* technically feasible path with this property. Indeed, all the divergent paths in Fig. 11.5 can be ruled out as equilibrium paths because they can be shown to violate the transversality condition of the household.

In the long run c and $y \equiv Y/L \equiv \tilde{y}\mathcal{T} = f(\tilde{k}^*)\mathcal{T}$ grow at the rate $\lambda n/(1-\lambda)$, which is positive if and only if $n > 0$. This is an example of *endogenous growth* in the sense that the positive long-run per capita growth rate is generated through an internal mechanism (learning) in the model (in contrast to exogenous technology growth as in the Ramsey model with exogenous technical progress).

Two types of endogenous growth

One may distinguish between two types of endogenous growth. One is called *fully endogenous* growth which occurs when the long-run growth rate of c is positive without the support by growth in any exogenous factor (for example exogenous growth in the labor force); the Romer case, to be considered in the next section, provides an example. The other type is called *semi-endogenous growth* and is present if growth is endogenous but a positive per capita growth rate can not be maintained in the long run without the support by growth in some exogenous factor (for example growth in the labor force). Clearly, in the Arrow model of learning by investing, growth is "only" semi-endogenous. The technical reason for this is the assumption that the learning parameter λ is below 1, which implies diminishing returns to capital at the aggregate level. If and only if $n > 0$, do we have $\dot{c}/c > 0$ in the long run.[11] In line with this, $\partial g_y^*/\partial n > 0$.

The key role of population growth derives from the fact that although there are diminishing marginal returns to capital at the aggregate level, there are increasing returns to scale w.r.t. capital *and* labor. For the increasing returns to be exploited, growth in the labor force is needed. To put it differently: when there are increasing returns to K and L together, growth in the labor force not only counterbalances the falling marginal productivity of aggregate capital (this

[11] Note, however, that the model, and therefore (11.38), presupposes $n \geq 0$. If $n < 0$, then K would tend to be decreasing and so, by (11.25), the level of technical knowledge would be decreasing, which is implausible, at least for a modern industrialized economy.

counter-balancing role reflects the complementarity between K and L), but also upholds sustained productivity growth.

Note that in the semi-endogenous growth case $\partial g_y^*/\partial\lambda = n/(1-\lambda)^2 > 0$ for $n > 0$. That is, a higher value of the learning parameter implies higher per capita growth in the long run, when $n > 0$. Note also that $\partial g_y^*/\partial\rho = 0 = \partial g_y^*/\partial\theta$, that is, in the semi-endogenous growth case preference parameters do not matter for long-run growth. As indicated by (11.38), the long-run growth rate is tied down by the learning parameter, λ, and the rate of population growth, n. But, like in the simple Ramsey model, it can be shown that preference parameters matter for the *level* of the growth path. This suggests that taxes and subsidies do not have long-run growth effects, but "only" *level* effects (see Exercise 11.??).

11.2.3 Romer's limiting case: $\lambda = 1$, $n = 0$

We now consider the limiting case $\lambda = 1$. We should think of it as a thought experiment because, by most observers, the value 1 is considered an unrealistically high value for the learning parameter. To avoid a forever rising growth rate we have to add the restriction $n = 0$.

The resulting model turns out to be extremely simple and at the same time it gives striking results (both circumstances have probably contributed to its popularity).

First, with $\lambda = 1$ we get $\mathcal{T} = K$ and so the equilibrium interest rate is, by (11.29),

$$r = F_1(k, K) - \delta = F_1(1, L) - \delta \equiv \bar{r},$$

where we have divided the two arguments of $F_1(k, K)$ by $k \equiv K/L$ and again used Euler's theorem. Note that the interest rate is constant "from the beginning" and independent of the historically given initial value of K, K_0. The aggregate production function is now

$$Y = F(K, KL) = F(1, L)K, \qquad L \text{ constant}, \tag{11.45}$$

and is thus *linear* in the aggregate capital stock. In this way the general neoclassical presumption of diminishing returns to capital has been suspended and replaced by exactly constant returns to capital. So the Romer model belongs to a class of models known as *AK models*, that is, models where in general equilibrium the interest rate and the output-capital ratio are necessarily constant over time whatever the initial conditions.

The method for analyzing an AK model is different from the one used for a diminishing returns model as above.

Dynamics

The Keynes-Ramsey rule now takes the form

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(\bar{r} - \rho) = \frac{1}{\theta}(F_1(1, L) - \delta - \rho) \equiv \gamma, \tag{11.46}$$

which is also constant "from the beginning". To ensure positive growth, we assume

$$F_1(1, L) - \delta > \rho. \tag{A1'}$$

And to ensure bounded intertemporal utility (and existence of equilibrium), it is assumed that

$$\rho > (1 - \theta)\gamma \text{ and therefore } \gamma < \theta\gamma + \rho = \bar{r}. \tag{A2'}$$

Solving the linear differential equation (11.46) gives

$$c_t = c_0 e^{\gamma t}, \tag{11.47}$$

where c_0 is unknown so far (because c is not a predetermined variable). We shall find c_0 by applying the households' transversality condition

$$\lim_{t\to\infty} a_t e^{-\bar{r}t} = \lim_{t\to\infty} k_t e^{-\bar{r}t} = 0. \tag{TVC}$$

First, note that the dynamic resource constraint for the economy is

$$\dot{K} = Y - cL - \delta K = F(1, L)K - cL - \delta K,$$

or, in per-capita terms,

$$\dot{k} = [F(1, L) - \delta]\, k - c_0 e^{\gamma t}. \tag{11.48}$$

In this equation it is important that $F(1, L) - \delta - \gamma > 0$. To understand this inequality, note that, by (A2'), $F(1, L) - \delta - \gamma > F(1, L) - \delta - \bar{r} = F(1, L) - F_1(1, L)$ $= F_2(1, L)L > 0$, where the first equality is due to $\bar{r} = F_1(1, L) - \delta$ and the second is due to the fact that since F is homogeneous of degree 1, we have, by Euler's theorem, $F(1, L) = F_1(1, L) \cdot 1 + F_2(1, L)L > F_1(1, L) > \delta$, in view of (A1'). The key property $F(1, L) - F_1(1, L) > 0$ is illustrated in Fig. 11.6.

The solution of a linear differential equation of the form $\dot{x}(t) + ax(t) = ce^{ht}$, with $h \neq -a$, is

$$x(t) = (x(0) - \frac{c}{a + h})e^{-at} + \frac{c}{a + h}e^{ht}. \tag{11.49}$$

Thus the solution to (11.48) is

$$k_t = (k_0 - \frac{c_0}{F(1, L) - \delta - \gamma})e^{(F(1,L)-\delta)t} + \frac{c_0}{F(1, L) - \delta - \gamma}e^{\gamma t}. \tag{11.50}$$

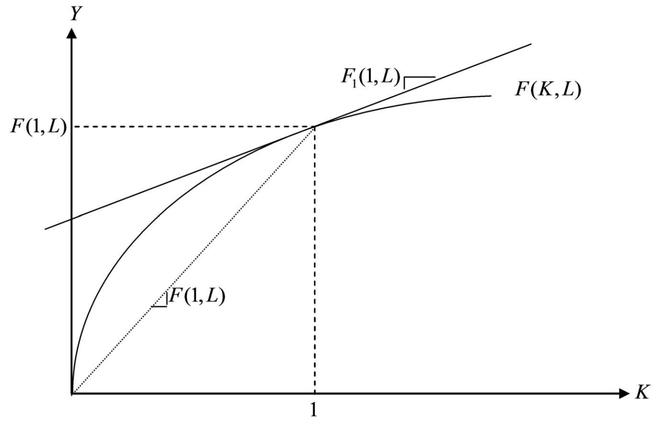

Figure 11.6: Illustration of the fact that for L given, $F(1,L) > F_1(1,L)$.

To check whether (TVC) is satisfied we consider

$$\begin{aligned} k_t e^{-\bar{r}t} &= (k_0 - \frac{c_0}{F(1,L)-\delta-\gamma})e^{(F(1,L)-\delta-\bar{r})t} + \frac{c_0}{F(1,L)-\delta-\gamma}e^{(\gamma-\bar{r})t} \\ &\rightarrow (k_0 - \frac{c_0}{F(1,L)-\delta-\gamma})e^{(F(1,L)-\delta-\bar{r})t} \text{ for } t \rightarrow \infty, \end{aligned}$$

since $\bar{r} > \gamma$, by (A2'). But $\bar{r} = F_1(1,L) - \delta < F(1,L) - \delta$, and so (TVC) is only satisfied if

$$c_0 = (F(1,L) - \delta - \gamma)k_0. \tag{11.51}$$

If c_0 is less than this, there will be over-saving and (TVC) is violated ($a_t e^{-\bar{r}t} \rightarrow \infty$ for $t \rightarrow \infty$, since $a_t = k_t$). If c_0 is higher than this, both the NPG and (TVC) are violated ($a_t e^{-\bar{r}t} \rightarrow -\infty$ for $t \rightarrow \infty$).

Inserting the solution for c_0 into (11.50), we get

$$k_t = \frac{c_0}{F(1,L)-\delta-\gamma}e^{\gamma t} = k_0 e^{\gamma t},$$

that is, k grows at the same constant rate as c "from the beginning". Since $y \equiv Y/L = F(1,L)k$, the same is true for y. Hence, from start the system is in balanced growth (there is no transitional dynamics).

This is a case of *fully endogenous growth* in the sense that the long-run growth rate of c is positive without the support by growth in any exogenous factor. This outcome is due to the absence of diminishing returns to aggregate capital, which is implied by the assumed high value of the learning parameter. The empirical foundation for being in a neighborhood of this high value is weak, however, cf. Literature notes. A further problem with this special version of the learning model is that the results are *non-robust*. With λ slightly less than 1, we are back in the Arrow case and growth peters out, since $n = 0$. With λ slightly above 1, it can be shown that growth becomes explosive (infinite output in finite time).[12]

The Romer case, $\lambda = 1$, is thus a *knife-edge* case in a double sense. First, it imposes a particular value for a parameter which *apriori* can take any value within an interval. Second, the imposed value leads to theoretically non-robust results; values in a hair's breadth distance result in qualitatively different behavior of the dynamic system. Still, whether the Romer case - or, more generally, a fully-endogenous growth case - can be used as an empirical approximation to its semi-endogenous "counterpart" for a sufficiently long time horizon to be of interest, is a debated question within growth analysis.

It is noteworthy that the *causal structure* in the long run in the diminishing returns case is different than in the AK-case of Romer. In the diminishing returns case the steady-state growth rate is determined first, as g_c^* in (11.38), and then r^* is determined through the Keynes-Ramsey rule; finally, Y/K is determined by the technology, given r^*. In contrast, the Romer case has Y/K and r directly given as $F(1, L)$ and $\bar{r}$, respectively. In turn, $\bar{r}$ determines the (constant) equilibrium growth rate through the Keynes-Ramsey rule.

Economic policy in the Romer case

In the AK case, that is, the fully endogenous growth case, we have $\partial\gamma/\partial\rho < 0$ and $\partial\gamma/\partial\theta < 0$. Thus, preference parameters *matter* for the long-run growth rate and not "only" for the *level* of the growth path. This suggests that taxes and subsidies can have *long-run* growth effects. In any case, in this model there is a motivation for government intervention due to the positive externality of private investment. This motivation is present whether $\lambda < 1$ or $\lambda = 1$. Here we concentrate on the latter case, which is the simpler one. We first find the social planner's solution.

The social planner The social planner faces the aggregate production function $Y_t = F(1, L)K_t$ or, in per capita terms, $y_t = F(1, L)k_t$. The social planner's problem is to choose $(c_t)_{=0}^{\infty}$ to maximize

$$\begin{aligned} U_0 &= \int_0^\infty \frac{c_t^{1-\theta}}{1-\theta} e^{-\rho t} dt \quad \text{s.t.} \\ c_t &\geq 0, \\ \dot{k}_t &= F(1, L)k_t - c_t - \delta k_t, \qquad k_0 > 0 \text{ given}, & (11.52) \\ k_t &\geq 0 \text{ for all } t > 0. & (11.53) \end{aligned}$$

[12] See Solow (1997).

The current-value Hamiltonian is

$$H(k, c, \eta, t) = \frac{c^{1-\theta}}{1-\theta} + \eta \left(F(1, L)k - c - \delta k \right),$$

where $\eta = \eta_t$ is the adjoint variable associated with the state variable, which is capital per unit of labor. Necessary first-order conditions for an interior optimal solution are

$$\frac{\partial H}{\partial c} = c^{-\theta} - \eta = 0, \text{ i.e., } c^{-\theta} = \eta, \tag{11.54}$$

$$\frac{\partial H}{\partial k} = \eta(F(1, L) - \delta) = -\dot{\eta} + \rho\eta. \tag{11.55}$$

We guess that also the transversality condition,

$$\lim_{t\to\infty} k_t \eta_t e^{-\rho t} = 0, \tag{11.56}$$

must be satisfied by an optimal solution. This guess will be of help in finding a candidate solution. Having found a candidate solution, we shall invoke a theorem on *sufficient* conditions to ensure that our candidate solution *is* really a solution.

Log-differentiating w.r.t. t in (11.54) and combining with (11.55) gives the social planner's Keynes-Ramsey rule,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(F(1, L) - \delta - \rho) \equiv \gamma_{SP}. \tag{11.57}$$

We see that $\gamma_{SP} > \gamma$. This is because the social planner internalizes the economy-wide learning effect associated with capital investment, that is, the social planner takes into account that the "social" marginal productivity of capital is $\partial y_t/\partial k_t = F(1, L) > F_1(1, L)$. To ensure bounded intertemporal utility we sharpen (A2') to

$$\rho > (1 - \theta)\gamma_{SP}. \tag{A2''}$$

To find the time path of k_t, note that the dynamic resource constraint (11.52) can be written

$$\dot{k}_t = (F(1, L) - \delta)k_t - c_0 e^{\gamma_{SP} t},$$

in view of (11.57). By the general solution formula (11.49) this has the solution

$$k_t = (k_0 - \frac{c_0}{F(1, L) - \delta - \gamma_{SP}})e^{(F(1,L)-\delta)t} + \frac{c_0}{F(1, L) - \delta - \gamma_{SP}} e^{\gamma_{SP} t}. \tag{11.58}$$

In view of (11.55), in an interior optimal solution the time path of the adjoint variable η is

$$\eta_t = \eta_0 e^{-[(F(1,L)-\delta-\rho]t},$$

where $\eta_0 = c_0^{-\theta} > 0$, by (11.54). Thus, the conjectured transversality condition (11.56) implies

$$\lim_{t\to\infty} k_t e^{-(F(1,L)-\delta)t} = 0, \tag{11.59}$$

where we have eliminated η_0. To ensure that this is satisfied, we multiply k_t from (11.58) by $e^{-(F(1,L)-\delta)t}$ to get

$$\begin{aligned} k_t e^{-(F(1,L)-\delta)t} &= k_0 - \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} + \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} e^{[\gamma_{SP}-(F(1,L)-\delta)]t} \\ &\to k_0 - \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} \text{ for } t\to\infty, \end{aligned}$$

since, by (A2"), $\gamma_{SP} < \rho + \theta\gamma_{SP} = F(1,L) - \delta$ in view of (11.57). Thus, (11.59) is only satisfied if

$$c_0 = (F(1,L) - \delta - \gamma_{SP})k_0. \tag{11.60}$$

Inserting this solution for c_0 into (11.58), we get

$$k_t = \frac{c_0}{F(1,L)-\delta-\gamma_{SP}} e^{\gamma_{SP}t} = k_0 e^{\gamma_{SP}t},$$

that is, k grows at the same constant rate as c "from the beginning". Since $y \equiv Y/L = F(1,L)k$, the same is true for y. Hence, our candidate for the social planner's solution is from start in balanced growth (there is no transitional dynamics).

The next step is to check whether our candidate solution satisfies a set of *sufficient* conditions for an optimal solution. Here we can use *Mangasarian's theorem*. Applied to a continuous-time optimization problem like this, with one control variable and one state variable, the theorem says that the following conditions are sufficient:

(a) Concavity: For all $t \geq 0$ the Hamiltonian is jointly concave in the control and state variables, here c and k.

(b) Non-negativity: There is for all $t \geq 0$ a non-negativity constraint on the state variable; in addition, the co-state variable, η, is non-negative for all $t \geq 0$ along the optimal path.

(c) TVC: The candidate solution satisfies the transversality condition $\lim_{t\to\infty} k_t\eta_t e^{-\rho t} = 0$, where $\eta_t e^{-\rho t}$ is the discounted co-state variable.

In the present case we see that the Hamiltonian is a sum of concave functions and therefore is itself concave in (k, c). Further, from (11.53) we see that condition (b) is satisfied. Finally, our candidate solution is constructed so as to satisfy condition (c). The conclusion is that our candidate solution *is* an optimal solution. We call it an *SP allocation.*

Implementing the SP allocation in the market economy Returning to the competitive market economy, we assume there is a policy maker, the government, with only two activities. These are (i) paying an investment subsidy, s, to the firms so that their capital costs are reduced to

$$(1-s)(r+\delta)$$

per unit of capital per time unit; (ii) financing this subsidy by a constant consumption tax rate τ.

Let us first find the size of s needed to establish the SP allocation. Firm i now chooses K_i such that

$$\frac{\partial Y_i}{\partial K_i}|_{K \text{ fixed}} = F_1(K_i, KL_i) = (1-s)(r+\delta).$$

By Euler's theorem this implies

$$F_1(k_i, K) = (1-s)(r+\delta) \quad \text{for all } i,$$

so that in equilibrium we must have

$$F_1(k, K) = (1-s)(r+\delta),$$

where $k \equiv K/L$, which is pre-determined from the supply side. Thus, the equilibrium interest rate must satisfy

$$r = \frac{F_1(k,K)}{1-s} - \delta = \frac{F_1(1,L)}{1-s} - \delta, \tag{11.61}$$

again using Euler's theorem.

It follows that s should be chosen such that the "right" r arises. What is the "right" r? It is that net rate of return which is implied by the production technology at the aggregate level, namely $\partial Y/\partial K - \delta = F(1,L) - \delta$. If we can obtain $r = F(1,L) - \delta$, then there is no wedge between the intertemporal rate of transformation faced by the consumer and that implied by the technology. The required s thus satisfies

$$r = \frac{F_1(1,L)}{1-s} - \delta = F(1,L) - \delta,$$

so that

$$s = 1 - \frac{F_1(1,L)}{F(1,L)} = \frac{F(1,L) - F_1(1,L)}{F(1,L)} = \frac{F_2(1,L)L}{F(1,L)}.$$

It remains to find the required consumption tax rate τ. The tax revenue will be τcL, and the *required* tax revenue is

$$\mathcal{T} = s(r + \delta)K = (F(1, L) - F_1(1, L))\,K = \tau cL.$$

Thus, with a balanced budget the required tax rate is

$$\tau = \frac{\mathcal{T}}{cL} = \frac{F(1, L) - F_1(1, L)}{c/k} = \frac{F(1, L) - F_1(1, L)}{F(1, L) - \delta - \gamma_{SP}} > 0, \tag{11.62}$$

where we have used that the proportionality in (11.60) between c and k holds for all $t \geq 0$. Substituting (11.57) into (11.62), the solution for τ can be written

$$\tau = \frac{\theta\,[F(1, L) - F_1(1, L)]}{(\theta - 1)(F(1, L) - \delta) + \rho} = \frac{\theta F_2(1, L)L}{(\theta - 1)(F(1, L) - \delta) + \rho}.$$

The required tax rate on consumption is thus a constant. It therefore does not distort the consumption/saving decision on the margin, cf. Appendix B.

It follows that the allocation obtained by this subsidy-tax policy *is* the SP allocation. A policy, here the policy (s, τ), which in a decentralized system induces the SP allocation, is called a *first-best policy.* In a situation where for some reason it is impossible to obtain an SP allocation in a decentralized way (because of adverse selection and moral hazard problems, say), a government's optimization problem would involve additional constraints to those given by technology and initial resources. A decentralized implementation of the solution to such a problem is called a *second-best policy.*

11.3 Concluding remarks

(not yet available)

11.4 Literature notes

(incomplete)

As to empirical evidence of learning-by-doing and learning-by-investing, see ...

As noted in Section 11.2.1, the citation of Arrow indicates that it was experience from cumulative *gross* investment, rather than net investment, he had in mind as the basis for learning. Yet the hypothesis in (11.25) is the more popular one - seemingly for no better reason than that it leads to simpler dynamics.

Another way in which (11.25) deviates from Arrow's original ideas is by assuming that technical progress is disembodied rather than embodied, a distinction we touched upon in Chapter 2. Moreover, we have assumed a neoclassical technology whereas Arrow assumed fixed technical coefficients.

11.5 Appendix

A. The golden-rule capital intensity in Arrow's growth model

In our discussion of Arrow's learning-by-investing model in Section 11.2.2 (where $0 < \lambda < 1$), we claimed that the golden-rule capital intensity, $\tilde{k}_{GR}$, will be that effective capital-labor ratio at which the social net marginal productivity of capital equals the steady-state growth rate of output. In this respect the Arrow model with endogenous technical progress is similar to the standard neoclassical growth model with exogenous technical progress. This claim corresponds to a very general theorem, valid also for models with many capital goods and non-existence of an aggregate production function. This theorem says that the highest sustainable path for consumption per unit of labor in the economy will be that path which results from those techniques which profit maximizing firms choose under perfect competition when the real interest rate equals the steady-state growth rate of GNP (see Gale and Rockwell, 1975).

To prove our claim, note that in steady state, (11.37) holds whereby consumption per unit of labor (here the same as per capita consumption as L = labor force = population) can be written

$$\begin{aligned} c_t &\equiv \tilde{c}_t \mathcal{T}_t = \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k} \right] K_t^{\lambda} \\ &= \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k} \right] \left(K_0 e^{\frac{n}{1-\lambda}t} \right)^{\lambda} \quad (\text{by } g_K^* = \frac{n}{1-\lambda}) \\ &= \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k} \right] \left((\tilde{k}L_0)^{\frac{1}{1-\lambda}} e^{\frac{n}{1-\lambda}t} \right)^{\lambda} \quad (\text{from } \tilde{k} = \frac{K_t}{K_t^{\lambda} L_t} = \frac{K_0^{1-\lambda}}{L_0}) \\ &= \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k} \right] \tilde{k}^{\frac{\lambda}{1-\lambda}} L_0^{\frac{\lambda}{1-\lambda}} e^{\frac{\lambda n}{1-\lambda}t} \equiv \varphi(\tilde{k}) L_0^{\frac{\lambda}{1-\lambda}} e^{\frac{\lambda n}{1-\lambda}t}, \end{aligned}$$

defining $\varphi(\tilde{k})$ in the obvious way.

We look for that value of $\tilde{k}$ at which this steady-state path for c_t is at the highest technically feasible level. The positive coefficient, $L_0^{\frac{\lambda}{1-\lambda}} e^{\frac{\lambda n}{1-\lambda}t}$, is the only time dependent factor and can be ignored since it is exogenous. The problem is thereby reduced to the static problem of maximizing $\varphi(\tilde{k})$ with respect to $\tilde{k} > 0$.

We find

$$\begin{aligned}
\varphi'(\tilde{k}) &= \left[f'(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\right]\tilde{k}^{\frac{\lambda}{1-\lambda}} + \left[f(\tilde{k}) - (\delta + \frac{n}{1-\lambda})\tilde{k}\right]\frac{\lambda}{1-\lambda}\tilde{k}^{\frac{\lambda}{1-\lambda}-1} \\
&= \left[f'(\tilde{k}) - (\delta + \frac{n}{1-\lambda}) + \left(\frac{f(\tilde{k})}{\tilde{k}} - (\delta + \frac{n}{1-\lambda})\right)\frac{\lambda}{1-\lambda}\right]\tilde{k}^{\frac{\lambda}{1-\lambda}} \\
&= \left[(1-\lambda)f'(\tilde{k}) - (1-\lambda)\delta - n + \lambda\frac{f(\tilde{k})}{\tilde{k}} - \lambda(\delta + \frac{n}{1-\lambda})\right]\frac{\tilde{k}^{\frac{\lambda}{1-\lambda}}}{1-\lambda} \\
&= \left[(1-\lambda)f'(\tilde{k}) - \delta + \lambda\frac{f(\tilde{k})}{\tilde{k}} - \frac{n}{1-\lambda}\right]\frac{\tilde{k}^{\frac{\lambda}{1-\lambda}}}{1-\lambda} \equiv \psi(\tilde{k})\frac{\tilde{k}^{\frac{\lambda}{1-\lambda}}}{1-\lambda}, \qquad (11.63)
\end{aligned}$$

defining $\psi(\tilde{k})$ in the obvious way. The first-order condition for the problem, $\varphi'(\tilde{k}) = 0$, is equivalent to $\psi(\tilde{k}) = 0$. After ordering this gives

$$f'(\tilde{k}) + \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}} - \delta = \frac{n}{1-\lambda}. \qquad (11.64)$$

We see that

$$\varphi'(\tilde{k}) \gtreqless 0 \quad \text{for} \quad \psi(\tilde{k}) \gtreqless 0,$$

respectively. Moreover,

$$\psi'(\tilde{k}) = (1-\lambda)f''(\tilde{k}) - \lambda\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{k}^2} < 0,$$

in view of $f'' < 0$ and $f(\tilde{k})/\tilde{k} > f'(\tilde{k})$. So a $\tilde{k} > 0$ satisfying $\psi(\tilde{k}) = 0$ is the unique maximizer of $\varphi(\tilde{k})$. By (A1) and (A3) in Section 11.2.2 such a $\tilde{k}$ exists and is thereby the same as the $\tilde{k}_{GR}$ we were looking for.

The left-hand side of (11.64) equals the social marginal productivity of capital and the right-hand side equals the steady-state growth rate of output. At $\tilde{k} = \tilde{k}_{GR}$ it therefore holds that

$$\frac{\partial Y}{\partial K} - \delta = \left(\frac{\dot{Y}}{Y}\right)^*.$$

This confirms our claim in Section 11.2.2 about $\tilde{k}_{GR}$.

Remark about the absence of a golden rule in the Romer case. In the Romer case the golden rule is not a well-defined concept for the following reason. Along any balanced growth path we have from (11.52),

$$g_k \equiv \frac{\dot{k}_t}{k_t} = F(1, L) - \delta - \frac{c_t}{k_t} = F(1, L) - \delta - \frac{c_0}{k_0},$$

because g_k ($= g_K$) is by definition constant along a balanced growth path, whereby also c_t/k_t must be constant. We see that g_k is decreasing linearly from $F(1, L)-\delta$ to $-\delta$ when c_0/k_0 rises from nil to $F(1, L)$. So choosing among alternative technically feasible balanced growth paths is inevitably a choice between starting with low consumption to get high growth forever or starting with high consumption to get low growth forever. Given any $k_0 > 0$, the alternative possible balanced growth paths will therefore sooner or later cross each other in the $(t, \ln c)$ plane. Hence, for the given k_0, there exists no balanced growth path which for all $t \geq 0$ has c_t higher than along any other technically feasible balanced growth path.

B. Consumption taxation

Is a consumption tax distortionary - always? never? sometimes?

The answer is the following.

1. Suppose labor supply is *elastic* (due to leisure entering the utility function). Then a consumption tax (whether constant or time-dependent) is generally distortionary (not neutral). This is because it reduces the effective opportunity cost of leisure by reducing the amount of consumption forgone by working one hour less. Indeed, the tax makes consumption goods more expensive and so the amount of consumption that the agent can buy for the hourly wage becomes smaller. The substitution effect on leisure of a consumption tax is thus positive, while the income and wealth effects will be negative. Generally, the net effect will not be zero, but can be of any sign; it may be small in absolute terms.

2. Suppose labor supply is *inelastic* (no trade-off between consumption and leisure). Then, at least in the type of growth models we consider in this course, a constant (time-independent) consumption tax acts as a lump-sum tax and is thus non-distortionary. If the consumption tax is *time-dependent*, however, a distortion of the *intertemporal* aspect of household decisions tends to arise.

To understand answer 2, consider a Ramsey household with inelastic labor supply. Suppose the household faces a time-varying consumption tax rate $\tau_t > 0$. To obtain a consumption level per time unit equal to c_t per capita, the household has to spend

$$\bar{c}_t = (1 + \tau_t)c_t$$

units of account (in real terms) per capita. Thus, spending $\bar{c}_t$ per capita per time unit results in the per capita consumption level

$$c_t = (1 + \tau_t)^{-1}\bar{c}_t. \tag{11.65}$$

In order to concentrate on the consumption tax as such, we assume the tax revenue is simply given back as lump-sum transfers and that there are no other government activities. Then, with a balanced government budget, we have

$$x_t L_t = \tau_t c_t L_t,$$

where x_t is the per capita lump-sum transfer, exogenous to the household, and L_t is the size of the representative household.

Assuming CRRA utility with parameter $\theta > 0$, the instantaneous per capita utility can be written

$$u(c_t) = \frac{c_t^{1-\theta}}{1-\theta} = \frac{(1+\tau_t)^{\theta-1}\bar{c}_t^{1-\theta}}{1-\theta}.$$

In our standard notation the household's intertemporal optimization problem is then to choose $(\bar{c}_t)_{t=0}^{\infty}$ so as to maximize

$$\begin{aligned} U_0 &= \int_0^{\infty} \frac{(1+\tau_t)^{\theta-1}\bar{c}_t^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt \quad \text{s.t.} \\ \bar{c}_t &\geq 0, \\ \dot{a}_t &= (r_t - n)a_t + w_t + x_t - \bar{c}_t, \quad a_0 \text{ given}, \\ \lim_{t\to\infty} a_t e^{-\int_0^{\infty}(r_s-n)ds} &\geq 0. \end{aligned}$$

From now, we let the timing of the variables be implicit unless needed for clarity. The current-value Hamiltonian is

$$H = \frac{(1+\tau)^{\theta-1}\bar{c}^{1-\theta}}{1-\theta} + \lambda\left[(r-n)a + w + x - \bar{c}\right],$$

where λ is the co-state variable associated with financial per capita wealth, a. An interior optimal solution will satisfy the first-order conditions

$$\frac{\partial H}{\partial \bar{c}} = (1+\tau)^{\theta-1}\bar{c}^{-\theta} - \lambda = 0, \text{ so that } (1+\tau)^{\theta-1}\bar{c}^{-\theta} = \lambda, \quad \text{(FOC1)}$$

$$\frac{\partial H}{\partial a} = \lambda(r-n) = -\dot{\lambda} + (\rho - n)\lambda, \quad \text{(FOC2)}$$

and a transversality condition which amounts to

$$\lim_{t\to\infty} a_t e^{-\int_0^{\infty}(r_s-n)ds} = 0. \quad \text{(TVC)}$$

We take logs in (FOC1) to get

$$(\theta - 1)\log(1+\tau) - \theta \log \bar{c} = \log \lambda.$$

Differentiating w.r.t. time, taking into account that $\tau = \tau_t$, gives

$$(\theta-1)\frac{\dot{\tau}}{1+\tau} - \theta\frac{\dot{\bar{c}}}{\bar{c}} = \frac{\dot{\lambda}}{\lambda} = \rho - r.$$

By ordering, we find the growth rate of consumption spending,

$$\frac{\dot{\bar{c}}}{\bar{c}} = \frac{1}{\theta}\left[r + (\theta - 1)\frac{\dot{\tau}}{1+\tau} - \rho\right].$$

Using (11.65), this gives the growth rate of consumption,

$$\frac{\dot{c}}{c} = \frac{\dot{\bar{c}}}{\bar{c}} - \frac{\dot{\tau}}{1+\tau} = \frac{1}{\theta}\left[r + (\theta - 1)\frac{\dot{\tau}}{1+\tau} - \rho\right] - \frac{\dot{\tau}}{1+\tau} = \frac{1}{\theta}(r - \frac{\dot{\tau}}{1+\tau} - \rho).$$

Assuming firms maximize profit under perfect competition, in equilibrium the real interest rate will satisfy

$$r = \frac{\partial Y}{\partial K} - \delta. \tag{11.66}$$

But the *effective* real interest rate, $\hat{r}$, faced by the consuming household, is

$$\hat{r} = r - \frac{\dot{\tau}}{1+\tau} \lesseqgtr r \text{ for } \dot{\tau} \gtreqless 0,$$

respectively. If for example the consumption tax is increasing, then the effective real interest rate faced by the consumer is smaller than the market real interest rate, given in (11.66), because saving implies postponing consumption and future consumption is more expensive due to the higher consumption tax rate.

The conclusion is that a time-varying consumption tax rate is distortionary. It implies a wedge between the intertemporal rate of transformation faced by the consumer, reflected by $\hat{r}$, and the intertemporal rate of transformation offered by the technology of society, indicated by r in (11.66). On the other hand, *if* the consumption tax rate is constant, the consumption tax is non-distortionary when there is no utility from leisure.

A remark on tax smoothing

Outside steady state it is often so that maintaining constant tax rates is inconsistent with maintaining a balanced government budget. Is the implication of this that we should recommend the government to let tax rates be continually adjusted so as to maintain a forever balanced budget? No! As the above example as well as business cycle theory suggest, maintaining tax rates constant ("tax smoothing"), and thereby allowing government deficits and surpluses to arise, will generally make more sense. In itself, a budget deficit is not worrisome. It only becomes worrisome if it is not accompanied later by sufficient budget surpluses to avoid an exploding government debt/GDP ratio to arise. This requires that the tax rates taken together have a *level* which in the long run matches the level of government expenses.

11.6 Exercises

Chapter 14

Fixed capital investment and Tobin's q

The models considered so far (the OLG models as well as the representative agent models) have ignored capital adjustment costs. In the closed-economy version of the models aggregate investment is merely a reflection of aggregate saving and appears in a "passive" way as just the residual of national income after households have chosen their consumption. We can describe what is going on by telling a story in which firms just rent capital goods owned by the households and households save by purchasing additional capital goods. In these models only households solve intertemporal decision problems. Firms merely demand labor and capital services with a view to maximizing current profits. This may be a legitimate abstraction in some contexts within long-run analysis. In short- and medium-run analysis, however, the dynamics of fixed capital investment is important. So a more realistic approach is desirable.

In the real world the capital goods used by a production firm are usually owned by the firm itself rather than rented for single periods on rental markets. One reason for this is that capital goods are often firm-specific, designed and adapted to the firm in which they are an integrated part. The capital goods are therefore generally worth more to the user than to others.

Tobin's *q-theory of investment* (after the American Nobel laureate James Tobin, 1918-2002) is an attempt to model these features. In this theory,

(a) *firms* make the *investment decisions* and *install* the purchased capital goods in their own businesses with the aim of maximizing discounted expected earnings in the future;

(b) there are certain *adjustment costs* associated with this investment: before acquiring new capital goods there are planning and design costs, and along with the implementation of the investment decisions there are costs of installation of the new equipment, costs of reorganizing the plant, costs of retraining workers to operate the new machines etc.;

(c) the adjustment costs are *strictly convex* so that marginal adjustment costs are increasing in the level of investment − think of constructing a plant in a month rather than a year.

The strict convexity of adjustment costs is the crucial constituent of the theory. It is that element which assigns investment decisions an *active* role in the model. There will be both a well-defined saving decision and a well-defined investment decision, separate from each other. Households decide the saving, firms the physical capital investment; households accumulate financial assets, firms accumulate physical capital. As a result, in a closed economy the current and expected future interest rates have to adjust for aggregate demand for goods (consumption plus investment) to match aggregate supply of goods. The role of interest rate changes is no longer to clear a rental market for capital goods.

To fix the terminology, from now on the different adjustment costs associated with investment will be subsumed under the term *capital installation costs*. When faced with strictly convex installation costs, the optimizing firm has to take the *future* into account, that is, firms' forward-looking *expectations* become important. To smooth out the adjustment costs, the firm will adjust its capital stock only *gradually* when new information arises. From an analytical point of view, we thereby avoid the counter-factual implication from earlier chapters that the capital stock in a small open economy with perfect mobility of goods and financial capital is *instantaneously* adjusted when the interest rate in the world financial market changes. Moreover, sluggishness in investment is exactly what the data show. Some empirical studies conclude that only a third of the difference between the current and the "desired" capital stock tends to be covered within a year (Clark 1979).

The q-theory of investment constitutes one approach to the explanation of this sluggishness in investment. Under certain conditions, to be described below, the theory gives a remarkably simple operational macroeconomic investment function, in which the key variable explaining aggregate investment is the valuation of the firms by the stock market relative to the replacement value of the firms' physical capital. This link between asset markets and firms' aggregate investment is an appealing feature of Tobin's q-theory.

14.1 Convex capital installation costs

Let the technology of a single firm be given by

$$\tilde{Y} = F(K, L),$$

where $\tilde{Y}$, K, and L are "potential output" (to be explained), capital input, and labor input per time unit, respectively, while F is a concave neoclassical production function. So we allow decreasing as well as constant returns to scale (or a combination of locally CRS and locally DRS), whereas increasing returns to scale is ruled out. Until further notice technological change is ignored for simplicity. Time is continuous. The dating of the variables will not be explicit unless needed for clarity. The increase per time unit in the firm's capital stock is given by

$$\dot{K} = I - \delta K, \qquad \delta > 0, \tag{14.1}$$

where I is gross fixed capital investment per time unit and δ is the rate of wearing down of capital (physical capital depreciation). To fix ideas, we presume the realistic case with positive capital depreciation, but most of the results go through even for $\delta = 0$.

Let J denote the firm's capital installation costs (measured in units of output) per time unit. The installation costs imply that a part of the potential output, $\tilde{Y}$, is "used up" in transforming investment goods into installed capital (possibly simply forgone due to interruptions of production during the process of installation). Only $\tilde{Y} - J$ is output available for sale.

Assuming the price of investment goods is one (the same as that of output goods), then total investment outlay per time unit are $I + J$, i.e., the direct purchase price, $1 \cdot I$, plus the indirect cost, J, associated with installation. The q-theory of investment assumes that the installation cost is a strictly convex function of gross investment and a non-increasing function of the current capital stock. Thus,

$$J = G(I, K),$$

where the installation cost function, G, satisfies

$$G(0, K) = 0,\ G_I(0, K) = 0,\ G_{II}(I, K) > 0,\ \text{and}\ G_K(I, K) \leq 0 \tag{14.2}$$

for all K and all (I, K), respectively. For fixed $K = \bar{K}$ the graph is as shown in Fig. 14.1. Also negative gross investment, i.e., sell off of capital equipment, involves costs (for dismantling, reorganization etc.). Therefore $G_I < 0$ for $I < 0$. The important assumption is that $G_{II} > 0$ (strict convexity in I), implying that the marginal installation cost is increasing in the level of gross investment. If the firm wants to accomplish a given installation project in only half the time, then

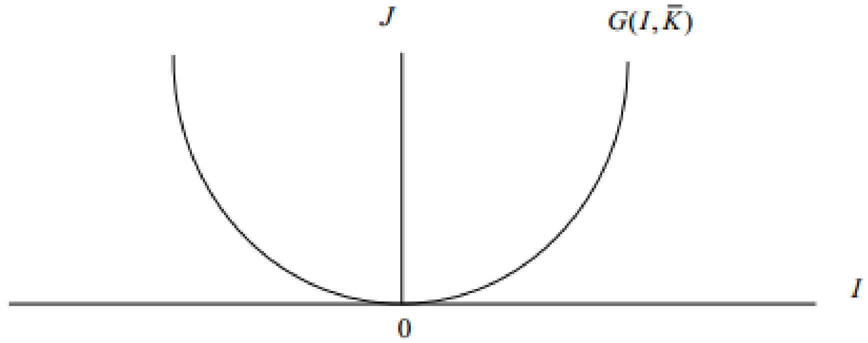

Figure 14.1: Installation costs as a function of gross investment when $K = \bar{K}$.

the installation costs are more than doubled (the risk of mistakes is larger, the problems with reorganizing work routines are larger etc.).

The strictly convex graph in Fig. 14.1 illustrates the essence of the matter. Assume the current capital stock in the firm is $\bar{K}$ and that the firm wants to increase it by a given amount $\overline{\Delta K}$. If the firm chooses the investment level $\bar{I} > 0$ per time unit in the time interval $[t, t + \Delta t)$, then, in view of (14.1), $\Delta K \approx (\bar{I} - \delta\bar{K})\Delta t$. So it takes $\Delta t \approx \overline{\Delta K}/(\bar{I} - \delta\bar{K})$ units of time to accomplish the desired increase $\overline{\Delta K}$. If, however, the firm slows down the adjustment and invests only half of $\bar{I}$ per time unit, then it takes approximately twice as long time to accomplish $\overline{\Delta K}$. Total costs of the two alternative courses of action are approximately $G(\bar{I}, \bar{K})\Delta t$ and $G(\frac{1}{2}\bar{I}, \bar{K})2\Delta t$, respectively (assuming, for simplicity, that $G_K(I, K) = 0$, and ignoring discounting). By drawing a few straight line segments in Fig. 14.1 the reader will be convinced that the last-mentioned cost is smaller than the first-mentioned due to strict convexity of installation costs (see Exercise 14.1). *Haste is waste.*

On the other hand, there are of course limits to how slow the adjustment to the desired capital stock should be. Slower adjustment means postponement of the potential benefits of a higher capital stock. So the firm faces a trade-off between fast adjustment to the desired capital stock and low adjustment costs.

In addition to the strict convexity of G with respect to I, (14.2) imposes the condition $G_K(I, K) \leq 0$. Indeed, it often seems realistic to assume that $G_K(I, K) < 0$ for $I \neq 0$. A given amount of investment may require more reorganization in a small firm than in a large firm (size here being measured by K). Owing to indivisibilities, when installing a new machine, a small firm has to stop production altogether, whereas a large firm can to some extent continue its production by shifting some workers to another production line. A further argument is that the more a firm has invested historically, the more experienced it is now concerning how to avoid large adjustment costs. So, for a given I today, the associated installation costs are lower, given a larger accumulated K.

14.1.1 The decision problem of the firm

In the absence of tax distortions, asymmetric information, and problems with enforceability of financial contracts, the Modigliani-Miller theorem (Modigliani and Miller, 1958) entails that the financial structure of the firm is both indeterminate and irrelevant for production decisions (see Appendix A). Although the conditions required for validity of this theorem are quite idealized, the q-theory of investment accepts them because they allow the analyst to concentrate on the production aspects in a first approach.

With the output good as unit of account, let the operating cash flow (the net payment stream to the firm before interest payments on debt, if any) at time t be denoted R_t (for "receipts"). Then

$$R_t \equiv F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t, \tag{14.3}$$

where the wage rate at time t is denoted w_t, and the market price of the investment good in terms of the output good is and remains 1. As mentioned, the installation cost $G(I_t, K_t)$ implies that a part of production, $F(K_t, L_t)$, is used up in transforming investment goods into installed capital. Only the difference $F(K_t, L_t) - G(I_t, K_t)$ is available for sale.

We ignore uncertainty and assume the firm is a price taker. The interest rate is r_t, which we assume to be positive, at least in the long run. The decision problem, as seen from time 0, is to choose a plan $(L_t, I_t)_{t=0}^{\infty}$ so as to maximize the firm's *market value*, i.e., the present value of the future stream of expected cash flows:

$$\max_{(L_t,I_t)_{t=0}^{\infty}} V_0 = \int_0^{\infty} R_t e^{-\int_0^t r_s ds} dt \quad \text{s.t. (14.3) and} \tag{14.4}$$

$$L_t \geq 0, I_t \text{ free (i.e., no restriction on } I_t), \tag{14.5}$$

$$\dot{K}_t = I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \tag{14.6}$$

$$K_t \geq 0 \text{ for all } t. \tag{14.7}$$

There is no specific terminal condition but we have posited the feasibility condition (14.7) saying that the firm can never have a negative capital stock.[1]

[1] It is assumed that w_t is a piecewise continuous function. At points of discontinuity (if any) in investment, we will consider investment to be a *right-continuous* function of time. That is, $I_{t_0} = \lim_{t \to t_0^+} I_t$. Likewise, at such points of discontinuity, by the "time derivative" of the corresponding state variable, K, we mean the *right-hand* time derivative, i.e., $\dot{K}_{t_0} = \lim_{t \to t_0^+}(K_t - K_{t_0})/(t - t_0)$. Mathematically, these conventions are inconsequential, but they help the intuition.

In the previous chapters the firm was described as solving a series of static profit maximization problems. Such a description is no longer valid, however, when there is dependence across time, as is the case here. When installation costs are present, current decisions depend on the expected future circumstances. The firm makes a plan for the whole future so as to maximize the value of the firm, which is what matters for the owners. This is the general neoclassical hypothesis about firms' behavior. As shown in Appendix A, when strictly convex installation costs or similar dependencies across time are absent, then value maximization is equivalent to solving a sequence of static profit maximization problems, and we are back in the previous chapters' description.

To solve the problem (14.4) − (14.7), where R_t is given by (14.3), we apply the Maximum Principle. The problem has two control variables, L and I, and one state variable, K. We set up the current-value Hamiltonian:

$$H(K, L, I, q, t) \equiv F(K, L) - wL - I - G(I, K) + q(I - \delta K), \tag{14.8}$$

where q (to be interpreted economically below) is the adjoint variable associated with the dynamic constraint (14.6). For each $t \geq 0$ we maximize H w.r.t. the control variables. Thus, $\partial H/\partial L = F_L(K, L) - w = 0$, i.e.,

$$F_L(K, L) = w, \tag{14.9}$$

and $\partial H/\partial I = -1 - G_I(I, K) + q = 0$, i.e.,

$$1 + G_I(I, K) = q. \tag{14.10}$$

Next, we partially differentiate H w.r.t. the state variable and set the result equal to $rq - \dot{q}$, where r is the discount rate in (14.4):

$$\frac{\partial H}{\partial K} = F_K(K, L) - G_K(I, K) - q\delta = rq - \dot{q}. \tag{14.11}$$

Then, the Maximum Principle says that for an interior optimal path (K_t, L_t, I_t) there exists an adjoint variable q, which is a continuous function of t, written q_t, such that for all $t \geq 0$ the conditions (14.9), (14.10), and (14.11) hold along the path. Moreover, it can be shown that the path will satisfy the "standard" infinite horizon transversality condition

$$\lim_{t\to\infty} K_t q_t e^{-\int_0^t r_s ds} = 0. \tag{14.12}$$

The optimality condition (14.9) is the usual employment condition equalizing the marginal productivity of labor to the real wage. In the present context with strictly convex capital installation costs, this condition attains a distinct role as labor will in the short run be the only variable input. Indeed, the firm's installed capital is in the short run a fixed production factor due to the strictly convex capital installation costs. So, effectively there are diminishing returns (equivalent to rising marginal costs) in the short run even though the production function might have CRS.

The left-hand side of (14.10) gives the total marginal investment cost at time t, that is, the sum of the purchase price of the investment good and the cost of its installation. So the left-hand side is the marginal cost, MC, of increasing the capital stock in the firm. Since (14.10) is a necessary condition for optimality, the right-hand side of (14.10) must then be the marginal benefit, MB, of increasing the capital stock. Hence, q_t must represent the value to the optimizing firm of having one more unit of (installed) capital at time t. To put it differently: the adjoint variable q_t can be interpreted as the shadow price (measured in current output units) of capital along the optimal path.[2]

As to the interpretation of the differential equation (14.11), a condition for optimality must be that the firm acquires capital up to the point where the "marginal productivity of capital", $F_K - G_K$, equals the marginal "capital cost", $r_t q_t + (\delta q_t - \dot{q}_t)$. The first term in the latter expression represents interest costs and the second economic depreciation. In (14.11), the "marginal productivity of capital" appears as $F_K - G_K$. This expression takes into account the potential reduction, $-G_K$, of installation costs in the next instant brought about by the marginal unit of installed capital. The shadow price q_t appears as the "overall" price at which the firm can acquire the marginal unit of installed capital. At the margin, q_t can also be seen as the "overall" cost saving associated with reducing the investment by one unit. In this situation the firm recovers q_t by saving both on installation costs and the purchase in the investment goods market.

In accordance with this line of thought, by reordering in (14.11), we get the "no-arbitrage" condition

$$\frac{F_K(K,L) - G_K(I,K) - \delta q + \dot{q}}{q} = r, \tag{14.13}$$

saying that along the optimal path the rate of return on the marginal unit of installed capital must equal the interest rate.

The transversality condition (14.12) says that the present value of the capital stock "left over" at infinity must be zero. That is, the capital stock should not in the long run grow too fast, given the evolution of its discounted shadow price. In addition to necessity of (14.12) it can be shown[3] that the discounted shadow

[2] Recall that a *shadow price,* measured in some unit of account, of a good, from the point of view of the buyer, is the maximum number of units of account that he or she is willing to offer for one extra unit of the good.

[3] See Appendix B.

price itself in the far future must along an optimal path be asymptotically nil, i.e.,

$$\lim_{t\to\infty} q_t e^{-\int_0^t r_s ds} = 0. \tag{14.14}$$

If along the optimal path, K_t grows without bound, then not only must (14.14) hold but, in view of (14.12), the discounted shadow price must in the long run approach zero *faster* than K_t grows. Intuitively, otherwise the firm would be "over-accumulating". The firm would gain by reducing the capital stock "left over" for eternity (which is like "money left on the table"). Reducing the ultimate investment and installation costs would raise the present value of the firm's expected cash flow.

In connection with (14.10) we claimed that q_t can be interpreted as the shadow price (measured in current output units) of capital along the optimal path. A confirmation of this interpretation is obtained by solving the differential equation (14.11). Indeed, multiplying by $e^{-\int_0^t (r_s+\delta)ds}$ on both sides of (14.11), we get by integration and application of (14.14),[4]

$$q_t = \int_t^\infty \left[F_K(K_\tau, L_\tau) - G_K(I_\tau, K_\tau)\right] e^{-\int_t^\tau (r_s+\delta)ds} d\tau. \tag{14.15}$$

The right-hand side of (14.15) is the present value, as seen from time t, of the expected future increases of the firm's cash-flow that would result if one extra unit of capital were installed at time t. Indeed, $F_K(K_\tau, L_\tau)$ is the direct contribution to output of one extra unit of capital, while $-G_K(I_\tau, K_\tau) \geq 0$ represents the potential reduction of installation costs in the next instant brought about by the marginal unit of installed capital. Note that the marginal future increases of cash-flow in (14.15) are discounted at a rate equal to the interest rate *plus* the capital depreciation rate. The reason is that from one extra unit of capital at time t there are only $e^{-\delta(\tau-t)}$ units left at time τ.

To concretize our interpretation of q_t as representing the value to the optimizing firm at time t of having one extra unit of installed capital, let us make a thought experiment. Assume that a extra units of installed capital at time t drops down from the sky. At time $\tau > t$ there are $a \cdot e^{-\delta(\tau-t)}$ units of these still in operation so that the stock of installed capital is

$$K'_\tau = K_\tau + a \cdot e^{-\delta(\tau-t)}, \tag{14.16}$$

where K_τ denotes the stock of installed capital as it would have been without this "injection". Now, in (14.3) replace t by τ and consider the optimizing firm's

[4] For details, see Appendix A.

cash-flow R_τ as a function of $(K_\tau, L_\tau, I_\tau, \tau, t, a)$. Taking the partial derivative of R_τ w.r.t. a at the point $(K_\tau, L_\tau, I_\tau, \tau, t, 0)$, we get

$$\frac{\partial R_\tau}{\partial a}_{|a=0} = \left[F_K(K_\tau, L_\tau) - G_K(I_\tau, K_\tau)\right] e^{-\delta(\tau-t)}. \tag{14.17}$$

Considering the value of the optimizing firm at time t as a function of installed capital, K_t, and t itself, we denote this function $V^*(K_t, t)$. Then at any point where V^* is differentiable, we have

$$\begin{aligned}\frac{\partial V^*(K_t,t)}{\partial K_t} &= \int_t^\infty \left(\frac{\partial R_\tau}{\partial a}_{|a=0}\right) e^{-\int_t^\tau r_s ds} d\tau \\ &= \int_t^\infty [F_K(K_\tau, L_\tau) - G_K(I_\tau, K_\tau)] e^{-\int_t^\tau (r_s+\delta)ds} d\tau = q_t \end{aligned} \tag{14.18}$$

when the firm moves along the optimal path. The second equality sign comes from (14.17) and the third is implied by (14.15). So the value of the adjoint variable, q, at time t equals the contribution to the firm's maximized value of a fictional marginal "injection" of installed capital at time t. This is just another way of saying that q_t represents the benefit to the firm of the marginal unit of installed capital along the optimal path.

This story facilitates the understanding that the control variables at any point in time should be chosen so that the Hamiltonian function is maximized. Thereby one maximizes the properly weighted sum of the current direct contribution to the criterion function and the indirect contribution, which is the benefit (as measured approximately by $q_t \Delta K_t$) of having a higher capital stock in the future.

As we know, the Maximum Principle gives only necessary conditions for an optimal path, not sufficient conditions. We use the principle as a tool for finding candidates for a solution. Having found in this way a candidate, one way to proceed is to check whether Mangasarian's sufficient conditions are satisfied. Given the transversality condition (14.12) and the non-negativity of the state variable, K, the only additional condition to check is whether the Hamiltonian function is jointly concave in the endogenous variables (here K, L, and I). If it is jointly concave in these variables, then the candidate *is* an optimal solution. Owing to concavity of $F(K, L)$, inspection of (14.8) reveals that the Hamiltonian function is jointly concave in (K, L, I) if $-G(I, K)$ is jointly concave in (I, K). This condition is equivalent to $G(I, K)$ being jointly convex in (I, K), an assumption allowed within the confines of (14.2); for example, $G(I, K) = (\frac{1}{2})\beta I^2/K$ as well as the simpler $G(I, K) = (\frac{1}{2})\beta I^2$ (where in both cases $\beta > 0$) will do. Thus, assuming joint convexity of $G(I, K)$, the first-order conditions and the transversality condition are not only necessary, but also sufficient for an optimal solution.

14.1.2 The implied investment function

From condition (14.10) we can derive an investment function. Rewriting (14.10), we have that an optimal path satisfies

$$G_I(I_t, K_t) = q_t - 1. \tag{14.19}$$

Combining this with the assumption (14.2) on the installation cost function, we see that

$$I_t \gtreqless 0 \text{ for } q_t \gtreqless 1, \text{ respectively,} \tag{14.20}$$

cf. Fig. 14.2.[5] By the implicit function theorem, in view of $G_{II} \neq 0$, (14.19) defines optimal investment, I_t, as an implicit function of the shadow price, q_t, and the state variable, K_t,

$$I_t = \mathcal{M}(q_t, K_t), \tag{14.21}$$

with partial derivatives

$$\frac{\partial I_t}{\partial q_t} = \frac{1}{G_{II}(\mathcal{M}(q_t, K_t), K_t)} > 0, \qquad \text{and} \qquad \frac{\partial I_t}{\partial K_t} = -\frac{G_{IK}(\mathcal{M}(q_t, K_t), K_t)}{G_{II}(\mathcal{M}(q_t, K_t), K_t)},$$

where the latter cannot be signed without further specification. In view of (14.20), $\mathrm{M}(1, K_t) = 0$.

It follows that optimal investment is an increasing function of the shadow price of installed capital. In view of (14.20), $\mathcal{M}(1, K) = 0$. Not surprisingly, the investment rule is: invest now, if and only if the value to the firm of the marginal unit of installed capital is larger than the price of the capital good (which is 1, excluding installation costs). At the same time, the rule says that, because of the convex installation costs, invest only up to the point where the marginal installation cost, $G_I(I_t, K_t)$, equals $q_t - 1$, cf. (14.19).

Condition (14.21) shows the remarkable information content that the shadow price q_t has. As soon as q_t is known (along with the current capital stock K_t), the firm can decide the optimal level of investment through knowledge of the installation cost function G alone (since, when G is known, so is in principle the inverse of G_I w.r.t. I, the investment function $\mathcal{M}$). All the information about the production function, input prices, and interest rates now and in the future that is relevant to the investment decision is summarized in one number, q_t. The form of the investment function, $\mathcal{M}$, depends only on the installation cost function G. These are very useful properties in theoretical and empirical analysis.

[5] From the assumptions made in (14.2), we only know that the graph of $G_I(I, \bar{K})$ is an upward-sloping curve going through the origin. Fig. 14.2 shows the special case where this curve happens to be linear.

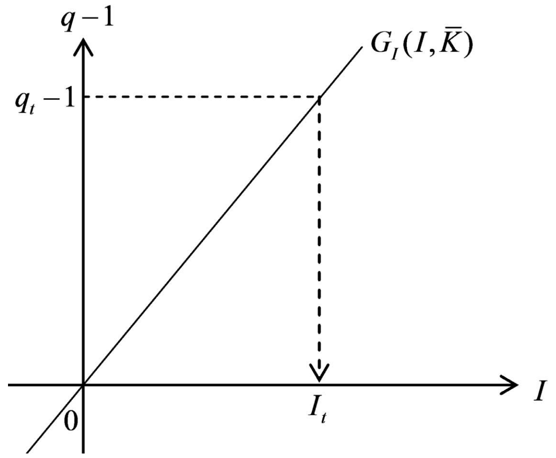

Figure 14.2: Marginal installation costs as a function of the gross investment level, I, for a given amount, $\bar{K}$, of installed capital. The optimal gross investment, I_t, when $q = q_t$ is indicated.

14.1.3 A not implausible special case

We now introduce the convenient case where the installation function G is homogeneous of degree one w.r.t. I and K so that we can, for $K > 0$, write

$$\begin{aligned} J &= G(I,K) = G(\frac{I}{K},1)K \equiv g(\frac{I}{K})K, \qquad \text{or} \qquad (14.22) \\ \frac{J}{K} &= g(\frac{I}{K}), \end{aligned}$$

where $g(\cdot)$ represents the installation cost-capital ratio and $g(0) \equiv G(0,1) = 0$, by (14.2).

LEMMA 1 The function $g(\cdot)$ has the following properties:
(i) $g'(I/K) = G_I(I,K)$;
(ii) $g''(I/K) = G_{II}(I,K)K > 0$ for $K > 0$; and
(iii) $g(I/K) - g'(I/K)I/K = G_K(I,K) < 0$ for $I \neq 0$.

Proof. (i) $G_I = Kg'/K = g'$; (ii) $G_{II} = g''/K$; (iii) $G_K = \partial(g(I/K)K)/\partial K = g(I/K) - g'(I/K)I/K < 0$ for $I \neq 0$ since, in view of $g'' > 0$ and $g(0) = 0$, we have $g(x)/x < g'(x)$ for all $x \neq 0$. □

The graph of $g(I/K)$ is qualitatively the same as that in Fig. 14.1 (imagine we have $\bar{K} = 1$ in that graph). The installation cost relative to the existing capital stock is now a strictly convex function of the investment-capital ratio, I/K.

EXAMPLE 1 Let $J = G(I,K) = \frac{1}{2}\beta I^2/K$, where $\beta > 0$. Then G is homogeneous of degree one w.r.t. I and K and gives $J/K = \frac{1}{2}\beta(I/K)^2 \equiv g(I/K)$. □

A further important property of (14.22) is that the cash-flow function in (14.3) becomes homogeneous of degree one w.r.t. K, L, and I in the "normal" case where the production function has CRS. This has two implications. First, Hayashi's theorem applies (see below). Second, the q-theory can easily be incorporated into a model of economic growth.[6]

Does the hypothesis of linear homogeneity of the cash flow in K, L, and I make economic sense? According to the replication argument it does. Suppose a given firm has K units of installed capital and produces Y units of output with L units of labor. When at the same time the firm invests I units of account in new capital, it obtains the cash flow R after deducting the installation costs, $G(I, K)$. Then it makes sense to assume that the firm could do the same thing at another place, hereby doubling its cash-flow. (Of course, owing to the possibility of indivisibilities, this reasoning does not take us all the way to linear homogeneity. Moreover, the argument ignores that also land is a necessary input. As discussed in Chapter 2, the empirical evidence on linear homogeneity is mixed.)

In view of (i) of Lemma 1, the linear homogeneity assumption for G allows us to write (14.19) as

$$g'(I/K) = q - 1. \tag{14.23}$$

This equation defines the investment-capital ratio, I/K , as an implicit function, m, of q :

$$\frac{I}{K} = m(q), \quad \text{where} \quad m(1) = 0 \quad \text{and} \quad m'(q) = \frac{1}{g''(m(q))} > 0, \tag{14.24}$$

by implicit differentiation in (14.23). In this case q encompasses all information that is of relevance to the decision about the investment-capital ratio.

In Example 1 above we have $g(I/K) = \frac{1}{2}\beta(I/K)^2$, in which case (14.23) gives $I/K = (q-1)/\beta$. So in this case we have $m(q) = q/\beta - 1/\beta$, a linear investment function, as illustrated in Fig. 14.3. The parameter β can be interpreted as the degree of sluggishness in the capital adjustment. The degree of sluggishness reflects the degree of convexity of installation costs.[7] Generally the graph of the investment function is positively sloped, but not necessarily linear. The interpretation of the stippled lines and q^* and n in Fig. 14.3 is as follows. Suppose the firm's employment grows at a constant rate n. Then a constant capital-labor ratio, K/L, requires $\dot{K}/K = n$, hence $I/K - \delta = m(q) - \delta = n$. The value of q satisfying this equation is denoted q^*.

[6] The relationship between the function g and other ways of formulating the theory is commented on in Appendix C.

[7] For a twice differentiable function, $f(x)$, with $f'(x) \neq 0$, we define the *degree of convexity* in the point x by $f''(x)/f'(x)$. So the degree of convexity of $g(I/K)$ is $g''/g' = (I/K)^{-1} = \beta(q-1)^{-1}$ and thereby we have $\beta = (q-1)g''/g'$. So, for given q, the degree of sluggishness is proportional to the degree of convexity of adjustment costs.

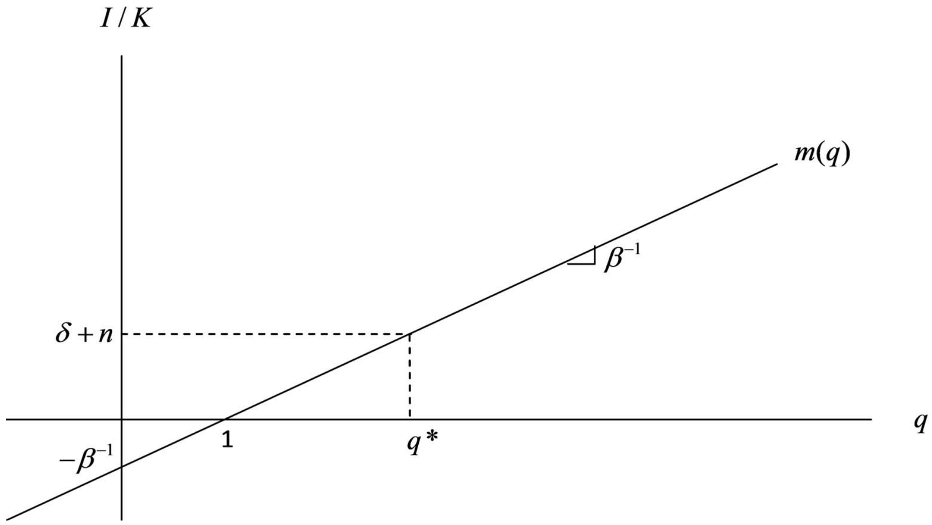

Figure 14.3: Optimal investment-capital ratio as a function of the shadow price of installed capital when $g(I/K) = \frac{1}{2}\beta(I/K)^2$.

To see how the shadow price q changes over time along the optimal path, we rearrange (14.11):

$$\dot{q}_t = (r_t + \delta)q_t - F_K(K_t, L_t) + G_K(I_t, K_t). \tag{14.25}$$

Recall that $-G_K(I_t, K_t)$ indicates how much *lower* the installation costs are as a result of the marginal unit of installed capital. In the special case (14.22), we have, from Lemma 1,

$$G_K(I, K) = g(\frac{I}{K}) - g'(\frac{I}{K})\frac{I}{K} = g(m(q)) - (q - 1)m(q),$$

using (14.24) and (14.23).

Inserting this into (14.25) gives

$$\dot{q}_t = (r_t + \delta)q_t - F_K(K_t, L_t) + g(m(q_t)) - (q_t - 1)m(q_t). \tag{14.26}$$

This differential equation is very useful in macroeconomic analysis, as we will soon see, cf. Fig. 14.4 below.

In a macroeconomic context, for steady state to be achievable, gross investment must be large enough to match not only capital depreciation, but also growth in the labor input. Otherwise a constant capital-labor ratio can not be sustained. That is, the investment-capital ratio, I/K, must be equal to the sum of the depreciation rate and the growth rate of the labor force, i.e., $\delta + n$. The level of q which is required to motivate such an investment-capital ratio is called q^* in Fig. 14.3.

14.2 Marginal q and average q

Our q above, determining investment, should be distinguished from what is usually called Tobin's q or average q. In a more general context, let p_{It} denote the current purchase price (in terms of output units) per unit of the investment good (before installment). Then *Tobin's* q or *average* q, q_t^a, is defined as $q_t^a \equiv V_t/(p_{It}K_t)$, that is, Tobin's q is the ratio of the market value of the firm to the replacement value of the firm in the sense of the "reacquisition value of the capital goods before installment costs" (the top index "a" stands for "average"). In our simplified context we have $p_{It} \equiv 1$ (the price of the investment good is the same as that of the output good). Therefore Tobin's q can be written

$$q_t^a \equiv \frac{V_t}{K_t} = \frac{V^*(K_t, t)}{K_t}, \tag{14.27}$$

where the equality holds for an optimizing firm. Conceptually this is different from the firm's internal shadow price on capital, i.e., what we have denoted q_t in the previous sections. In the language of the q-theory of investment, this q_t is the *marginal* q, representing the value to the firm of one *extra* unit of installed capital relative to the price of un-installed capital equipment. The term marginal q is natural since along the optimal path, as a slight generalization of (14.18), we must have $q_t = (\partial V^*/\partial K_t)/p_{It}$. Letting q_t^m ("m" for "marginal") be an alternative symbol for this q_t, we have in our model above, where we consider the special case $p_{It} \equiv 1$,

$$q_t^m \equiv q_t = \frac{\partial V^*(K_t, t)}{\partial K_t}. \tag{14.28}$$

The two concepts, average q and marginal q, have not always been clearly distinguished in the literature. What is directly relevant to the investment decision is marginal q. Indeed, the analysis above showed that optimal investment is an increasing function of q^m. Further, the analysis showed that a "critical" value of q^m is 1 and that only if $q^m > 1$, is positive gross investment warranted.

The importance of q^a is that it can be measured empirically as the ratio of the sum of the share market value of the firm and its debt to the current acquisition value of its total capital before installment. Since q^m is much harder to measure than q^a, it is important to know the relationship between q^m and q^a. Fortunately, we have a simple theorem giving conditions under which $q^m = q^a$.

THEOREM (Hayashi, 1982) Assume the firm is a price taker, that the production function F is jointly concave in (K, L), and that the installation cost function G is jointly convex in (I, K).[8] Then, along an optimal path we have:

[8] That is, in addition to (14.2), we assume $G_{KK} \geq 0$ and $G_{II}G_{KK} - G_{IK}^2 \geq 0$. The specification in Example 1 above satisfies this.

(i) $q_t^m = q_t^a$ for all $t \geq 0$, if F and G are homogeneous of degree 1.

(ii) $q_t^m < q_t^a$ for all t, if F is strictly concave in (K, L) and/or G is strictly convex in (I, K).

Proof. See Appendix D.

The assumption that the firm is a price taker may, of course, seem critical. The Hayashi theorem has been generalized, however. Also a monopolistic firm, facing a downward-sloping demand curve and setting its own price, may have a cash flow which is homogeneous of degree one in the three variables K, L, and I. If so, then the condition $q_t^m = q_t^a$ for all $t \geq 0$ still holds (Abel 1990). Abel and Eberly (1994) present further generalizations.

In any case, when q^m is approximately equal to (or just proportional to) q^a, the theory gives a remarkably simple operational investment function, $I = m(q^a)K$, cf. (14.24). At the macro level we interpret q^a as the market valuation of the firms relative to the replacement value of their total capital stock. This market valuation is an indicator of the expected future earnings potential of the firms. Under the conditions in (i) of the Hayashi theorem the market valuation also indicates the marginal earnings potential of the firms, hence, it becomes a determinant of their investment. This establishment of a relationship between the stock market and firms' aggregate investment is the basic point in Tobin (1969).

14.3 Applications

Capital installation costs in a closed economy

Allowing for convex capital installation costs in the economy has far-reaching implications for the causal structure of a model of a closed economy. Investment decisions attain an active role in the economy and forward-looking expectations become important for these decisions. Expected future market conditions and announced future changes in corporate taxes and depreciation allowance will affect firms' investment already today.

The essence of the matter is that current and expected future interest rates have to adjust for aggregate saving to equal aggregate investment, that is, for the output market to clear. Given full employment ($L_t = \bar{L}_t$), the output market clears when aggregate supply equals aggregate demand, i.e.,

$$F(K_t, \bar{L}_t) - G(I_t, K_t) \ (= \text{value added} \equiv GDP_t) = C_t + I_t,$$

where C_t is determined by the intertemporal utility maximization of the forward-looking households, and I_t is determined by the intertemporal value maximization of the forward-looking firms facing strictly convex installation costs. Like in the determination of C_t, current and expected future interest rates now also matter for the determination of I_t. This is the first time in this book where *clearing in the output market* is assigned an *active* role. In the earlier models investment was just a passive reflection of household saving. Desired investment was automatically equal to the residual of national income left over after consumption decisions had taken place. Nothing had to adjust to clear the output market, neither interest rates nor output. In contrast, in the present framework adjustments in interest rates and/or the output level are needed for the continuous clearing in the output market and these adjustments are decisive for the macroeconomic dynamics.

A related implication of the theory is that we have to discard the simple conception from our previous models that the real interest rate is the variable which adjusts so as to clear a rental market for capital goods. The interest rate will no longer be tied down by a requirement that such markets clear, and will, even under perfect competition, no longer in equilibrium equal the net marginal productivity of capital. This is seen for instance in the formula (14.13).

In actual economies there may of course exist "secondary markets" for used capital goods and markets for renting capital goods owned by others. In view of installation costs and similar, however, shifting capital goods from one plant to another is generally costly. Therefore the turnover in that kind of markets tends to be limited (with the exception of rental markets for cars, trucks, air planes, and similar). And, importantly for our theory, the effective capital cost per time unit for a firm that hire its capital goods, rather than buying them, will still consist not only of the simple rental rate (interest plus depreciation costs, $r + \delta$) but also costs associated with installation and now presumably also later dismantling.

In for instance Abel and Blanchard (1983), a Ramsey-style model integrating the q-theory of investment and Hayashi's theorem is presented. The authors study the two-dimensional general equilibrium dynamics resulting from the adjustment of current and expected future interest rates *needed for the output market to clear*. Adjustments of the whole structure of interest rates (the yield curve) take place and constitute the equilibrating mechanism in the output and asset markets.

By having output market equilibrium playing this role in the model, a first step is taken toward medium- and short-run macroeconomic theory. We take further steps in later chapters, by allowing imperfect competition and nominal price rigidities to enter the picture. Then the demand side gets an active role both in the determination of q (and thereby investment) and in the determination of aggregate output and employment. This is what Keynesian theory (old and new) deals with.

In the remainder of this chapter we will still assume perfect competition in all markets including the labor market. In this sense we will stay within the neoclassical framework (supply-dominated models) where, by instantaneous adjustment of the real wage, labor demand continuously matches labor supply. The next two subsections present simple examples of how Tobin's q-theory of investment can be integrated into the neoclassical framework. To avoid the complications arising from an endogenous interest rate, the focus is on a small open economy. In that context, households financial wealth is distinct from the market value of the capital stock, and the theoretical analysis is not dependent on Hayashi's theorem.

A small open economy with capital installation costs

By introducing convex capital installation costs in a model of a small open economy (SOE), we avoid the counterfactual outcome that the capital stock adjusts *instantaneously* when the interest rate in the world financial market changes. In the standard neoclassical growth model for a small open economy, without convex capital installation costs, a rise in the interest rate leads immediately to a complete adjustment of the capital stock so as to equalize the net marginal productivity of capital to the new higher interest rate. Moreover, in that model expected *future* changes in the interest rate or in corporate taxes and depreciation allowances do *not* trigger an investment response until these changes actually happen. In contrast, when convex installation costs are present, expected future changes tend to influence firms' investment already today.

We assume:

1. Perfect mobility across borders of goods and financial capital.
2. Domestic and foreign financial claims are perfect substitutes.
3. No mobility across borders of labor.
4. Labor supply is inelastic and constant and there is no technological progress.
5. The capital installation cost function $G(I, K)$ is homogeneous of degree 1.

In this setting the SOE faces an exogenous interest rate, r, given from the world financial market. We assume r is a positive constant. The aggregate production function, $F(K_t, L_t)$, is neoclassical and concave as in the previous sections. Suppose markets are competitive. Let $\bar{L} > 0$ denote the constant labor supply. With profit maximizing firms and continuous clearing in the labor market we have, for all $t \geq 0$,

$$w_t = F_L(K_t, \bar{L}) \equiv w(K_t), \tag{14.29}$$

since $L_t = \bar{L}$. At any time t, K_t is predetermined in the sense that due to the convex installation costs, changes in K take time. Thus (14.29) *determines* the market real wage w_t.

To pin down the evolution of the economy, we now derive two coupled differential equations in K and q. By (14.24) we get

$$\dot{K}_t = I_t - \delta K_t = (m(q_t) - \delta)K_t, \qquad K_0 > 0 \text{ given.} \tag{14.30}$$

As to the dynamics of q, we have (14.26). Since the capital installation cost function $G(I, K)$ is assumed to be homogeneous of degree 1, point (iii) of Lemma 1 applies and we can write (14.26) as

$$\dot{q}_t = (r + \delta)q_t - F_K(K_t, \bar{L}) + g(m(q_t)) - (q_t - 1)m(q_t). \tag{14.31}$$

As r and $\bar{L}$ are exogenous, the capital stock, K, and its shadow price, q, are the only endogenous variables in the differential equations (14.30) and (14.31). In addition, we have an initial condition for K and a necessary transversality condition involving q, namely

$$\lim_{t\to\infty} K_t q_t e^{-rt} = 0. \tag{14.32}$$

Fig. 14.4 shows the phase diagram for these two coupled differential equations. Let q^* be defined as the value of q satisfying the equation $m(q) = \delta$. Since $m' > 0$, q^* is unique. Suppressing for convenience the explicit time subscripts, we then have

$$\dot{K} = 0 \text{ for } m(q) = \delta, \text{ i.e., for } q = q^*.$$

As $\delta > 0$, we have $q^* > 1$. This is so because also mere reinvestment to offset capital depreciation requires an incentive, namely that the marginal value to the firm of replacing worn-out capital is larger than the purchase price of the investment good (since the installation cost must also be compensated). From (14.30) is seen that

$$\dot{K} \gtreqless 0 \text{ for } m(q) \gtreqless \delta, \text{ respectively, i.e., for } q \gtreqless q^*, \text{ respectively,}$$

cf. the horizontal arrows in Fig. 14.4.

From (14.31) we have

$$\dot{q} = 0 \text{ for } 0 = (r + \delta)q - F_K(K, \bar{L}) + g(m(q)) - (q - 1)m(q). \tag{14.33}$$

If, in addition $\dot{K} = 0$ (hence, $q = q^*$ and $m(q) = m(q^*) = \delta$), this gives

$$0 = (r + \delta)q^* - F_K(K, \bar{L}) + g(\delta) - (q^* - 1)\delta, \tag{14.34}$$

where the right-hand-side is increasing in K, in view of $F_{KK} < 0$. Hence, there exists at most one value of K such that the steady state condition (14.34) is

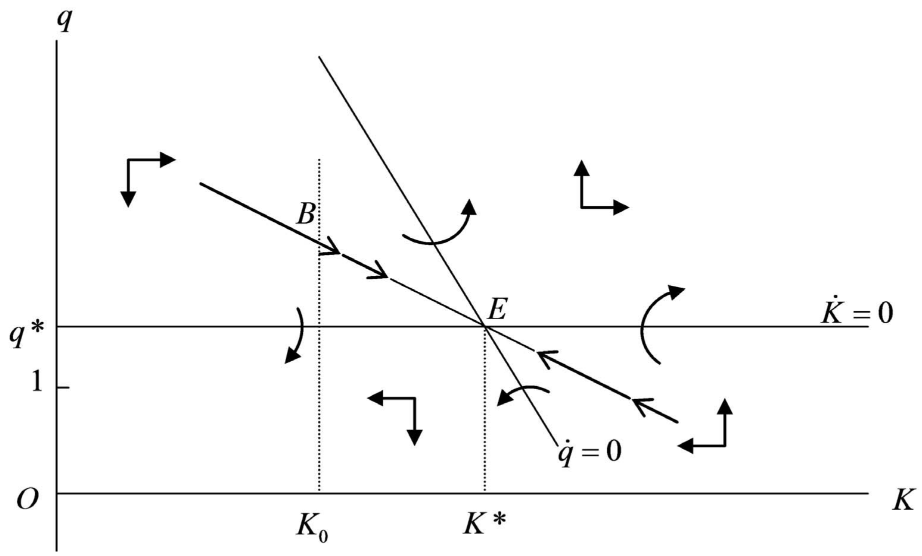

Figure 14.4: Phase diagram for investment dynamics in a small open economy (a case where $\delta > 0$).

satisfied;[9] this value is denoted K^*, corresponding to the steady state point E in Fig. 14.4. The question is now: what is the slope of the $\dot{q} = 0$ locus? In Appendix E it is shown that at least in a neighborhood of the steady state point E, this slope is negative in view of the assumption $r > 0$ and $F_{KK} < 0$. From (14.31) we see that

$$\dot{q} \lessgtr 0 \text{ for points to the left and to the right, respectively, of the } \dot{q} = 0 \text{ locus,}$$

since $F_{KK}(K_t, \bar{L}) < 0$. The vertical arrows in Fig. 14.4 show these directions of movement.

Altogether the phase diagram shows that the steady state E is a saddle point, and since there is one predetermined variable, K, and one jump variable, q, and the saddle path is not parallel to the jump variable axis, the steady state is saddle-point stable. At time 0 the economy will be at the point B in Fig. 14.4 where the vertical line $K = K_0$ crosses the saddle path. Then the economy will move along the saddle path towards the steady state. This solution satisfies the transversality condition (14.32) and is the unique solution to the model (for details, see Appendix F).

The effect of an unanticipated rise in the interest rate Suppose that until time 0 the economy has been in the steady state E in Fig. 14.4. Then,

[9] And assuming that F satisfies the Inada conditions, we are sure that such a value exists since (14.34) gives $F_K(K, \bar{L}) = rq^* + g(\delta) + \delta > 0$.

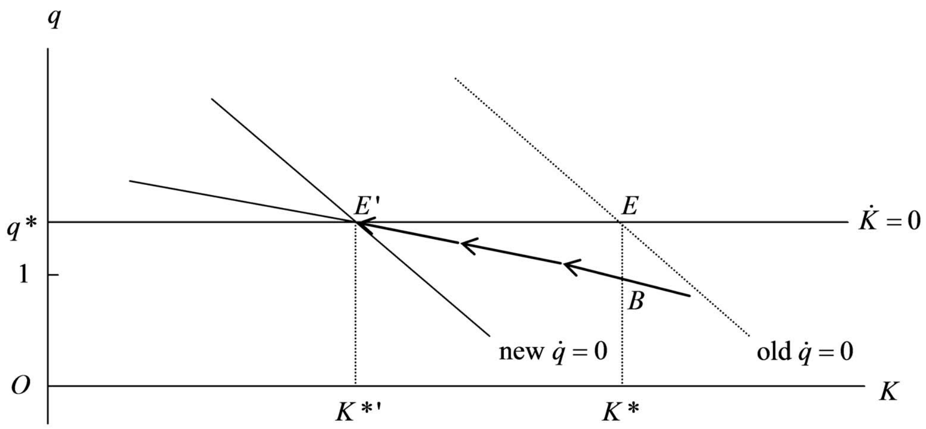

Figure 14.5: Phase portrait of an unanticipated rise in r (the case $\delta > 0$).

an unexpected shift in the interest rate occurs so that the new interest rate is a constant $r' > r$. We assume that the new interest rate is rightly expected to remain at this level forever. From (14.30) we see that q^* is not affected by this shift, hence, the $\dot{K} = 0$ locus is not affected. However, (14.33) implies that the $\dot{q} = 0$ locus and K^* shift to the left, in view of $F_{KK}(K, \bar{L}) < 0$.

Fig. 14.5 illustrates the situation for $t > 0$. At time $t = 0$ the shadow price q jumps down to a level corresponding to the point B in Fig. 14.5. There is now a heavier discounting of the future benefits that the marginal unit of capital can provide. As a result the incentive to invest is diminished and gross investment will not even compensate for the depreciation of capital. Hence, the capital stock decreases gradually. This is where we see a crucial role of convex capital installation costs in an open economy. For now, the installation costs are the costs associated with disinvestment (dismantling and selling out of machines). If these convex costs were not present, we would get the same counterfactual prediction as from the previous open-economy models in this book, namely that the new steady state is attained immediately after the shift in the interest rate.

As the capital stock is diminished, the marginal productivity of capital rises and so does q. The economy moves along the new saddle path and approaches the new steady state E' as time goes by.

Suppose that for some reason such a decrease in the capital stock is not desirable from a social point of view. This could be because of positive external effects of capital and investment, e.g., a kind of "learning by doing". Then the government could decide to implement an investment subsidy $\sigma \in (0, 1)$ so that to attain an investment level I, purchasing the investment goods involves a cost of $(1 - \sigma)I$. Assuming the subsidy is financed by some tax not affecting firms' behavior, investment is increased again and the economy may in the long run end up at the old steady-state level of K (but the new q^* will be lower than the old).

In a similar way the effect of a depreciation allowance and a corporate tax can be studied.

A growing small open economy with capital installation costs*

The basic assumptions are the same as in the previous section except that now labor supply, $\bar{L}_t$, grows at the constant rate $n \geq 0$, while the technology level, T, grows at the constant rate $\gamma \geq 0$ (both rates exogenous and constant) and the production function is neoclassical with CRS. We assume that the world market real interest rate, r, is a constant and satisfies $r > \gamma + n$. Still assuming full employment, we have $L_t = \bar{L}_t = \bar{L}_0 e^{nt}$.

In this setting the production function on intensive form is useful:

$$\tilde{Y} = F(K, T\bar{L}) = F(\frac{K}{T\bar{L}}, 1)T\bar{L} \equiv f(\tilde{k})T\bar{L},$$

where $\tilde{k} \equiv K/(T\bar{L})$ and f satisfies $f' > 0$ and $f'' < 0$. Still assuming perfect competition, the market-clearing real wage at time t is determined as

$$w_t = F_2(K_t, T_t\bar{L}_t)T_t = \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t)\right] T_t \equiv \tilde{w}(\tilde{k}_t)T_t,$$

where both $\tilde{k}_t$ and T_t are predetermined. By log-differentiation of $\tilde{k} \equiv K/(T\bar{L})$ w.r.t. time we get $\dot{\tilde{k}}_t/\tilde{k}_t = \dot{K}_t/K_t - (\gamma + n)$. Substituting (14.30), we get

$$\dot{\tilde{k}}_t = [m(q_t) - (\delta + \gamma + n)]\,\tilde{k}_t. \tag{14.35}$$

The change in the shadow price of capital is now described by

$$\dot{q}_t = (r + \delta)q_t - f'(\tilde{k}_t) + g(m(q_t)) - (q_t - 1)m(q_t), \tag{14.36}$$

from (14.26). In addition, the transversality condition,

$$\lim_{t\to\infty} \tilde{k}_t q_t e^{-(r-\gamma-n)t} = 0, \tag{14.37}$$

must hold.

The differential equations (14.35) and (14.36) constitute our new dynamic system. Fig. 14.6 shows the phase diagram, which is qualitatively similar to that in Fig. 14.4. We have

$$\dot{\tilde{k}} = 0 \quad \text{for} \quad m(q) = \delta + \gamma + n, \text{ i.e., for } q = q^*,$$

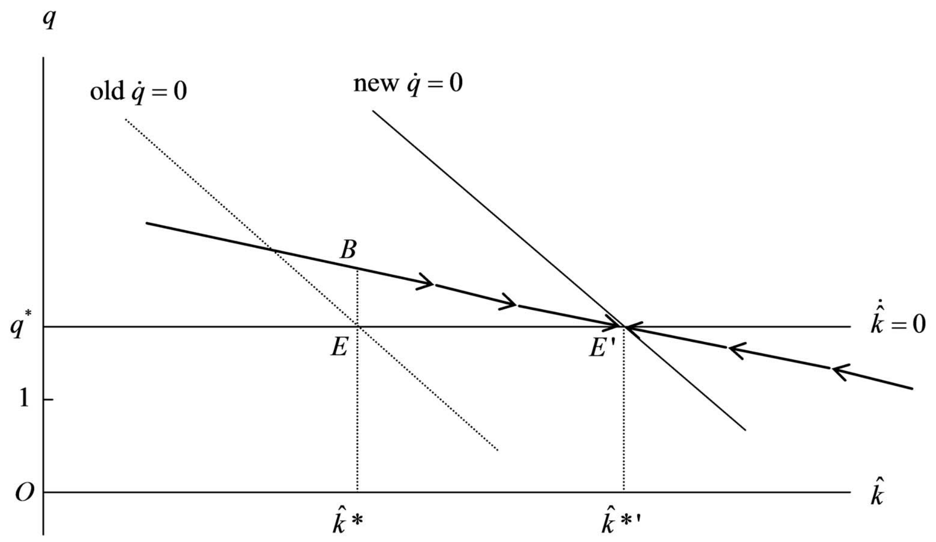

Figure 14.6: Phase portrait of an unanticipated fall in r (a growing economy with $\delta + \gamma + n \geq \gamma + n > 0$).

where q^* now is defined by the requirement $m(q^*) = \delta + \gamma + n$. Notice, that when $\gamma + n > 0$, we get a larger steady state value q^* than in the previous section. This is so because now a higher investment-capital ratio is required for a steady state to be possible. Moreover, the transversality condition (14.12) is satisfied in the steady state.

From (14.36) we see that $\dot{q} = 0$ now requires

$$0 = (r + \delta)q - f'(\tilde{k}) + g(m(q)) - (q - 1)m(q).$$

If, in addition $\dot{\tilde{k}} = 0$ (hence, $q = q^*$ and $m(q) = m(q^*) = \delta + \gamma + n$), this gives

$$0 = (r + \delta)q^* - f'(\tilde{k}) + g(\delta + \gamma + n) - (q^* - 1)(\delta + \gamma + n).$$

Here, the right-hand-side is increasing in $\tilde{k}$ (in view of $f''(\tilde{k}) < 0$). Hence, the steady state value $\tilde{k}^*$ of the effective capital-labor ratio is unique, cf. the steady state point E in Fig. 14.6.

By the assumption $r > \gamma + n$ we have, at least in a neighborhood of E in Fig. 14.6, that the $\dot{q} = 0$ locus is negatively sloped (see Appendix E).[10] Again the steady state is a saddle point, and the economy moves along the saddle path towards the steady state.

[10] In our perfect foresight model we in fact *have* to assume $r > \gamma + n$ for the firm's maximization problem to be well-defined. If instead $r \leq \gamma + n$, the market value of the representative firm would be infinite, and maximization would loose its meaning.

In Fig. 14.6 it is assumed that until time 0, the economy has been in the steady state E. Then, an unexpected shift in the interest rate to a *lower* constant level, r', takes place. The $\dot{q} = 0$ locus is shifted to the right, in view of $f'' < 0$. The shadow price, q, immediately jumps up to a level corresponding to the point B in Fig. 14.6. The economy moves along the new saddle path and approaches the new steady state E' with a higher effective capital-labor ratio as time goes by. In Exercise 14.2 the reader is asked to examine the analogue situation where an unanticipated downward shift in the rate of technological progress takes place.

14.4 Concluding remarks

Tobin's q-theory of investment gives a remarkably simple operational macroeconomic investment function, in which the variable explaining aggregate investment is the valuation of the firms by the stock market relative to the replacement value of the firms' physical capital. This link between asset markets and firms' aggregate investment is an appealing feature of Tobin's q-theory.

When faced with strictly convex installation costs, the firm has to take the *future* into account to invest optimally. Therefore, the firm's *expectations* become important. Owing to the strictly convex installation costs, the firm adjusts its capital stock only *gradually* when new information arises.

By incorporating these features, Tobin's q-theory helps explaining the sluggishness in investment we see in the empirical data. And the theory avoids the counterfactual outcome from earlier chapters that the capital stock in a small open economy with perfect mobility of goods and financial capital is *instantaneously* adjusted when the interest rate in the world market changes. So the theory takes into account the time lags in capital adjustment in real life. Possibly, this feature can be abstracted from in long-run analysis and models of economic growth, but not in short- and medium-run analysis.

Many econometric tests of the q theory of investment have been made, often with critical implications. Movements in q^a, even taking account of changes in taxation, seemed capable of explaining only a minor fraction of the movements in investment. And the estimated equations relating fixed capital investment to q^a typically give strong auto-correlation in the residuals. Other variables, in particular availability of current corporate profits for internal financing, seem to have explanatory power independently of q^a (see Abel 1990, Chirinko 1993, Gilchrist and Himmelberg, 1995). So there is reason to be sceptical towards the notion that *all* information of relevance for the investment decision is reflected by the market valuation of firms. Or we might question the validity of the assumption in Hayashi's theorem (and its generalizations), that firms' cash flow tends to be homogeneous of degree one w.r.t. K, L, and I.

Further circumstances are likely to relax the link between q^a and investment. In the real world with many production sectors, physical capital is heterogeneous. If for example a sharp unexpected rise in the price of energy takes place, a firm with energy-intensive technology will loose in market value. At the same time it has an incentive to invest in energy-saving capital equipment. Hence, we might observe a fall in q^a at the same time as investment increases.

Imperfections in credit markets are ignored by the q-theory. Their presence further loosens the relationship between q^a and investment and may help explain the observed positive correlation between investment and corporate profits.

We might also question that capital installation costs really have the hypothesized *strictly convex* form. It is one thing that there are costs associated with installation, reorganizing and retraining etc., when new capital equipment is procured. But should we expect these costs to be strictly convex in the volume of investment? To think about this, let us for a moment ignore the role of the existing capital stock. Hence, we write total installation costs $J = G(I)$ with $G(0) = 0$. It does not seem problematic to assume $G'(I) > 0$ for $I > 0$. The question concerns the assumption $G''(I) > 0$. According to this assumption the average installation cost $G(I)/I$ must be increasing in I.[11] But against this speaks the fact that capital installation may involve indivisibilities, fixed costs, acquisition of new information etc. All these features tend to imply *decreasing* average costs. In any case, at least at the microeconomic level one should expect unevenness in the capital adjustment process rather than the above smooth adjustment.

Because of the mixed empirical success of the convex installation cost hypothesis other theoretical approaches that can account for sluggish and sometimes non-smooth and lumpy capital adjustment have been considered: uncertainty, investment irreversibility, indivisibility, or financial problems due to bankruptcy costs (Nickell 1978, Zeira 1987, Dixit and Pindyck 1994, Caballero 1999, Adda and Cooper 2003). These approaches notwithstanding, it turns out that the q-theory of investment has recently been somewhat rehabilitated from both a theoretical and an empirical point of view. At the theoretical level Wang and Wen (2010) show that financial frictions in the form of collateralized borrowing at the firm level can give rise to strictly convex adjustment costs at the aggregate level yet at the same time generate lumpiness in plant-level investment. For large firms, unlikely to be much affected by financial frictions, Eberly et al. (2008) find that the theory does a good job in explaining investment behavior.

In any case, the q-theory of investment is in different versions widely used in short- and medium-run macroeconomics because of its simplicity and the ap-

[11] Indeed, for $I \neq 0$ we have $d[G(I)/I]/dI = [IG'(I) - G(I)]/I^2 > 0$, when G is strictly convex ($G'' > 0$) and $G(0) = 0$.

pealing link it establishes between asset markets and firms' investment. And the q-theory has also had an important role in studies of the housing market and the role of housing prices for household wealth and consumption, a theme to which we return in the next chapter.

14.5 Literature notes

A first sketch of the q-theory of investment is contained in Tobin (1969). Later advances of the theory took place through the contributions of Hayashi (1982) and Abel (1982), as surveyed in (1990).

Both the Ramsey model and the Blanchard OLG model for a closed market economy may be extended by adding strictly convex capital installation costs, see Abel and Blanchard (1983) and Lim and Weil (2003). Adding a public sector, such a framework is useful for the study of how different subsidies, taxes, and depreciation allowance schemes affect investment in physical capital as well as housing, see, e.g., Summers (1981), Abel and Blanchard (1983), and Dixit (1990).

Groth and Madsen (2016) study medium-term fluctuations in a closed economy, arising in a Tobin's q framework extended by sluggishness in real wage adjustments.

14.6 Appendix

A. When value maximization is - and is not - equivalent to continuous static profit maximization

For the idealized case where tax distortions, asymmetric information, and problems with enforceability of financial contracts are absent, the Modigliani-Miller theorem (Modigliani and Miller, 1958) says that the market value (debt plus equity) does not depend on the level of the debt. So the financial structure of the firm is both indeterminate and irrelevant for production outcomes. Considering the firm described in Section 14.1, the implied separation of the financing decision from the production and investment decision can be exposed in the following way.

The Modigliani-Miller theorem in action Although the theorem allows for risk, we here ignore risk. Let the real debt of the firm be denoted B_t and the real dividends, X_t. We then have the accounting relationship

$$\dot{B}_t = X_t - (F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t - r_t B_t).$$

A positive X_t represents dividends in the usual meaning (payout to the owners of the firm), whereas a negative X_t can be interpreted as emission of new shares of stock. Since we assume perfect competition, the time path of w_t and r_t is exogenous to the firm.

Consider first the firm's combined financing and production-investment problem, which we call *Problem I.* Assume (realistically) that those who own the firm at time 0 want it to maximize its net worth, i.e., the present value of expected future dividends:

$$\begin{aligned}
\max_{(L_t, I_t, X_t)_{t=0}^{\infty}} \tilde{V}_0 &= \int_0^{\infty} X_t e^{-\int_0^t r_s ds} dt \quad \text{s.t.} \\
L_t &\geq 0, I_t \text{ free}, \\
\dot{K}_t &= I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \quad K_t \geq 0 \text{ for all } t, \\
\dot{B}_t &= X_t - (F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t - r_t B_t), \\
& \quad \text{where } B_0 \text{ is given}, \qquad (14.38) \\
\lim_{t \to \infty} B_t e^{-\int_0^t r_s ds} &\leq 0. \qquad \text{(NPG)}
\end{aligned}$$

The last constraint is the firm's No-Ponzi-Game condition, saying that a positive debt should in the long run at most grow at a rate which is *less* than the interest rate.

In Section 14.1 we considered another problem, namely a separate investment-production problem:

$$\begin{aligned}
\max_{(L_t, I_t)_{t=0}^{\infty}} V_0 &= \int_0^{\infty} R_t e^{-\int_0^t r_s ds} dt \quad \text{s.t.}, \\
R_t &\equiv F(K_t, L_t) - G(I_t, K_t) - w_t L_t - I_t, \\
L_t &\geq 0, I_t \text{ free}, \\
\dot{K}_t &= I_t - \delta K_t, \qquad K_0 > 0 \text{ given}, \quad K_t \geq 0 \text{ for all } t.
\end{aligned}$$

Let this problem, where the financing aspects are ignored, be called *Problem II.* When considering the relationship between Problem I and Problem II, the following mathematical fact is useful.

LEMMA A1 Consider a continuous function $a(t)$ and a differentiable function $f(t)$. Then

$$\int_{t_0}^{t_1} (f'(t) - a(t)f(t)) e^{-\int_{t_0}^t a(s)ds} dt = f(t_1) e^{-\int_{t_0}^{t_1} a(s)ds} - f(t_0).$$

Proof. Integration by parts from time t_0 to time t_1 yields

$$\int_{t_0}^{t_1} f'(t) e^{-\int_{t_0}^t a(s)ds} dt = f(t) e^{-\int_{t_0}^t a(s)ds} \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} f(t) a(t) e^{-\int_{t_0}^t a(s)ds} dt.$$

Hence,

$$\int_{t_0}^{t_1} (f'(t) - a(t)f(t))e^{-\int_{t_0}^{t} a(s)ds} dt$$
$$= f(t_1)e^{-\int_{t_0}^{t_1} a(s)ds} - f(t_0). \qquad \square$$

CLAIM 1 If $(K_t^*, B_t^*, L_t^*, I_t^*, X_t^*)_{t=0}^{\infty}$ is a solution to Problem I, then $(K_t^*, L_t^*, I_t^*)_{t=0}^{\infty}$ is a solution to Problem II.

Proof. By (14.38) and the definition of R_t, $X_t = R_t + \dot{B}_t - r_t B_t$ so that

$$\tilde{V}_0 = \int_0^{\infty} X_t e^{-\int_0^t r_s ds} dt = V_0 + \int_0^{\infty} (\dot{B}_t - r_t B_t) e^{-\int_0^t r_s ds} dt. \tag{14.39}$$

In Lemma A1, let $f(t) = B_t$, $a(t) = r_t$, $t_0 = 0$, $t_1 = T$ and consider $T \to \infty$. Then

$$\lim_{T\to\infty} \int_0^T (\dot{B}_t - r_t B_t) e^{-\int_0^t r_s ds} dt = \lim_{T\to\infty} B_T e^{-\int_0^T r_s ds} - B_0 \leq -B_0,$$

where the weak inequality is due to (NPG). Substituting this into (14.39), we see that maximum of net worth $\tilde{V}_0$ is obtained by maximizing V_0 and ensuring $\lim_{T\to\infty} B_T e^{-\int_0^T r_s ds} = 0$, in which case net worth equals ((maximized V_0) $- B_0$), where B_0 is given. So a plan that maximizes net worth of the firm must also maximize V_0 in Problem II. $\square$

In view of Claim 1, it does not matter for the firm's production and investment behavior whether the firm's investment is financed by issuing new debt or by issuing shares of stock. Moreover, if we assume investors do not care about whether they receive the firm's earnings in the form of dividends or valuation gains on the shares, the firm's dividend policy is also irrelevant. Hence, from now on we can concentrate on the investment-production problem, Problem II above.

The case with no capital installation costs Suppose the firm has no capital installation costs. Then the cash flow reduces to $R_t = F(K_t, L_t) - w_t L_t - I_t$.

CLAIM 2 When there are no capital installation costs, Problem II can be reduced to a series of static profit maximization problems.

Proof. Current (pure) profit is defined as

$$\Pi_t = F(K_t, L_t) - w_t L_t - (r_t + \delta)K_t \equiv \Pi(K_t, L_t).$$

It follows that R_t can be written

$$R_t = F(K_t, L_t) - w_t L_t - (\dot{K}_t + \delta K_t) = \Pi_t + (r_t + \delta)K_t - (\dot{K}_t + \delta K_t). \quad (14.40)$$

Hence,

$$V_0 = \int_0^\infty \Pi_t e^{-\int_0^t r_s ds} dt + \int_0^\infty (r_t K_t - \dot{K}_t) e^{-\int_0^t r_s ds} dt. \quad (14.41)$$

The first integral on the right-hand side of this expression is independent of the second. Indeed, the firm can maximize the first integral by *renting* capital and labor, K_t and L_t, at the going factor prices, $r_t + \delta$ and w_t, respectively, such that $\Pi_t = \Pi(K_t, L_t)$ is maximized at each t. The factor costs are accounted for in the definition of Π_t.

The second integral on the right-hand side of (14.41) is the present value of net revenue from renting capital out to others. In Lemma A1, let $f(t) = K_t$, $a(t) = r_t$, $t_0 = 0$, $t_1 = T$ and consider $T \to \infty$. Then

$$\lim_{T\to\infty} \int_0^T (r_t K_t - \dot{K}_t) e^{-\int_0^t r_s ds} dt = K_0 - \lim_{T\to\infty} K_T e^{-\int_0^T r_s ds} = K_0, \quad (14.42)$$

where the last equality comes from the fact that maximization of V_0 requires maximization of the left-hand side of (14.42) which in turn, since K_0 is given, requires minimization of $\lim_{T\to\infty} K_T e^{-\int_0^T r_s ds}$. The latter expression is always non-negative and can be made zero by choosing any time path for K_t such that $\lim_{T\to\infty} K_T = 0$. (We may alternatively put it this way: it never pays the firm to accumulate costly capital so fast in the long run that $\lim_{T\to\infty} K_T e^{-\int_0^T r_s ds} > 0$, that is, to maintain accumulation of capital at a rate equal to or higher than the interest rate.) Substituting (14.42) into (14.41), we get $V_0 = \int_0^\infty \Pi_t e^{-\int_0^t r_s ds} dt + K_0$.

The conclusion is that, given K_0,[12] V_0 is maximized if and only if K_t and L_t are at each t chosen such that $\Pi_t = \Pi(K_t, L_t)$ is maximized. □

The case with strictly convex capital installation costs Now we reintroduce the capital installation cost function $G(I_t, K_t)$, satisfying in particular the condition $G_{II}(I, K) > 0$ for all (I, K). Then, as shown in the text, the firm adjusts to a change in its environment, say a downward shift in r, by a *gradual*

[12] Note that in the absence of capital installation costs, the historically given K_0 is no more "given" than the firm may instantly let it jump to a lower or higher level. In the first case the firm would immediately sell a bunch of its machines and in the latter case it would immediately buy a bunch of machines. Indeed, without convex capital installation costs nothing rules out jumps in the capital stock. But such jumps just reflect an immediate jump, in the opposite direction, in another asset item in the balance sheet and leave the maximized net worth of the firm unchanged.

adjustment of K, in this case upward, rather than attempting an instantaneous maximization of $\Pi(K_t, L_t)$. The latter would entail an instantaneous upward jump in K_t of size $\Delta K_t = a > 0$, requiring $I_t \cdot \Delta t = a$ for $\Delta t = 0$. This would require $I_t = \infty$, which implies $G(I_t, K_t) = \infty$, which may interpreted either as such a jump being impossible or at least so costly that no firm will pursue it.

Proof that q_t satisfies (14.15) along an interior optimal path Rearranging (14.11) and multiplying through by the integrating factor $e^{-\int_0^t (r_s+\delta)ds}$, we get

$$[(r_t + \delta)q_t - \dot{q}_t]\, e^{-\int_0^t (r_s+\delta)ds} = (F_{Kt} - G_{Kt})\, e^{-\int_0^t (r_s+\delta)ds}, \tag{14.43}$$

where $F_{Kt} \equiv F_K(K_t, L_t)$ and $G_{Kt} \equiv G_K(I_t, K_t)$. In Lemma A1, let $f(t) = q_t$, $a(t) = r_t + \delta$, $t_0 = 0$, $t_1 = T$. Then

$$\begin{aligned}\int_0^T [(r_t + \delta)q_t - \dot{q}_t]\, e^{-\int_0^t (r_s+\delta)ds} dt &= q_0 - q_T e^{-\int_0^T (r_s+\delta)ds} \\ &= \int_0^T (F_{Kt} - G_{Kt})\, e^{-\int_0^t (r_s+\delta)ds} dt,\end{aligned}$$

where the last equality comes from (14.43). Letting $T \to \infty$, we get

$$q_0 - \lim_{T\to\infty} q_T e^{-\int_0^T (r_s+\delta)ds} = q_0 = \int_0^\infty (F_{Kt} - G_{Kt})\, e^{-\int_0^t (r_s+\delta)ds} dt, \tag{14.44}$$

where the first equality follows from the transversality condition (14.14), which we repeat here:

$$\lim_{t\to\infty} q_t e^{-\int_0^t r_s ds} = 0. \tag{*}$$

Indeed, since $\delta \geq 0$, $\lim_{T\to\infty}(e^{-\int_0^T r_s ds} e^{-\delta T}) = 0$, when (*) holds. Initial time is arbitrary, and so we may replace 0 and t in (14.44) by t and τ, respectively. The conclusion is that (14.15) holds along an interior optimal path, given the transversality condition (*). A proof of necessity of the transversality condition (*) is given in Appendix B.[13]

B. Transversality conditions

In view of (14.44), a qualified conjecture is that the condition $\lim_{t\to\infty} q_t e^{-\int_0^t (r_s+\delta)ds}$ $= 0$ is necessary for optimality. This is indeed true, since this condition follows from the stronger transversality condition (*) in Appendix A, the necessity of which along an optimal path we will now prove.

[13] An equivalent approach to derivation of (14.15) can be based on applying the transversality condition (*) to the general solution formula for linear inhomogeneous first-order differential equations. Indeed, the first-order condition (14.11) provides such a differential equation in q_t.

Proof of necessity of (14.14) As the transversality condition (14.14) is the same as (*) in Appendix A, from now we refer to (*).

Rearranging (14.11) and multiplying through by the integrating factor $e^{-\int_0^t r_s ds}$, we have

$$(r_t q_t - \dot{q}_t)e^{-\int_0^t r_s ds} = (F_{Kt} - G_{Kt} - \delta q_t)\, e^{-\int_0^t r_s ds}.$$

In Lemma A1, let $f(t) = q_t$, $a(t) = r_t$, $t_0 = 0$, $t_1 = T$. Then

$$\int_0^T (r_t q_t - \dot{q}_t)e^{-\int_0^t r_s ds} dt = q_0 - q_T e^{-\int_0^T r_s ds} = \int_0^T (F_{Kt} - G_{Kt} - \delta q_t)\, e^{-\int_0^t r_s ds} dt.$$

Rearranging and letting $T \to \infty$, we see that

$$q_0 = \int_0^\infty (F_{Kt} - G_{Kt} - \delta q_t)\, e^{-\int_0^t r_s ds} dt + \lim_{T\to\infty} q_T e^{-\int_0^T r_s ds}. \tag{14.45}$$

If, contrary to (*), $\lim_{T\to\infty} q_T e^{-\int_0^T r_s ds} > 0$ along the optimal path, then (14.45) shows that the firm is over-investing. By reducing initial investment by one unit, the firm would save approximately $1 + G_I(I_0, K_0) = q_0$, by (14.10), which would be more than the present value of the stream of potential net gains coming from this marginal unit of installed capital (the first term on the right-hand side of (14.45)).

Suppose instead that $\lim_{T\to\infty} q_T e^{-\int_0^T r_s ds} < 0$. Then, by a symmetric argument, the firm has under-invested initially.

Necessity of (14.12) In cases where along an optimal path, K_t remains bounded from above for $t \to \infty$, the transversality condition (14.12) is implied by (*). In cases where along an optimal path, K_t is not bounded from above for $t \to \infty$, the transversality condition (14.12) is stronger than (*). A proof of the necessity of (14.12) in this case can be based on Weitzman (2003) and Long and Shimomura (2003).

C. On different specifications of the q-theory

The simple relationship we have found between I and q can easily be generalized to the case where the purchase price on the investment good, p_{It}, is allowed to differ from 1 (its value in the main text) and the capital installation cost is $p_{It}G(I_t, K_t)$. In this case it is convenient to replace q in the Hamiltonian function by, say, λ. Then the first-order condition (14.10) becomes $p_{It} + p_{It}G_I(I_t, K_t) = \lambda_t$, implying

$$G_I(I_t, K_t) = \frac{\lambda_t}{p_{It}} - 1,$$

and we can proceed, defining as before q_t by $q_t \equiv \lambda_t / p_{It}$.

Sometimes in the literature installation costs, J, appear in a somewhat different form than in the above exposition. But applied to a model with economic growth this will result in installation costs that rise faster than output and ultimately swallow the total produce.

Abel and Blanchard (1983), followed by Barro and Sala-i-Martin (2004, p. 152-160), introduce a function, ϕ, representing capital installation costs *per unit of investment* as a function of the investment-capital ratio. That is, total installation cost is $J = \phi(I/K)I$, where $\phi(0) = 0, \phi' > 0$. This implies that $J/K = \phi(I/K)(I/K)$. The right-hand side of this equation may be called $g(I/K)$, and then we are back at the formulation in Section 14.1. Indeed, defining $x \equiv I/K$, we have installation costs per unit of capital equal to $g(x) = \phi(x)x$, and assuming $\phi(0) = 0, \phi' > 0$, it holds that

$$\begin{aligned} g(x) &= 0 \text{ for } x = 0, \ g(x) > 0 \text{ for } x \neq 0, \\ g'(x) &= \phi(x) + x\phi'(x) \gtreqless 0 \text{ for } x \gtreqless 0, \text{ respectively, and} \\ g''(x) &= 2\phi'(x) + x\phi''(x). \end{aligned}$$

Clearly, $g''(x)$ must be positive for the theory to work. But the assumptions $\phi(0) = 0, \phi' > 0$, and $\phi'' \geq 0$, imposed in p. 153 and again in p. 154 in Barro and Sala-i-Martin (2004), are *not* sufficient for this (as $x < 0$ is possible). Since in macroeconomics $x < 0$ is seldom, this is a minor point, however.

It is sometimes convenient to let the capital installation cost $G(I, K)$ appear, not as a reduction in output, but as a reduction in capital formation so that

$$\dot{K} = I - \delta K - G(I, K). \tag{14.46}$$

This approach is used in Hayashi (1982) and Heijdra and Ploeg (2002, p. 573 ff.; see also Uzawa (1969). For example, Heijdra and Ploeg write the rate of capital accumulation as $\dot{K}/K = \psi(I/K) - \delta$, where the "capital installation function" $\psi(I/K)$ has the properties $\psi' > 0$ and $\psi'' < 0$ and can be interpreted as $\psi(I/K) \equiv [I - G(I, K)]/K = I/K - g(I/K)$. The latter equality comes from assuming G is homogeneous of degree 1. To maintain the likely desired property that $\psi'(I/K) > 0$ (though small) even for large I/K, the G function should not be "too convex". That is, for instance $g(I/K) = (\beta/2)(I/K)^2$ would *not* do.

In one-sector models, as we usually consider in this text, this need not change anything of importance. In more general models the installation function approach (14.46) may have some analytical advantages; what gives the best fit empirically is an open question.

Finally, some analysts assume that installation costs are a strictly convex function of *net* investment, $I - \delta K$. This agrees well with intuition if mere replacement investment occurs in a smooth way not involving new technology, work interruption, and reorganization. To the extent capital investment involves indivisibilities and embodies new technology, it may seem more plausible to specify the installation costs as a convex function of *gross* investment.

D. Proof of Hayashi's theorem

For convenience we repeat:

THEOREM (Hayashi) Assume the firm is a price taker, that the production function F is jointly concave in (K, L), and that the installation cost function G is jointly convex in (I, K). Then, along the optimal path we have:

(i) $q_t^m = q_t^a$ for all $t \geq 0$, if F and G are homogeneous of degree 1.

(ii) $q_t^m < q_t^a$ for all t, if F is strictly concave in (K, L) and/or G is strictly convex in (I, K).

Proof. The value of the firm as seen from time t is

$$V_t = \int_t^\infty (F(K_\tau, L_\tau) - G(I_\tau, K_\tau) - w_\tau L_\tau - I_\tau)e^{-\int_t^\tau r_s ds} d\tau. \tag{14.47}$$

We introduce the functions

$$A = A(K, L) \equiv F(K, L) - F_K(K, L)K - F_L(K, L)L, \tag{14.48}$$

$$B = B(I, K) \equiv G_I(I, K)I + G_K(I, K)K - G(I, K). \tag{14.49}$$

Then the cash-flow of the firm at time τ can be written

$$\begin{aligned} R_\tau &= F(K_\tau, L_\tau) - F_{L\tau}L_\tau - G(I_\tau, K_\tau) - I_\tau \\ &= A(K_\tau, L_\tau) + F_{K\tau}K_\tau + B(I_\tau, K_\tau) - G_{I\tau}I_\tau - G_{K\tau}K_\tau - I_\tau, \end{aligned}$$

where we have used first $F_{L_\tau} = w$ and then the definitions of A and B above. Consequently, when moving along the optimal path,

$$\begin{aligned} V_t &= V^*(K_t, t) = \int_t^\infty (A(K_\tau, L_\tau) + B(I_\tau, K_\tau))\, e^{-\int_t^\tau r_s ds} d\tau \qquad (14.50) \\ &\quad + \int_t^\infty [(F_{K\tau} - G_{K\tau})K_\tau - (1 + G_{I\tau})I_\tau]e^{-\int_t^\tau r_s ds} d\tau \\ &= \int_t^\infty (A(K_\tau, L_\tau) + B(I_\tau, K_\tau))e^{-\int_t^\tau r_s ds} d\tau + q_t K_t, \end{aligned}$$

cf. Lemma D1 below. Isolating q_t, it follows that

$$q_t^m \equiv q_t = \frac{V_t}{K_t} - \frac{1}{K_t}\int_t^\infty [A(K_\tau, L_\tau) + B(I_\tau, K_\tau)]e^{-\int_t^\tau r_s ds} d\tau, \tag{14.51}$$

when moving along the optimal path.

Since F is concave and $F(0,0) = 0$, we have for all K and L, $A(K,L) \geq 0$ with equality sign, if and only if F is homogeneous of degree one. Similarly, since G is convex and $G(0,0) = 0$, we have for all I and K, $B(I,K) \geq 0$ with equality sign, if and only if G is homogeneous of degree one. Now the conclusions (i) and (ii) follow from (14.51) and the definition of q^a in (14.27). □

LEMMA D1 The last integral on the right-hand side of (14.50) equals $q_t K_t$, when investment follows the optimal path.

Proof. We want to characterize a given optimal path $(K_\tau, I_\tau, L_\tau)_{\tau=t}^{\infty}$. Keeping t fixed and using z as our varying time variable, we have

$$(F_{Kz} - G_{Kz})K_z - (1 + G_{Iz})I_z = [(r_z + \delta)q_z - \dot{q}_z]K_z - (1 + G_{Iz})I_z$$

$$= [(r_z + \delta)q_z - \dot{q}_z]K_z - q_z(\dot{K}_z + \delta K_z) = r_z q_z K_z - (\dot{q}_z K_z + q_z \dot{K}_z) = r_z u_z - \dot{u}_z,$$

where we have used (14.11), (14.10), (14.6), and the definition $u_z \equiv q_z K_z$. We look at this as a differential equation: $\dot{u}_z - r_z u_z = \varphi_z$, where $\varphi_z \equiv -[(F_{Kz} - G_{Kz})K_z - (1 + G_{Iz})I_z]$ is considered as some given function of z. The solution of this linear differential equation is

$$u_z = u_t e^{\int_t^z r_s ds} + \int_t^z \varphi_\tau e^{\int_\tau^z r_s ds} d\tau,$$

implying, by multiplying through by $e^{-\int_t^z r_s ds}$, reordering, and inserting the definitions of u and φ,

$$\begin{aligned} &\int_t^z [(F_{K\tau} - G_{K\tau})K_\tau - (1 + G_{I\tau})I_\tau] e^{-\int_t^\tau r_s ds} d\tau \\ = \;& q_t K_t - q_z K_z e^{-\int_t^z r_s ds} \to q_t K_t \quad \text{for} \quad z \to \infty, \end{aligned}$$

from the transversality condition (14.12) with t replaced by z and 0 replaced by t. □

A different – and perhaps more illuminating – way of understanding (i) in Hayashi's theorem is the following.

Suppose F and G are homogeneous of degree one. Then $A = B = 0$, $G_I I + G_K K = G = g(I/K)K$, and $F_K = f'(k)$, where f is the production function in intensive form. Consider an optimal path $(K_\tau, I_\tau, L_\tau)_{\tau=t}^{\infty}$ and let $k_\tau \equiv K_\tau / L_\tau$ and $x_\tau \equiv I_\tau / K_\tau$ along this path which we now want to characterize. As the path is assumed optimal, from (14.47) follows

$$V_t = V^*(K_t, t) = \int_t^\infty [f'(k_\tau) - g(x_\tau) - x_\tau] K_\tau e^{-\int_t^\tau r_s ds} d\tau. \tag{14.52}$$

From $\dot{K}_t = (x_t - \delta)K_t$ follows $K_\tau = K_t e^{-\int_t^\tau (x_s - \delta)ds}$. Substituting this into (14.52) yields

$$V^*(K_t, t) = K_t \int_t^\infty [f'(k_\tau) - g(x_\tau) - x_\tau] e^{-\int_t^\tau (r_s - x_s + \delta)ds} d\tau.$$

In view of (14.24), with t replaced by τ, the optimal investment *ratio* x_τ depends, for all τ, only on q_τ, not on K_τ, hence not on K_t. Therefore,

$$\partial V^*/\partial K_t = \int_t^\infty [f'(k_\tau) - g(x_\tau) - x_\tau] e^{-\int_t^\tau (r_s - x_s + \delta)ds} d\tau = V_t/K_t.$$

Hence, from (14.28) and (14.27), we conclude $q_t^m = q_t^a$.

Remark. We have assumed throughout that G is strictly convex in I. This does not imply that G is jointly strictly convex in (I, K). For example, the function $G(I, K) = I^2/K$ is strictly convex in I (since $G_{II} = 2/K > 0$). But at the same time this function has $B(I, K) = 0$ and is therefore homogeneous of degree one. Hence, it is not jointly strictly convex in (I, K).

E. The slope of the $\dot{q} = 0$ locus in the SOE case

First, we shall determine the sign of the slope of the $\dot{q} = 0$ locus in the case $g + n = 0$, considered in Fig. 14.4. Taking the total differential in (14.33) w.r.t. K and q gives

$$\begin{aligned} 0 &= -F_{KK}(K, \bar{L})dK + \{r + \delta + g'(m(q))m'(q) - [m(q) + (q-1)m'(q)]\} dq \\ &= -F_{KK}(K, \bar{L})dK + [r + \delta - m(q)] dq, \end{aligned}$$

since $g'(m(q)) = q - 1$, by (14.23) and (14.24). Therefore

$$\frac{dq}{dK}_{|\dot{q}=0} = \frac{F_{KK}(K, \bar{L})}{r + \delta - m(q)} \quad \text{for } r + \delta \neq m(q).$$

From this it is not possible to sign dq/dK at all points along the $\dot{q} = 0$ locus. But in a neighborhood of the steady state we have $m(q) \approx \delta$, hence $r + \delta - m(q) \approx r > 0$. And since $F_{KK} < 0$, this implies that at least in a neighborhood of E in Fig. 14.4, the $\dot{q} = 0$ locus is negatively sloped.

Second, consider the case $g + n > 0$, illustrated in Fig. 14.6. Here we get in a similar way

$$\frac{dq}{d\tilde{k}}_{|\dot{q}=0} = \frac{f''(\tilde{k}^*)}{r + \delta - m(q)} \quad \text{for } r + \delta \neq m(q).$$

From this it is not possible to sign $dq/d\tilde{k}$ at all points along the $\dot{q} = 0$ locus. But in a small neighborhood of the steady state, we have $m(q) \approx \delta + \gamma + n$, hence $r + \delta - m(q) \approx r - \gamma - n > 0$, where the inequality was assumed in the text. Since $f'' < 0$, then, at least in a small neighborhood of E in Fig. 14.6, the $\dot{q} = 0$ locus is negatively sloped, when $r > \gamma + n$.

F. The divergent paths

Text not yet available.

14.7 Exercises

14.1 *(induced sluggish capital adjustment).* Consider a firm with capital installation costs $J = G(I, K)$, satisfying

$$G(0, K) = 0, \; G_I(0, K) = 0, \; G_{II}(I, K) > 0, \quad \text{and} \quad G_K(I, K) \leq 0.$$

a) Can we from this conclude anything as to strict concavity or strict convexity of the function G? If yes, with respect to what argument or arguments?

b) For two values of K, $\underline{K}$ and $\bar{K}$, illustrate graphically the capital installation costs J in the (I, J) plane. Comment.

c) By drawing a few straight line segments in the diagram, illustrate that $G(\frac{1}{2}I, \bar{K})2 < G(I, \bar{K})$ for any given $I > 0$.

14.2 (see end of Section 14.3)

Note 2.

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Uncertainty, expectations, and asset price bubbles

This lecture note provides a framework for addressing themes where expectations in *uncertain* situations are important elements. Our previous models have not taken seriously the problem of uncertainty. Where agent's expectations about future variables were involved and these expectations were assumed to be model-consistent ("rational"), we only considered a special case: perfect foresight. Shocks were treated in a peculiar (almost self-contradictory) way: they might occur, but only as a complete surprise, a one-off event. Agents' expectations and actions never incorporated that new shocks could arrive.

We will now allow recurrent shocks to take place. The environment in which the economic agents act will be considered inherently uncertain. How can this be modeled and how can we solve the resultant models? Since it is easier to model uncertainty in discrete rather than continuous time, we examine uncertainty and expectations in a discrete time framework.

Our emphasis will be on the hypothesis that when facing uncertainty a dominating fraction of the economic agents form "rational expectations" in the sense of making probabilistic forecasts which coincide with the forecast calculated on the basis of the "relevant economic model". But we begin with simple mechanistic expectation formation hypotheses that have been used to describe day-to-day expectations of people who do not think much about the probabilistic properties of their economic environment.

1 Simple expectation formation hypotheses

One simple supposition is that expectations change gradually to correct past expectation errors. Let P_t denote the general price level in period t and $\pi_t \equiv (P_t - P_{t-1})/P_{t-1}$ the corresponding inflation rate. Further, let $\pi^e_{t-1,t}$ denote the "subjective expectation", formed in period $t-1$, of π_t, i.e., the inflation rate from period $t-1$ to period t. We may think of the "subjective expectation" as the expected value in a vaguely defined subjective conditional probability distribution.

The hypothesis of *adaptive expectations* (the AE hypothesis) says that the expectation is revised in proportion to the past expectation error,

$$\pi^e_{t-1,t} = \pi^e_{t-2,t-1} + \lambda(\pi_{t-1} - \pi^e_{t-2,t-1}), \; 0 < \lambda \leq 1, \tag{1}$$

where the parameter λ is called the adjustment speed. If $\lambda = 1$, the formula reduces to

$$\pi^e_{t-1,t} = \pi_{t-1}. \tag{2}$$

This limiting case is known as *static expectations* or *myopic expectations;* the subjective expectation is that the inflation rate will remain the same. As we shall see, *if* inflation follows a random walk, this subjective expectation is in fact the "rational expectation".

We may write (1) on the alternative form

$$\pi^e_{t-1,t} = \lambda\pi_{t-1} + (1-\lambda)\pi^e_{t-2,t-1}. \tag{3}$$

This says that the expected value concerning this period (period t) is a weighted average of the actual value for the last period and the expected value for the last period. By backward substitution we find

$$\begin{aligned}
\pi^e_{t-1,t} &= \lambda\pi_{t-1} + (1-\lambda)[\lambda\pi_{t-2} + (1-\lambda)\pi^e_{t-3,t-2}] \\
&= \lambda\pi_{t-1} + (1-\lambda)\lambda\pi_{t-2} + (1-\lambda)^2[\lambda\pi_{t-3} + (1-\lambda)\pi^e_{t-4,t-3}] \\
&= \lambda\sum_{i=1}^{n}(1-\lambda)^{i-1}\pi_{t-i} + (1-\lambda)^n\pi^e_{t-n-1,t-n}.
\end{aligned}$$

Since $(1-\lambda)^n \to 0$ for $n \to \infty$, we have (for $\pi^e_{t-n-1,t-n}$ bounded as $n \to \infty$),

$$\pi^e_{t-1,t} = \lambda\sum_{i=1}^{\infty}(1-\lambda)^{i-1}\pi_{t-i}. \tag{4}$$

Thus, according to the AE hypothesis with $0 < \lambda < 1$, the expected inflation rate is a weighted average of the historical inflation rates back in time. The weights are geometrically declining with increasing time distance from the current period. The weights sum to one (in that $\sum_{i=1}^{\infty}\lambda(1-\lambda)^{i-1} = \lambda(1-(1-\lambda))^{-1} = 1$).

The formula (4) can be generalized to the *general backward-looking expectations* formula,

$$\pi^e_{t-1,t} = \sum_{i=1}^{\infty} w_i\pi_{t-1-i}, \qquad \text{where } \sum_{i=1}^{\infty} w_i = 1. \tag{5}$$

If the weights w_i in (5) satisfy $w_i = \lambda(1-\lambda)^{i-1}$, $i = 1, 2, \ldots$, we get the AE formula (4). If the weights are

$$w_1 = 1 + \beta,\ w_2 = -\beta,\ w_i = 0 \text{ for } i = 3, 4, \ldots,$$

we get

$$\pi^e_{t-1,t} = (1+\beta)\pi_{t-1} - \beta\pi_{t-2} = \pi_{t-1} + \beta(\pi_{t-1} - \pi_{t-2}). \tag{6}$$

This is called the hypothesis of *extrapolative expectations* and says:

if $\beta > 0$, then the recent direction of change in π is expected to continue;

if $\beta < 0$, then the recent direction of change in π is expected to be reversed;

if $\beta = 0$, then expectations are static as in (2).

As hinted, there *are* cases where for instance myopic expectations *are* "rational" (in a sense to be defined below). Exercise 1 provides an example. But in many cases purely backward-looking formulas are too rigid, too mechanistic. They will often lead to systematic expectation errors to one side or the other. It seems implausible that people should not then respond to their experience and revise their expectations formula. When expectations are about things that really matter for them, people are likely to listen to professional forecasters who build their forecasting on statistical or econometric *models.* Such models are based on a formal probabilistic framework, take the interaction between different variables into account, and incorporate new information about future possible events.

2 The rational expectations hypothesis

2.1 Preliminaries

We first recapitulate a few concepts from statistics. A sequence $\{X_t\}$ of random variables indexed by time is called a *stochastic process.* A stochastic process $\{X_t\}$ is called *white noise* if for all t, X_t has zero expected value, constant variance, and zero covariance across time.[1] A stochastic process $\{X_t\}$ is called a *first-order autoregressive process,* abbreviated AR(1), if $X_t = \beta_0 + \beta_1 X_{t-1} + \varepsilon_t$, where β_0 and β_1 are constants, and $\{\varepsilon_t\}$ is white noise.

[1] The expression white noise derives from electrotechnics. In electrotechnical systems signals will often be subject to noise. If this noise is arbitrary and has no dominating frequence, it looks like white light. The various colours correspond to a certain wave length, but white light is light which has all frequences (no dominating frequence).

If $|\beta_1| < 1$, then $\{X_t\}$ is called a *stationary* AR(1) process. A stochastic process $\{X_t\}$ is called a *random walk* if $X_t = X_{t-1} + \varepsilon_t$, where $\{\varepsilon_t\}$ is white noise.

Before defining the term rational expectation, it is useful to clarify a distinction between two ways in which expectations, whatever their nature, may enter a macroeconomic model.

2.1.1 Two model types

Type A: models with past expectations of current endogenous variables

Suppose a given macroeconomic model can be reduced to two equations, the first being

$$Y_t = a\, Y^e_{t-1,t} + c\, X_t, \qquad t = 0, 1, 2, ..., \tag{7}$$

where Y_t is some endogenous variable (not necessarily GDP), a and c are given constant coefficients, and X_t is an exogenous random variable which follows some specified stochastic process. In line with the notation from Section 1, $Y^e_{t-1,t}$ is the subjective expectation formed in period $t-1$, of the value of the variable Y in period t. The economic agents are in simple models assumed to have the same expectations. Or, at least there is a dominating expectation, $Y^e_{t-1,t}$, in the society. What the equation (7) claims is that the endogenous variable, Y_t, depends, in the specified linear way, on the "generally held" expectation of Y_t, formed in the previous period. It is natural to think of the outcome Y_t as being the aggregate result of agents' decisions and market mechanisms, the decisions being made at discrete points in time $\ldots, t-2, t-1, t, \ldots$, immediately after the uncertainty concerning the period in question is resolved.

The second equation specifies how the subjective expectation is formed. To fix ideas, let us assume myopic expectations,

$$Y^e_{t-1,t} = Y_{t-1}, \tag{8}$$

as in (2) above. A *solution* to the model is a stochastic process for Y_t such that (7) holds, given the expectation formation (8) and the stochastic process which X_t follows.

EXAMPLE 1 *(imported raw materials and the domestic price level)* Let the endogenous variable in (7) represent the domestic price level (the consumer price index) P_t, and let X_t be the price level of imported raw materials. Suppose the price level is determined through a markup on unit costs,

$$P_t = (\lambda W_t + \eta X_t)(1 + \mu), \qquad 0 < \lambda < \frac{1}{1+\mu}, \tag{*}$$

where W_t is the nominal wage level in period $t = 0, 1, 2, \ldots$, and λ and η are positive technical coefficients representing the assumed constant labor and raw materials requirements, respectively, per unit of output; μ is a constant markup. Assume further that workers in period $t-1$ negotiate next period's wage level, W_t, so as to achieve, in expected value, a certain target real wage which we normalize to 1, i.e.,

$$\frac{W_t}{P^e_{t-1,t}} = 1.$$

Inserting into (*), we have

$$P_t = a\ P^e_{t-1,t} + c\ X_t, \qquad 0 < \alpha = \lambda(1+\mu) < 1, 0 < c = \eta(1+\mu). \tag{9}$$

Suppose $X_t = \bar{x} + \varepsilon_t$, where $\bar{x}$ is a positive constant and $\{\varepsilon_t\}$ is white noise. Assuming myopic expectations,

$$P^e_{t-1,t} = P_{t-1}, \tag{10}$$

the solution for the evolution of the price level is

$$P_t = a\ P_{t-1} + c(\bar{x} + \varepsilon_t), \qquad t = 0, 1, 2, \ldots.$$

Without shocks, and starting from an arbitrary $P_{-1} > 0$, the time path of the price level would be $P_t = (P_{-1} - P^*)a^{t+1} + P^*$, where $P^* = c\bar{x}/(1-\alpha)$. Shocks to the price of imported raw materials result in transitory deviations from P^*. But as the shocks are only temporary and $|a| < 1$, the domestic price level gradually returns towards the constant level P^*. The intervening changes in wage demands in response to the changes in the price level changes prolong the time it takes to return to P^* in the absence of new shocks. □

Equation (7) can also be interpreted as a vector equation (such that Y_t and $Y^e_{t-1,t}$ are n-vectors, a is an $n \times n$ matrix, c an $n \times m$ matrix, and X an m-vector). The crucial feature is that the endogenous variables dated t *only* depend on previous expectations of date-t values of these variables and on the exogenous variables.

Models with past expectations of current endogenous variables will serve as our point of reference when introducing the concept of rational expectations below.

Type B: models with forward-looking expectations

Another way in which agents' expectations may enter is exemplified by

$$Y_t = a\ Y^e_{t,t+1} + c\ X_t, \qquad t = 0, 1, 2, ... \tag{11}$$

Here $Y^e_{t,t+1}$ is the subjective expectation, formed in period t, of the value of Y in period $t+1$. Example: the equity price today depends on what the equity price is expected to be tomorrow. Or more generally: the current expectation of a future value of an endogenous variable influences the current value of this variable. We name this the case of *forward-looking expectations*. (In "everyday language" also $Y_{t-1,t}^e$ in model type 1 can be said to be a forward-looking variable as seen from period $t-1$. But the dividing line between the two model types, (7) and (11), is whether *current* expectations of future values of the endogenous variables do or do not influence the current values of these.)

The complete model with forward-looking expectations will include an additional equation, specifying how the subjective expectation, $Y_{t,t+1}^e$, is formed. We might again impose myopic expectations, $Y_{t,t+1}^e = Y_t$. A *solution* to the model is a stochastic process for Y_t satisfying (11), given the stochastic process followed by X_t and given the specified expectation formation and perhaps some additional restrictions in the form of boundary conditions or similar. The case of forward-looking expectations is important in connection with many topics in macroeconomics, including the evolution of asset prices and issues of asset price bubbles. This case will be dealt with in sections 3 and 4 below.

In passing we note that in both model type 1 and model type 2, it is the mean (in the subjective probability distribution) of the random variable(s) that enters. This is typical of simple macroeconomic models which often ignore other measures such as the median, mode, or higher-order moments. The latter, say the variance of X_t, may be included in more advanced models where for instance behavior towards risk is important.

2.1.2 The concept of a model-consistent expectation

The concepts of a *rational expectation* and *model-consistent expectation* are closely related, but not the same. We start with the latter.

Let there be given a stochastic model represented by (7) combined with some given expectation formation (8), say. We put ourselves in the position of the investigator or model builder and ask what the *model-consistent expectation* of the endogenous variable Y_t is as seen from period $t-1$. It is the mathematical *conditional expectation* that can be calculated on the basis of the model and available relevant data revealed up to and including period $t-1$. Let us denote this expectation

$$E(Y_t|I_{t-1}), \tag{12}$$

where E is the expectation operator and I_{t-1} denotes the information available at time $t-1$. We think of period $t-1$ as the half-open time interval $[t-1,t)$ and imagine that the uncertainty concerning the exogenous random variable X_{t-1} is resolved at time $t-1$.

So I_{t-1} includes knowledge of X_{t-1} and thereby, via the model, also of Y_{t-1}.

The information I_{t-1} may comprise knowledge of the realized values of X and Y up until and including period $t-1$. Instead of (12) we could, for instance, write

$$E(Y_t|Y_{t-1} = y_{t-1}, \ldots, Y_{t-n} = y_{t-n}; X_{t-1} = x_{t-1}, \ldots, X_{t-n} = x_{t-n}).$$

Here information (some of which may be redundant) goes back to a given initial period, say period 0, in which case n equals t. Alternatively, perhaps information goes back to "ancient times", possibly represented by $n = \infty$. Anyway, as time proceeds, in general more and more realizations of the exogenous and endogenous variables become known and in *this* sense the information I_{t-1} *expands* with rising t. The information I_{t-1} may also be interpreted as "partial lack of uncertainty", so that an "increasing amount of information" and "reduced uncertainty" are seen as two sides of the same thing. The "reduced uncertainty" lies in the fact that the space of *possible* time paths $\{(X_t, Y_t)\}_{t-n}^{t+T}$ as of time t *shrinks* as time proceeds (T denotes the time horizon as seen from time t).[2] Indeed, this space shrinks precisely because more and more realizations of the variables take place (more information appears) and thereby rule out an increasing subset of paths that were earlier possible.[3]

In Example 1, as long as the subjective expectation is the myopic expectation (10), the model-consistent expectation is

$$E(P_t|I_{t-1}) = a\ P_{t-1} + c\bar{x}.$$

Inserting the investigator's estimated values of the coefficients a and c, the investigator's forecast of P_t is obtained.

2.2 The rational expectations hypothesis

Unsatisfied with mechanistic formulas like those of Section 1, the American economist John F. Muth (1961) introduced a radically different approach, the hypothesis of *rational expectations.* Muth stated the hypothesis the following way:

> I should like to suggest that expectations, since they are informed predictions of future events, are essentially the same as the predictions of the relevant

[2] By "possible" is meant "ex ante feasible according to a given model".

[3] We refer to I_{t-1} as the "available information" rather than the "information set" which is an alternative term used in the literature. The latter term is tricky because, as we have just exemplified, it is ambiguous what is meant by a "larger information set". Moreover, the term "information set" has different meanings in different branches of economics, hence we are hesitant to use it. More about subtleties relating to "information" in Appendix B, dealing with mathematical conditional expectations in general.

economic theory. At the risk of confusing this purely descriptive hypothesis with a pronouncement as to what firms ought to do, we call such expectations 'rational' (Muth 1961).

Muth applied this hypothesis to simple microeconomic problems. The hypothesis was subsequently extended and applied to general equilibrium theory and macroeconomics by what since the early 1970s became known as the New Classical Macroeconomics school. Nobel laureate Robert E. Lucas from the University of Chicago lead the way by a series of papers starting with Lucas (1972) and Lucas (1973). Assuming rational expectations in a model instead of, for instance, adaptive expectations may radically change the dynamics as well as the impact of economic policy.

2.2.1 The concept of rational expectations

Assuming the economic agents have *rational expectations* (RE) is to assume that their subjective expectation equals the model-consistent expectation, that is, the mathematical conditional expectation that can be calculated on the basis of the model and available relevant information about the exogenous stochastic variables. In connection with the model ingredient (7), assuming the agents have rational expectations thus means that

$$Y_{t-1,t}^{e} = E(Y_t|I_{t-1}), \tag{13}$$

i.e., agents' subjective conditional expectation coincides with the "objective" or "true" conditional expectation, given the model (7).

Together, the equations (7) and (13) constitute a simple *rational expectations model* (henceforth an RE model). We may write the model in compact form as

$$Y_t = aE(Y_t|I_{t-1}) + c\ X_t, \qquad t = 0, 1, 2, ... \tag{14}$$

The assumption of rational expectations thus relies on idealized conditions.

2.2.2 Solving a simple RE model

To solve the model means to find the stochastic process followed by Y_t, given the stochastic process followed by the exogenous variable X_t. For a linear RE model with past expectations of current endogenous variables, the solution procedure is the following.

1. By substitution, reduce the RE model (or the relevant part of the model) into a form like (14) expressing the endogenous variable in period t in terms of its past

expectation and the exogenous variable(s). (The case with multiple endogenous variables is treated similarly.)

2. Take the conditional expectation on both sides of the equation and solve for the conditional expectation of the endogenous variable.

3. Insert into the "reduced form" attained at 1.

In practice there is often a fourth step, namely to express *other* endogenous variables in the model in terms of those found in step 3. Let us see how the procedure works by way of the following example.

EXAMPLE 2 We modify Example 1 by replacing myopic expectations by rational expectations, i.e., (10) is replaced by $P^e_{t-1,t} = E(P_t|I_{t-1})$. Now "available information" includes that the subjective expectations are rational expectations. Step 1:

$$P_t = aE(P_t|I_{t-1}) + c\ X_t, \qquad 0 < \alpha < 1, c > 0. \tag{15}$$

Step 2: $E(P_t|I_{t-1}) = aE(P_t|I_{t-1}) + c\bar{x}$, implying

$$E(P_t\,|I_{t-1}) = c\frac{\bar{x}}{1-a}.$$

Step 3: Insert into (15) to get

$$P_t = c\frac{a\bar{x}}{1-a} + c(\bar{x} + \varepsilon_t).$$

This is the solution of the model in the sense of a specification of the stochastic process followed by P_t.

To compare with myopic expectations, suppose the event $\varepsilon_t \neq 0$ is relatively seldom and that at $t = 0,\ 1,\ ...,\ t_0 - 1$, it so happens that $\varepsilon_t = 0$, hence $P_t = c\bar{x}/(1-a) \equiv P^*$. Then, at $t = t_0$, $\varepsilon_{t_0} > 0$, so that $P_{t_0} = P^* + c\varepsilon_{t_0} > P^*$. But for $t = t_0 + 1,\ t_0 + 2,\ ...,\ t_0 + n$ there is again a sequence of periods with $\varepsilon_t = 0$. Then, under RE, domestic price level returns to P^* already in period $t_0 + 1$.

With myopic expectations, combined with $P_{-1} = P^*$, say, the positive shock to import prices at $t = t_0$ will imply $P_{t_0} = aP^* + c(\bar{x} + \varepsilon_{t_0}) = P^* + c\varepsilon_{t_0}$, $P_{t_0+1} = a(P^* + c\varepsilon_t) + c\bar{x}$ $= P^* + ac\varepsilon_t$, $P_{t_0+i} = P^* + a^i c\varepsilon_t$ for $i = 1, 2, ..., n$. After t_0 there is a systematic positive forecast error. This is because the mechanical expectation does not consider how the economy really functions. □

Returning to the general form (14), without specifying the process $\{X_t\}$, the second step gives

$$E(Y_t \,|I_{t-1}) = c\frac{E(X_t \,|I_{t-1})}{1-a}, \tag{16}$$

when $a \neq 1$.[4] Then, in the third step we get

$$Y_t = c\frac{aE(X_t \,|I_{t-1}) + (1-a)X_t}{1-a} = c\frac{X_t - a(X_t - E(X_t \,|I_{t-1}))}{1-a}. \tag{17}$$

For instance, let X_t follow the process $X_t = \bar{x} + \rho X_{t-1} + \varepsilon_t$, where $0 < \rho < 1$ and ε_t has zero expected value, given all observed past values of X and Y. Then (17) yields the solution

$$Y_t = c\frac{X_t - a\varepsilon_t}{1-a} = c\frac{\bar{x} + \rho X_{t-1} + (1-a)\varepsilon_t}{1-a}, \qquad t = 0, 1, 2, ...$$

In Exercise 2 you are asked to solve a simple Keynesian model of this form and compare the solution under rational expectations with the solution under static expectations.

Rational expectations should be viewed as a simplifying assumption that at best offers an approximation. *First*, the assumption entails essentially that the economic agents share one and the same understanding about how the economic system functions (and in this chapter they also share one and the same information, I_{t-1}). This is already a big mouthful. *Second*, this perception is assumed to *comply with the model* of the informed economic specialist. *Third*, this model is supposed to be the *true* model of the economic process, including the true parameter values as well as the true stochastic process which X_t follows. Indeed, by equalizing $Y^e_{t-1,t}$ with the true conditional expectation, $E(Y_t|I_{t-1})$, and not at most some econometric estimate of this, it is presumed that agents know the true values of the parameters a and c in the data-generating process which the model is supposed to mimic. In practice it is not possible to attain such precise knowledge, at least not unless the considered economic system has reached some kind of steady state and no structural changes occur (a condition which is hardly ever satisfied in macroeconomics).

Nevertheless, a model based on the rational expectations hypothesis can in many contexts be seen as a useful cultivation of a theoretical research question. The results that emerge cannot be due to *systematic* expectation errors from the economic agents' side. In this sense the assumption of rational expectations makes up a theoretically interesting *benchmark case*.

We shall stick to the term "rational expectation" because it is standard. The term can easily be misunderstood, however. Usually, in economists' terminology "rational"

[4] If $a = 1$, the model (14) is inconsistent unless $E(X_t \,|I_{t-1})) = 0$ in which case there are multiple solutions. Indeed, for any number $k \in (-\infty, +\infty)$, the process $Y_t = k + cX_t$ solves the model when $E(X_t \,|I_{t-1}) = 0$.

refers to behavior based on optimization subject to the constraints faced by the agent. So one might think that the RE hypothesis stipulates that economic agents try to get the most out of a situation with limited information, contemplating the benefits and costs of gathering more information and using adequate statistical estimation methods. But this is a misunderstanding. The RE hypothesis presumes that the true model is already known to the agents. The "rationality" refers to taking this assumed knowledge fully into account in the chosen actions.

2.2.3 The forecast error*

Let the forecast of some variable Y one period ahead be denoted $Y^e_{t-1,t}$. Suppose the forecast is determined by some given function, f , of realizations of Y and X up to and including period $t-1$, that is, $Y^e_{t-1,t} = f(y_{t-1},\, y_{t-2}, ...,\, x_{t-1},\, x_{t-2}, ...)$. Such a function is known as a *forecast function*. It might for instance be one of the mechanistic forecasting principles in Section 1. At the other extreme the forecast function might, at least theoretically, coincide with the a model-consistent conditional expectation. In the latter case it is a *model-consistent forecast function* and we can write

$$\begin{aligned} & f(y_{t-1}, y_{t-2}, ..., x_{t-1}, x_{t-2}, ...) = E(Y_t \,|I_{t-1}) \qquad (18) \\ = \; & E(Y_t \,|Y_{t-1} = y_{t-1}, Y_{t-2} = y_{t-2}, ..., x_{t-1} = x_{t-1}, x_{t-2} = x_{t-2}, ...)\,. \end{aligned}$$

The *forecast error* is the difference between the actually occurring future value, Y_t, of a variable and the forecasted value. So, for a given forecast, $Y^e_{t-1,t}$, the *forecast error* is $e_t \equiv Y_t - Y^e_{t-1,t}$ and is itself a stochastic variable.

If the forecast function in (18) complies with the true data-generating process (a big "if"), then the implied forecasts would have several ideal properties:

(a) the forecast error would have zero mean;

(b) the forecast error would be uncorrelated with any of the variables in the information I_{t-1} and therefore also with its own past values; and

(c) the expected squared forecast error would be minimized.

To see these properties, note that the model-consistent forecast error is $e_t = Y_t - E(Y_t \,|I_{t-1})$. From this follows that $E(e_t \,|I_{t-1}) = 0$, cf. (a). Also the unconditional expectation is nil, i.e., $E(e_t) = 0$. This is because $E(E(e_t \,|I_{t-1})) = E(0) = 0$ at the same time as

$E(E(e_t \,|I_{t-1})) = E(e_t)$, by the *law of iterated expectations* from statistics saying that the unconditional expectation of the conditional expectation of a stochastic variable Z is given by the unconditional expectation of Z, cf. Appendix B. Considering the specific model (7), the model-consistent-forecast error is $e_t = Y_t - E(Y_t \,|I_{t-1}) = c(X_t - E(X_t \,|I_{t-1}))$, by (16) and (17). An ex post error ($e_t \neq 0$) thus emerges if and only if the realization of the exogenous variable deviates from its conditional expectation as seen from the previous period.

As to property (b), for $i = 1, 2, ...$, let s_{t-i} be some variable value belonging to the information I_{t-i}. Then, property (b) is the claim that the (unconditional) covariance between e_t and s_{t-i} is zero, i.e., $\mathrm{Cov}(e_t s_{t-i}) = 0$, for $i = 1, 2, ...$ This follows from the *orthogonality property* of model-consistent expectations (see Appendix C). In particular, with $s_{t-i} = e_{t-i}$, we get $\mathrm{Cov}(e_t e_{t-i}) = 0$, i.e., the forecast errors exhibit *lack of serial correlation.* If the covariance were not zero, it would be possible to improve the forecast by incorporating the correlation into the forecast. In other words, under the assumption of rational expectations economic agents have no more to learn from past forecast errors. As remarked above, the RE hypothesis precisely refers to a fictional situation where learning has been completed and underlying mechanisms do not change.

Finally, a desirable property of a forecast function $f(\cdot)$ is that it maximizes "accuracy", i.e., minimizes an appropriate loss function. A popular loss function, L, in this context is the expected squared forecast error conditional on the information I_{t-1},

$$L = E((Y_t - f(y_{t-1}, y_{t-2}, ..., x_{t-1}, x_{t-2}, ...))^2 \,|I_{t-1}) \,.$$

Assuming $Y_t, Y_{t-1}, ..., X_{t-1}, X_{t-2}, ...$ are jointly normally distributed, then the solution to the problem of minimizing L is to set $f(\cdot)$ equal to the conditional expectation $E(Y_t \,|I_{t-1})$ based on the data-generating model as in (18).[5] This is what property (c) refers to.

EXAMPLE 3 Let $Y_t = aE(Y_t \,|I_{t-1}) + cX_t$, with $X_t = \bar{x} + \varepsilon_t$, where $\bar{x}$ is a constant and ε_t is white noise with variance σ^2. Then (17) applies, so that

$$Y_t = \frac{c\bar{x}}{1-a} + c\varepsilon_t, \qquad t = 0, 1, ...,$$

with variance $c^2\sigma^2$. The model-consistent forecast error is $e_t = Y_t - E(Y_t \,|I_{t-1}) = c\varepsilon_t$ with conditional expectation equal to $E(c\varepsilon_t \,|I_{t-1}) = 0$. This forecast error itself is white noise and is therefore uncorrelated with the information on which the forecast is based. □

[5] For proof, see Pesaran (1987). Under the restriction of only *linear* forecast functions, property (c) holds even without the joint normality assumption, see Sargent (1979).

It is worth emphasizing that the "true" conditional expectation usually can not be known − neither to the economic agents nor to the investigator. At best there can be a reasonable estimate, probably somewhat different across the agents because of differences in information and conceptions of how the economic system functions. A deeper model of expectations would give an account of the mechanisms through which agents *learn* about the economic environment. An important ingredient here would be how agents contemplate the costs and potential gains associated with further information search needed to reduce systematic expectation errors where possible. This contemplation is intricate because information search often means entering unknown territory. Moreover, for a significant subset of the agents the costs may be prohibitive. A further complicating factor involved in learning is that when the agents have obtained *some* knowledge about the statistical properties of the economic variables, the resulting behavior of the agents may *change* these statistical properties. The rational expectations hypothesis sets these problems aside. It is simply assumed that the structure of the economy remains unchanged and that the learning process has been completed.

2.2.4 Perfect foresight as a special case

The notion of *perfect foresight* corresponds to the limiting case where the variance of the exogenous variable(s) is zero so that with probability one, $X_t = E(X_t\,|I_{t-1})$ for all t. Then we have a non-stochastic model where rational expectations imply that agents' ex post forecast error with respect to Y_t is zero.[6] To put it differently: rational expectations in a non-stochastic model is equivalent to perfect foresight. Note, however, that perfect foresight necessitates the exogenous variable X_t to be known in advance. Real-world situations are usually not like that. If we want our model to take this into account, the model ought to be formulated in an explicit stochastic framework. And assumptions should be stated about how the economic agents respond to the uncertainty. The rational expectations assumption is one approach to the problem and has been much applied in macroeconomics in recent decades, perhaps due to lack of compelling tractable alternatives.

3 Models with rational forward-looking expectations

We here turn to models where current expectations of a future value of an endogenous variable have an influence on the current value of this variable, that is, the case exemplified

[6] Here we disregard zero probability events.

by equation (11). At the same time we introduce two simplifications in the notation. First, instead of using capital letters to denote the stochastic variables (as we did above and is common in mathematical statistics), we follow the tradition in macroeconomics to often use lower case letters. So a lower case letter may from now on represent a stochastic variable *or* a specific value of this variable, depending on the context.

An equation like (11) will now read $y_t = a\ y^e_{t,t+1} + c\ x_t$. Under rational expectations it takes the form $y_t = aE(y_{t+1}\,|I_t) + c\ x_t$, $t = 0, 1, 2, \ldots$. Second, from now on we write this equation as

$$y_t = aE_t y_{t+1} + c\ x_t, \ldots t = 0, 1, 2, \ldots, \ a \neq 0. \tag{19}$$

That is, the expected value of a stochastic variable, z_{t+i}, conditional on the information I_t, will be denoted $E_t z_{t+i}$.

A stochastic difference equation of the form (19) is called a linear *expectation difference equation of first order* with constant coefficient a.[7] A *solution* is a specified stochastic process $\{y_t\}$ which satisfies (19), given the stochastic process followed by x_t. In the economic applications usually no initial value, y_0, is given. On the contrary, the interpretation is that y_t depends, for all t, on expectations about the future.[8] So y_t is considered a *jump variable* that can immediately shift its value in response to the emergence of new information about the future x's. For example, a share price may immediately jump to a new value when the accounts of the firm become publicly known (often even before, due to sudden rumors).

Due to the lack of an initial condition for y_t, there can easily be infinitely many processes for y_t satisfying our expectation difference equation. We have an infinite forward-looking "regress", where a variable's value today depends on its expected value tomorrow, this value depending on the expected value the day after tomorrow and so on. Then usually there are infinitely many expected sequences which can be self-fulfilling in the sense that if only the agents expect a particular sequence, then the aggregate outcome of their behavior will be that the sequence is realized. It "bites its own tail" so to speak. Yet, when an equation like (19) is part of a larger model, there will often (but not always) be conditions that allow us to select *one* of the many solutions to (19) as the only *economically* relevant one. For example, an economy-wide transversality condition or another general

[7] To keep things simple, we let the coefficients a and c be constants, but a generalization to time-dependent coefficients is straightforward.

[8] The reason we say "depends on" is that it would be inaccurate to say that y_t is *determined* (in a one-way-sense) by expectations about the future. Rather there is *mutual dependence*. In view of y_t being an element in the information I_t, the expectation of y_{t+1} in (19) may depend on y_t just as much as y_t depends on the expectation of y_{t+1}.

equilibrium condition may rule out divergent solutions and leave a unique convergent solution as the final solution.

We assume $a \neq 0$, since otherwise (19) itself is already the unique solution. It turns out that the set of solutions to (19) takes a different form depending on whether $|a| < 1$ or $|a| > 1$:

The case $|a| < 1$. In general, there is a unique *fundamental solution* and infinitely many explosive solutions ("bubble solutions").

The case $|a| > 1$. In general, there is no fundamental solution but infinitely many non-explosive solutions. (The case $|a| = 1$ resembles this.)

In the case $|a| < 1$, the expected future has modest influence on the present. Here we will concentrate on this case, since it is the case most frequently appearing in macroeconomic models with rational expectations.

4 Solutions when $|a| < 1$

Various solution methods are available. *Repeated forward substitution* is the most easily understood method.

4.1 Repeated forward substitution

Repeated forward substitution consists of the following steps. We first shift (19) one period ahead:

$$y_{t+1} = a\ E_{t+1}y_{t+2} + c\ x_{t+1}.$$

Then we take the conditional expectation on both sides to get

$$E_t y_{t+1} = a\ E_t(E_{t+1}y_{t+2}) + c\ E_t x_{t+1} = a\ E_t y_{t+2} + c\ E_t x_{t+1}, \tag{20}$$

where the second equality sign is due to the *law of iterated expectations*, which says that

$$E_t(E_{t+1}y_{t+2}) = E_t y_{t+2}. \tag{21}$$

see Box 1. Inserting (20) into (19) then gives

$$y_t = a^2 E_t y_{t+2} + ac\ E_t x_{t+1} + c\ x_t. \tag{22}$$

The procedure is repeated by forwarding (19) two periods ahead; then taking the conditional expectation and inserting into (22), we get

$$y_t = a^3 E_t y_{t+3} + a^2 c\ E_t x_{t+2} + ac\ E_t x_{t+1} + c\ x_t.$$

We continue in this way and the general form (for $n = 0, 1, 2, ...$) becomes

$$y_{t+n} = a\ E_{t+n}(y_{t+n+1}) + c\ x_{t+n},$$

$$E_t y_{t+n} = a\ E_t y_{t+n+1} + c\ E_t x_{t+n},$$

$$y_t = a^{n+1} E_t y_{t+n+1} + c x_t + c \sum_{i=1}^{n} a^i E_t x_{t+i}. \tag{23}$$

Box 1. The law of iterated expectations

The method of repeated forward substitution is based on the law of iterated expectations which says that $E_t(E_{t+1} y_{t+2}) = E_t y_{t+2}$, as in (21). The logic is the following. Events in period $t+1$ are stochastic as seen from period t and so $E_{t+1} y_{t+2}$ (the expectation conditional on these events) is a stochastic variable. Then the law of iterated expectations says that the conditional expectation of this stochastic variable as seen from period t is the same as the conditional expectation of y_{t+2} itself as seen from period t. So, given that expectations are rational, then an earlier expectation of a later expectation of y is just the earlier expectation of y. Put differently: my best forecast today of how I am going to forecast tomorrow a share price the day after tomorrow, will be the same as my best forecast today of the share price the day after tomorrow. If beforehand we have good reasons to expect that we will revise our expectations upward, say, when next period's additional information arrives, the original expectation would be biased, hence not rational.[9]

4.2 The fundamental solution

PROPOSITION 1 Consider the expectation difference equation (19), where $a \neq 0$. If

$$\lim_{n \to \infty} \sum_{i=1}^{n} a^i E_t x_{t+i} \text{ exists,} \tag{24}$$

then

$$y_t = c \sum_{i=0}^{\infty} a^i E_t x_{t+i} = c x_t + c \sum_{i=1}^{\infty} a^i E_t x_{t+i} \equiv y_t^*, \qquad t = 0, 1, 2, ..., \tag{25}$$

is a solution to the equation.

[9] A formal account of conditional expectations and the law of iterated expectations is given in Appendix B.

Proof Assume (24). Then the formula (25) is meaningful. In view of (23), it satisfies (19) if and only if $\lim_{n\to\infty} a^{n+1}E_t y_{t+n+1} = 0$. Hence, it is enough to show that the process (25) satisfies this latter condition.

In (25), replace t by $t+n+1$ to get $y_{t+n+1} = c\sum_{i=0}^{\infty} a^i E_{t+n+1}x_{t+n+1+i}$. Using the law of iterated expectations, this yields

$$\begin{aligned} E_t y_{t+n+1} &= c\sum_{i=0}^{\infty} a^i E_t x_{t+n+1+i} \qquad \text{so that} \\ a^{n+1}E_t y_{t+n+1} &= c\, a^{n+1}\sum_{i=0}^{\infty} a^i E_t x_{t+n+1+i} = c\sum_{j=n+1}^{\infty} a^j E_t x_{t+j}. \end{aligned}$$

It remains to show that $\lim_{n\to\infty}\sum_{j=n+1}^{\infty} a^j E_t x_{t+j} = 0$. From the identity

$$\sum_{j=1}^{\infty} a^j E_t x_{t+j} = \sum_{j=1}^{n} a^j E_t x_{t+j} + \sum_{j=n+1}^{\infty} a^j E_t x_{t+j}$$

follows

$$\sum_{j=n+1}^{\infty} a^j E_t x_{t+j} = \sum_{j=1}^{\infty} a^j E_t x_{t+j} - \sum_{j=1}^{n} a^j E_t x_{t+j}.$$

Letting $n\to\infty$, this gives

$$\lim_{n\to\infty}\sum_{j=n+1}^{\infty} a^j E_t x_{t+j} = \sum_{j=1}^{\infty} a^j E_t x_{t+j} - \sum_{j=1}^{\infty} a^j E_t x_{t+j} = 0,$$

which was to be proved. □

The solution (25) is called the *fundamental solution* of (19), often marked by an asterisk *. The fundamental solution is (for $c \neq 0$) defined only when the condition (24) holds. In general this condition requires that $|a| < 1$. In addition, (24) requires that the absolute value of the expectation of the exogenous variable does not increase "too fast". More precisely, the requirement is that $|E_t x_{t+i}|$, when $i \to \infty$, has a growth factor less than $|a|^{-1}$. As an example, let $0 < a < 1$ and $g > 0$, and suppose that $E_t x_{t+i} > 0$ for $i = 0, 1, 2, ...$, and that $1+g$ is an upper bound for the growth factor of $E_t x_{t+i}$. Then

$$E_t x_{t+i} \leq (1+g)E_t x_{t+i-1} \leq (1+g)^i E_t x_t = (1+g)^i x_t.$$

Multiplying by a^i, we get $a^i E_t x_{t+i} \leq a^i(1+g)^i x_t$. By summing from $i=1$ to n,

$$\sum_{i=1}^{n} a^i E_t x_{t+i} \leq x_t \sum_{i=1}^{n} [a(1+g)]^i .$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty}\sum_{i=1}^{n} a^i E_t x_{t+i} \le x_t \lim_{n\to\infty}\sum_{i=1}^{n} [a(1+g)]^i = x_t \frac{a(1+g)}{1-a(1+g)} < \infty,$$

if $1 + g < a^{-1}$, using the sum rule for an infinite geometric series.

As noted in the proof of Proposition 1, the fundamental solution, (25), has the property that

$$\lim_{n\to\infty} a^n E_t y_{t+n} = 0. \tag{26}$$

That is, the expected value of y is not "explosive": its absolute value has a growth factor less than $|a|^{-1}$. Given $|a| < 1$, the fundamental solution is the only solution of (19) with this property. Indeed, it is seen from (23) that whenever (26) holds, (25) must also hold. In Example 1 below, y_t is interpreted as the market price of a share and x_t as dividends. Then the fundamental solution gives the share price as the present value of the expected future flow of dividends.

EXAMPLE 1 *(the fundamental value of an equity share)* Consider arbitrage between shares of stock and a riskless asset paying the constant rate of return $r > 0$. Let period t be the current period. Let p_{t+i} be the market price (in real terms, say) of the share at the beginning of period $t + i$ and d_{t+i} the dividend paid out at the end of that period, $t + i$, $i = 0, 1, 2, ...$ As seen from period t there is uncertainty about p_{t+i} and d_{t+i} for $i = 1, 2, ...$ An investor who buys n_t shares at time t (the beginning of period t) thus invests $V_t \equiv p_t n_t$ units of account at time t. At the end of the period the gross return comes out as the known dividend $d_t n_t$ and the potential sales value of the shares at the beginning of next period. This is unlike standard *accounting* and *finance* notation in discrete time, where V_t would be the end-of-period-t market value of the stock of shares that begins to yield dividends in period $t + 1$.[10]

Suppose investors have rational expectations and care only about expected return.

[10] Our use of p_t for the (real) price of a share bought at the beginning of period t is not inconsistent with our use, in earlier chapters, of P_t to denote the nominal price per unit of consumption in period t, but paid for at the *end* of the period. At the beginning of period t, after the uncertainty pertaining to period t has been resolved and available information thereby been updated, a consumer-investor will decide both the investment and the consumption flow for the period. But only the investment expence, p_t, is disbursed immediately.

It is convenient to think of the course of actions such that receipt of the previous period's dividend, d_{t-1}, and payment for that period's consumption, at the price P_{t-1}, occur right before period t begins and the new information arrives. Indeed, the resolution of uncertainty at discrete points in time motivates a *distinction* between "end of" period $t-1$ and "beginning of" period t, where the new information has just arrived.

Then the no-arbitrage condition reads

$$\frac{d_t + E_t p_{t+1} - p_t}{p_t} = r > 0. \tag{27}$$

This can be written

$$p_t = \frac{1}{1+r} E_t p_{t+1} + \frac{1}{1+r} d_t, \tag{28}$$

which is of the same form as (19) with $a = c = 1/(1+r) \in (0,1)$. Assuming dividends do not grow "too fast", we find the fundamental solution, denoted p_t^*, as

$$p_t^* = \frac{1}{1+r} d_t + \frac{1}{1+r} \sum_{i=1}^{\infty} \frac{1}{(1+r)^i} E_t d_{t+i} = \sum_{i=0}^{\infty} \frac{1}{(1+r)^{i+1}} E_t d_{t+i}. \tag{29}$$

The fundamental solution is simply the present value of expected future dividends.

If the dividend process is $d_{t+1} = d_t + \varepsilon_{t+1}$, where ε_{t+1} is white noise, then the dividend process is known as a *random walk* and $E_t d_{t+i} = d_t$ for $i = 1, 2, \ldots$. Thus $p_t^* = d_t/r$, by the sum rule for an infinite geometric series. In this case the fundamental value is thus itself a random walk. More generally, the dividend process could be a *martingale,* that is, a sequence of stochastic variables with the property that the expected value next period exists and equals the current actual value, i.e., $E_t d_{t+1} = d_t$; but in a martingale, $\varepsilon_{t+1} \equiv d_{t+1} - d_t$ need not be white noise; it is enough that $E_t \varepsilon_{t+1} = 0$.[11] Given the constant required return r, we still have $p_t^* = d_t/r$. So the fundamental value itself is in this case a martingale. □

In finance theory the present value of the expected future flow of dividends on an equity share is referred to as the *fundamental value* of the share. It is by analogy with this that the general designation *fundamental solution* has been introduced for solutions of form (25). We could also think of p_t as the market price of a house rented out and d_t as the rent. Or p_t could be the market price of an oil well and d_t the revenue (net of extraction costs) from the extracted oil in period t.

4.3 Bubble solutions

Other than the fundamental solution, the expectation difference equation (19) has infinitely many *bubble solutions.* In view of $|a| < 1$, these are characterized by violating the condition (26). That is, they are solutions whose expected value explodes over time.

[11] A random walk is thus a special case of a martingale.

It is convenient to first consider the *homogenous* expectation equation associated with (19). This is defined as the equation emerging when setting $c = 0$ in (19):

$$y_t = aE_t y_{t+1}. \tag{30}$$

Every stochastic process $\{b_t\}$ of the form

$$b_{t+1} = a^{-1}b_t + u_{t+1}, \quad \text{where } E_t u_{t+1} = 0, \tag{31}$$

has the property that

$$b_t = aE_t b_{t+1}, \tag{32}$$

and is thus a solution to (30). The "disturbance" u_{t+1} represents "new information" which may be related to movements in "fundamentals", x_{t+1}. But it does not have to. In fact, u_{t+1} may be related to conditions that *per se* have no economic relevance whatsoever.

For ease of notation, from now on we just write b_t even if we think of the whole process $\{b_t\}$ rather than the value taken by b in the specific period t. The meaning should be clear from the context. A solution to (30) is referred to as a *homogenous solution* associated with (19). Let b_t be a given homogenous solution and let K be an arbitrary constant. Then $B_t = Kb_t$ is also a homogenous solution (try it out for yourself). Conversely, any homogenous solution b_t associated with (19) can be written in the form (31). To see this, let b_t be a given homogenous solution, that is, $b_t = aE_t b_{t+1}$. Let $u_{t+1} = b_{t+1} - E_t b_{t+1}$. Then

$$b_{t+1} = E_t b_{t+1} + u_{t+1} = a^{-1}b_t + u_{t+1},$$

where $E_t u_{t+1} = E_t b_{t+1} - E_t b_{t+1} = 0$. Thus, b_t is of the form (31).

For convenience we here repeat our original expectation difference equation (19) and name it (*):

$$y_t = aE_t y_{t+1} + c\ x_t, \dots t = 0, 1, 2, \dots, \quad a \neq 0. \tag{*}$$

PROPOSITION 2 Consider the expectation difference equation (*), where $a \neq 0$. Let $\tilde{y}_t$ be a particular solution to the equation. Then:

(i) every stochastic process of the form

$$y_t = \tilde{y}_t + b_t, \tag{33}$$

where b_t satisfies (31), is a solution to (*);

(ii) every solution to (*) can be written in the form (33) with b_t being an appropriately chosen homogenous solution associated with (*).

Proof. Let some particular solution $\tilde{y}_t$ be given. (i) Consider $y_t = \tilde{y}_t + b_t$, where b_t satisfies (31). Since $\tilde{y}_t$ satisfies (*), we have $y_t = a\ E_t\tilde{y}_{t+1} + c\ x_t + b_t$. Consequently, by (30),

$$y_t = a\ E_t\tilde{y}_{t+1} + c\ x_t + a\ E_t b_{t+1} = a\ E_t(\tilde{y}_{t+1} + b_{t+1}) + c\ x_t = a\ E_t y_{t+1} + c\ x_t,$$

saying that (33) satisfies (*). (ii) Let Y_t be an arbitrary solution to (*). Define $b_t = Y_t - \tilde{y}_t$. Then we have

$$\begin{aligned} b_t &= Y_t - \tilde{y}_t = aE_tY_{t+1} + cx_t - (aE_t\tilde{y}_{t+1} + cx_t) \\ &= aE_t(Y_{t+1} - \tilde{y}_{t+1}) = aE_tb_{t+1}, \end{aligned}$$

where the second equality follows from the fact that both Y_t and $\tilde{y}_t$ are solutions to (*). This shows that b_t is a solution to the homogenous equation (30) associated with (*). Since $Y_t = \tilde{y}_t + b_t$, the proposition is hereby proved. □

Proposition 2 holds for any $a \neq 0$. In case the fundamental solution (25) exists and $|a| < 1$, it is convenient to choose this solution as the particular solution in (33). Thus, referring to the right-hand side of (25) as y_t^*, we can use the particular form,

$$y_t = y_t^* + b_t. \tag{34}$$

When the component b_t is different from zero, the solution (34) is called a *bubble solution* and b_t is called the *bubble component.* In the typical economic interpretation the bubble component shows up only because it is expected to show up next period, cf. (32). The name bubble springs from the fact that the expected value of b_t, conditional on the information available in period t, explodes over time when $|a| < 1$. To see this, as an example, let $0 < a < 1$. Then, from (30), by repeated forward substitution we get

$$b_t = a\ E_t(aE_{t+1}b_{t+2}) = a^2E_tb_{t+2} = ... = a^iE_tb_{t+i}, \quad i = 1, 2, ...$$

It follows that $E_tb_{t+i} = a^{-i}b_t$, and from this follows that the bubble, for t going to infinity, is unbounded in expected value:

$$\lim_{i\to\infty} E_tb_{t+i} = \begin{cases} \infty, \text{ if } b_t > 0 \\ -\infty, \text{ if } b_t < 0 \end{cases}. \tag{35}$$

Indeed, the absolute value of E_tb_{t+i} will for rising i grow *geometrically* towards infinity with a growth factor equal to $1/a > 1$.

Let us consider a special case of (*) that allows a simple graphical illustration of both the fundamental solution and some bubble solutions.

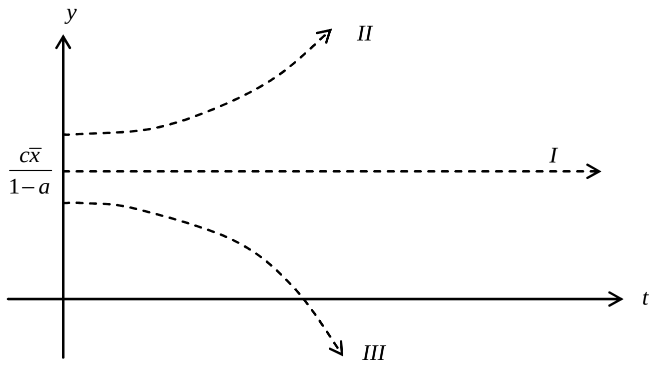

Figure 1: Deterministic bubbles (the case $0 < a < 1$, $c > 0$, and $x_t = \bar{x}$).

4.3.1 When x_t has constant mean

Suppose the stochastic process x_t (the "fundamentals") takes the form $x_t = \bar{x} + \varepsilon_t$, where $\bar{x}$ is a constant and ε_t is white noise. Then

$$y_t = a\ E_t y_{t+1} + c(\bar{x} + \varepsilon_t), \qquad 0 < |a| < 1. \tag{36}$$

The fundamental solution is

$$y_t^* = c\ x_t + c\sum_{i=1}^{\infty} a^i \bar{x} = c\bar{x} + c\varepsilon_t + c\frac{a\bar{x}}{1-a} = \frac{c\bar{x}}{1-a} + c\varepsilon_t.$$

Referring to (i) of Proposition 2,

$$y_t = \frac{c\bar{x}}{1-a} + c\varepsilon_t + b_t \tag{37}$$

is thus also a solution of (36) if b_t is of the form (31).

It may be instructive to consider the case where all stochastic features are eliminated. So we assume $u_t \equiv \varepsilon_t \equiv 0$. Then we have a model with perfect foresight; the solution (37) simplifies to

$$y_t = \frac{c\bar{x}}{1-a} + b_0 a^{-t}, \tag{38}$$

where we have used repeated *backward* substitution in (31). By setting $t = 0$ we see that $y_0 - \frac{c\bar{x}}{1-a} = b_0$. Inserting this into (38) gives

$$y_t = \frac{c\bar{x}}{1-a} + (y_0 - \frac{c\bar{x}}{1-a})a^{-t}. \tag{39}$$

In Fig. 1 we have drawn three trajectories for the case $0 < a < 1$, $c > 0$. Trajectory I has $y_0 = c\bar{x}/(1-a)$ and represents the fundamental solution. Trajectory II, with $y_0 > c\bar{x}/(1-a)$, and trajectory III, with $y_0 < c\bar{x}/(1-a)$, are bubble solutions. Since we have imposed no boundary condition apriori, one y_0 is as good as any other. The interpretation is that there are infinitely many trajectories with the property that if only the economic agents expect the economy will follow that particular trajectory, the aggregate outcome of their behavior will be that this trajectory is realized. This is the potential indeterminacy arising when y_t is not a predetermined variable. However, as alluded to above, in a complete economic model there will often be restrictions on the endogenous variable(s) not visible in the basic expectation difference equation(s), here (36). It may be that the economic meaning of y_t precludes negative values (a share certificate would be an example). In that case no-one can rationally expect a path such as III in Fig. 1. Or perhaps, for some reason, there is an upper bound on y_t (think of the full-employment ceiling for output in a situation where the "natural" growth factor for output is smaller than a^{-1}). Then no one can rationally expect a trajectory like II in the figure.

To sum up: in order for a solution of a first-order linear expectation difference equation with constant coefficient a, where $|a| < 1$, to differ from the fundamental solution, the solution must have the form (34) where b_t has the form described in (31). This provides a clue as to what asset price bubbles might look like.

4.3.2 Asset price bubbles

A stylized fact of stock markets is that stock price indices are quite volatile on a month-to-month, year-to-year, and especially decade-to-decade scale, cf. Fig. 2. There are different views about how these swings should be understood. According to the *Efficient Market Hypothesis* the swings just reflect unpredictable changes in the "fundamentals", that is, changes in the present value of rationally expected future dividends. This is for instance the view of Nobel laureate Eugene Fama (1970, 2003) from University of Chicago.

In contrast, Nobel laureate Robert Shiller (1981, 2003, 2005) from Yale University, and others, have pointed to the phenomenon of "excess volatility". The view is that asset prices tend to fluctuate more than can be rationalized by shifts in information about fundamentals (present values of dividends). Although in no way a verification, graphs like those in Fig. 2 and Fig. 3 are suggestive. Fig. 2 shows the monthly real Standard and Poors (S&P) composite stock prices and real S&P composite earnings for the period 1871-2008. The unusually large increase in real stock prices since the mid-90's, which ended with the collapse in 2000, is known as the "dot-com bubble". Fig. 3 shows, on a monthly basis, the ratio of real S&P stock prices to an average of the previous ten years' real S&P earnings along with the long-term real interest rate. It is seen that this ratio

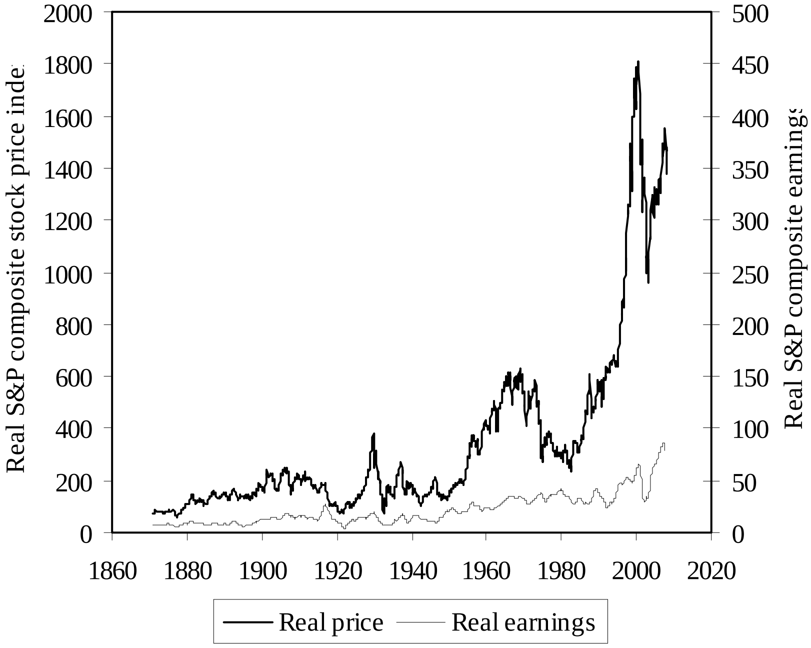

Figure 2: Monthly real S&P composite stock prices from January 1871 to January 2008 (left) and monthly real S&P composite earnings from January 1871 to September 2007 (right). Source: http://www.econ.yale.edu/~shiller/data.htm.

reached an all-time high in 2000, by many observers considered as "the year the dot-com bubble burst".

Shiller's interpretation of the large stock market swings is that they are due to fads, herding, and shifts in fashions and "animal spirits" (the latter being a notion from Keynes).

A third possible source of large stock market swings was pointed out by Blanchard (1979) and Blanchard and Watson (1982). They argued that bubble phenomena need not be due to irrational behavior and non-rational expectations. This lead to the theory of *rational bubbles* − the idea that excess volatility can be explained as speculative bubbles arising from self-fulfilling *rational* expectations.

Consider an asset which yields either dividends or services in production or consumption in every period in the future. The fundamental value of the asset is, at the theoretical level, defined as the present value of the expected future flow of dividends or services.[12] An *asset price bubble* is then defined as a systematic positive deviation of the market

[12]In practice there are many ambiguities involved in this definition of the fundamental value because it relates to a future which is often essentially unknown.

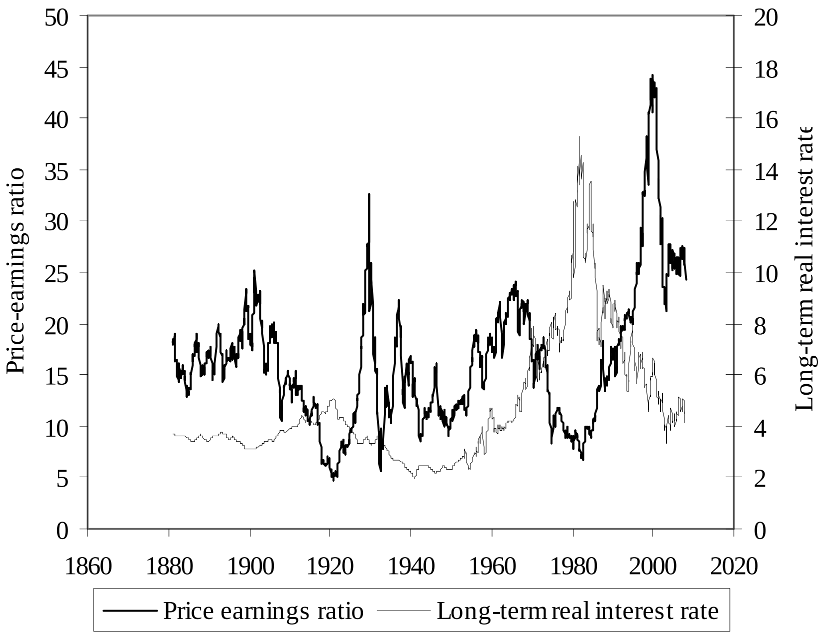

Figure 3: S&P price-earnings ratio and long-term real interest rates from January 1881 to January 2008. The earnings are calculated as a moving average over the preceding ten years. The long-term real interest rate is the 10-year Treasury rate from 1953 and government bond yields from Sidney Homer, "A History of Interest Rates" from before 1953. Source: http://www.econ.yale.edu/~shiller/data.htm.

price, p_t, of the asset from its fundamental value, p_t^*:

$$p_t = p_t^* + b_t. \tag{40}$$

An asset price bubble, $p_t - p_t^*$, that emerges in a setting where the no-arbitrage condition (27) holds under rational expectations, is called a *rational bubble*. It emerges only because there is in the market a self-fulfilling belief that it will appreciate at a rate high enough to warrant the overcharge involved.

EXAMPLE 2 *(an ever-expanding rational bubble)* Consider again an equity share for which the no-arbitrage condition is

$$\frac{d_t + E_t p_{t+1} - p_t}{p_t} = r > 0. \tag{41}$$

As in Example 1, the implied expectation difference equation is $p_t = aE_t p_{t+1} + cd_t$, with $a = c = 1/(1+r) \in (0,1)$. Let the price of the share at time t be $p_t = p_t^* + b_t$, where p_t^* is the fundamental value and $b_t > 0$ a bubble component following the deterministic process, $b_{t+1} = (1+r)b_t$, $b_0 > 0$, so that $b_t = b_0(1+r)^t$. This is called a *deterministic rational bubble*. The sum $p_t^* + b_t$ will satisfy the no-arbitrage condition (41) just as much as p_t^* itself, because we just add something which equals the discounted value of itself one period later.

Agents may be ready to pay a price over and above the fundamental value (whether or not they know the "true" fundamental value) if they expect they can sell at a sufficiently higher price later; trading with such motivation is called *speculative behavior.* If generally held and lasting for some time, this expectation may be self-fulfilling. Note that (41) implies that the asset price ultimately grows at the rate r. Indeed, let $d_t = d_0(1+\gamma)^t$, $\gamma < r$ (if $r \leq \gamma$, the asset price would be infinite). By the rule of the sum of an infinite geometric series, we then have $p_t^* = d_t/(r-\gamma)$, showing that the fundamental value grows at the rate γ. Consequently, $p_t/b_t = (p_t^* + b_t)/b_t = p_t^*/b_t + 1 \rightarrow 1$, as $\gamma < r$. It follows that the asset price in the long run grows at the same rate as the bubble, the rate r.

We are not acquainted with *ever*-expanding incidents of that caliber in real world situations, however. A deterministic rational bubble thus appears implausible. $\square$

In some contexts it may not matter whether or not we think of the "rational" market participants as actually knowing the probability distribution of the "fundamentals", hence knowing p_t^* (by "fundamentals" is meant any information relating to the future dividend or service capacity of an asset: a firm's technology, resources, market conditions etc.). All the same, it seems common to imply such a high level of information in the term "rational bubbles". Unless otherwise indicated, we shall let this implication be understood.

While a deterministic rational bubble was found implausible, let us now consider an example of a *stochastic* rational bubble which sooner or later *bursts.*

EXAMPLE 3 *(a bursting bubble)* Once again we consider the no-arbitrage condition is (41) where for simplicity we still assume the required rate of return is constant, though possibly including a risk premium. Following Blanchard (1979), we assume that the market price, p_t, of the share contains a stochastic bubble of the following form:

$$b_{t+1} = \begin{cases} \frac{1+r}{q_t} b_t & \text{with probability } q_t, \\ 0 & \text{with probability } 1 - q_t, \end{cases} \tag{42}$$

where $t = 0, 1, 2, ...$ and $b_0 > 0$. In addition we may assume that $q_t = f(p_t^*, b_t)$, $f_{p^*} \geq 0$, $f_b \leq 0$. If $f_{p^*} > 0$, the probability that the bubble persists at least one period ahead is higher the greater the fundamental value has become. If $f_b < 0$, the probability that the bubble persists at least one period ahead is less, the greater the bubble has already become. In this way the probability of a crash becomes greater and greater as the share price comes further and further away from fundamentals. As a compensation, the longer time the bubble has lasted, the higher is the expected growth rate of the bubble in the absence of a collapse.

This bubble satisfies the criterion for a rational bubble. Indeed, (42) implies

$$E_t\ b_{t+1} = (\frac{1+r}{q_{t+1}} b_t) q_{t+1} + 0 \cdot (1 - q_{t+1}) = (1+r) b_t.$$

This is of the form (31) with $a^{-1} = 1 + r$, and the bubble is therefore a stochastic rational bubble. The stochastic component is $u_{t+1} = b_{t+1} - E_t b_{t+1} = b_{t+1} - (1+r) b_t$ and has conditional expectation equal to zero. Although u_{t+1} must have zero conditional expectation, it need not be white noise (it can for instance have varying variance). □

As this example illustrates, a stochastic rational bubble does not have the implausible ever-expanding form of a deterministic rational bubble. Yet, under certain conditions even stochastic rational bubbles can be ruled out or at least be judged implausible. The next section reviews some cases.

4.4 When rational bubbles in asset prices can or can not be ruled out

We concentrate on assets whose services are valued independently of the price.[13] Let p_t be the market price and p_t^* the fundamental value of the asset as of time t. Even if the asset yields services rather than dividends, we think of p_t^* as in principle the same for all agents. This is because a user who, in a given period, values the service flow of the asset relatively low can hire it out to the one who values it highest (the one with the highest willingness to pay). Until further notice we assume p_t^* known to the market participants.

4.4.1 Partial equilibrium arguments

The principle of reasoning to be used is called *backward induction:* If we know something about an asset price in the future, we can conclude something about the asset price today.

(a) Assets which can be freely disposed of ("free disposal") Can a rational asset price bubble be *negative*? The answer is no. The logic can be illustrated on the basis of Example 2 above. For simplicity, let the dividend be the same constant $d > 0$ for all $t = 0, 1, 2, ...$ Then, from the formula (39) we have

$$p_t - p^* = (p_0 - p^*)(1+r)^t,$$

[13] This is in contrast to assets that serve as means of payment.

where $r > 0$ and $p^* = d/r$. Suppose there is a negative bubble in period 0, i.e., $p_0 - p^* < 0$. In period 1, since $1 + r > 1$, the bubble is greater in absolute value. The downward movement of p_t continues and sooner or later p_t is negative. The intuition is that the low p_0 in period 0 implies a high dividend-price ratio. Hence a negative capital gain ($p_{t+1} - p_t < 0$) is needed for the no-arbitrage condition (41) to hold. Thereby $p_1 < p_0$, and so on.

But in a market with self-interested rational agents, an object which can be freely disposed of can never have a negative price. A negative price means that the "seller" has to *pay* to dispose of the object. Nobody will do that if the object can just be thrown away. An asset which can be freely disposed of (share certificates for instance) can therefore never have a negative price. We conclude that a *negative* rational bubble can not be consistent with rational expectations. Similarly, with a stochastic dividend, a negative rational bubble would imply that in expected value the share price becomes negative at some point in time, cf. (35). Again, rational expectations rule this out.

Hence, if we imagine that for a short moment $p_t < p_t^*$, then everyone will want to *buy* the asset and hold it forever, which by own use or by hiring out will imply a discounted value equal to p_t^*. There is thus excess demand until p_t has risen to p_t^*.

When a negative rational bubble can be ruled out, then, if at the first date of trading of the asset there were no positive bubble, neither can a positive bubble arise later. Let us make this precise:

PROPOSITION 3 Assume free disposal of a given asset. Then, if a rational bubble in the asset price is present today, it must be positive and must have been present also yesterday and so on back to the first date of trading the asset. And if a rational bubble bursts, it will not restart later.

Proof As argued above, in view of free disposal, a negative rational bubble in the asset price can be ruled out. It follows that $b_t = p_t - p_t^* \geq 0$ for $t = 0, 1, 2, ...$, where $t = 0$ is the first date of trading the asset. That is, any rational bubble in the asset price must be a positive bubble. We now show by contradiction that if, for an arbitrary $t = 1, 2, ...$, it holds that $b_t > 0$, then $b_{t-1} > 0$. Let $b_t > 0$. Then, if $b_{t-1} = 0$, we have $E_{t-1}b_t = E_{t-1}u_t$ $= 0$ (from (31) with t replaced by $t-1$), implying, since $b_t < 0$ is not possible, that $b_t = 0$ with probability *one* as seen from period $t-1$. Ignoring zero probability events, this rules out $b_t > 0$ and we have arrived at a contradiction. Thus $b_{t-1} > 0$. Replacing t by $t-1$ and so on backward in time, we end up with $b_0 > 0$. This reasoning also implies that if a bubble bursts in period t, it can not restart in period $t+1$, nor, by extension, in any subsequent period. □

This proposition (due to Diba and Grossman, 1988) claims that a rational bubble in an asset price must have been there since trading of the asset began. Yet such a conclusion is not without ambiguities. If new information about radically new technology comes up at some point in time, is a share in the firm then the same asset as before? In a legal sense the firm is the same, but is the asset also the same? Even if an earlier bubble has crashed, cannot a new rational bubble arise later in case of an utterly new situation?

These ambiguities reflect the difficulty involved in the concepts of rational expectations and rational bubbles when we are dealing with uncertainties about future developments of the economy. The market's evaluation of many assets of macroeconomic importance, not the least shares in firms, depends on vague beliefs about future preferences, technologies, and societal circumstances. The fundamental value can not be determined in any objective way. There is no well-defined probability distribution over the potential future outcomes. *Fundamental uncertainty*, also called *Knightian uncertainty*,[14] is present.

(b) Bonds with finite maturity The finite maturity ensures that the value of the bond is given at some finite future date. Therefore, if there were a positive bubble in the market price of the bond, no rational investor would buy just before that date. Anticipating this, no one would buy the date before, and so on. Consequently, nobody will buy in the first place. By this backward-induction argument follows that a positive bubble cannot get started. And since there also is "free disposal", *all* rational bubbles can be precluded.

From now on we take as given that negative rational bubbles are ruled out. So, the discussion is about whether *positive* rational asset price bubbles may exist or not.

(c) Assets whose supply is elastic Real capital goods (including buildings) can be reproduced and have clearly defined costs of reproduction. This precludes rational bubbles on this kind of assets, since a potential buyer can avoid the overcharge by producing instead. Notice, however, that building sites with a specific amenity value and apartments in attractive quarters of a city are not easily reproducible. Therefore, rational bubbles on such assets are more difficult to rule out.

[14] After the Chicago of University economist Frank Knight who in his book, *Risk, Uncertainty, and Profit* (1921), coined the important distinction between *measurable risk* and *unmeasurable uncertainty.*

Here are a few intuitive remarks about bubbles on shares of stock in an established firm. An argument against a rational bubble might be that if there were a bubble, the firm would tend to exploit it by issuing more shares. But thereby market participants mistrust is raised and may pull market evaluation back to the fundamental value. On the other hand, the firm might anticipate this adverse response from the market. So the firm chooses instead to "fool" the market by steady financing behavior, calmly enjoying its solid equity and continuing as if no bubble were present. It is therefore not obvious that this kind of argument can rule out rational bubbles on shares of stock.

(d) Assets for which there exists a "backstop-technology" For some articles of trade there exists substitutes in elastic supply which will be demanded if the price of the article becomes sufficiently high. Such a substitute is called a "backstop-technology". For example oil and other fossil fuels will, when their prices become sufficiently high, be subject to intense competition from substitutes (renewable energy sources). This precludes an unbounded bubble process in the price of oil.

On account of the arguments (c) and (d), it seems more difficult to rule out rational bubbles when it comes to assets which are not reproducible or substitutable, let alone assets whose fundamentals are difficult to ascertain. For some assets the fundamentals are not easily ascertained. Examples are paintings of past great artists, rare stamps, diamonds, gold etc. Also new firms that introduce completely novel products and technologies are potential candidates. Think of the proliferation of radio broadcasting in the 1920s before the wall Street crash in 1929 and the internet in the 1990s before the dotcom bubble burst in 2000.

What these situations allow for may not be termed rational bubbles, if by definition this concept requires a well-defined fundamental. Then we may think of a broader class of real-world bubbly phenomena driven by self-reinforcing expectations.

4.4.2 Adding general equilibrium arguments

The above considerations are of a partial equilibrium nature. On top of this, *general equilibrium* arguments can be put forward to limit the possibility of rational bubbles. We may briefly give a flavour of two such general equilibrium arguments. We still consider assets whose services are valued independently of the price and which, as in (a) above, can be freely disposed of. A house, a machine, or a share in a firm yields a service in consumption or production or in the form of a dividend stream. Since such an asset has an intrinsic value, p_t^*, equal to the present value of the flow of services, one might believe that positive rational bubbles on such assets can be ruled out in general equilibrium. As we shall see, this is indeed true for an economy with a finite number of "neoclassical" households (to be defined below), but not necessarily in an overlapping generations model. Yet even there, rational bubbles can under certain conditions be ruled out.

(e) An economy with a finite number of infinitely-lived households Assume that the economy consists of a finite number of infinitely-lived agents − here called households − indexed $i = 1, 2, ..., N$. The households are "neoclassical" in the sense that they save only with a view to future consumption.

Under free disposal in point (a) we saw that $p_t < p_t^*$ can not be an equilibrium. We now consider the case of a positive bubble, i.e., $p_t > p_t^*$. All owners of the bubble asset who are users will in this case prefer to *sell* and then *rent*; this would imply excess supply and could thus not be an equilibrium. Hence, we turn to households that are not users, but speculators. Assuming "short selling" is legal, speculators may pursue "short selling", that is, they first rent the asset (for a contracted interval of time) and immediately sell it at p_t. This results in excess supply and so the asset price falls towards p_t^*. Within the contracted interval of time the speculators buy the asset back and return it to the original owners in accordance with the loan accord. So $p_t > p_t^*$ can not be an equilibrium.

Even ruling out "short selling" (which *is* sometimes outright forbidden), we can exclude positive bubbles in the present setup with a finite number of households. To assume that owners who are not users would want to hold the bubble asset forever as a permanent investment will contradict that these owners are "neoclassical". Indeed, their transversality condition would be violated because the value of their wealth would grow at a rate asymptotically equal to the rate of interest. This would allow them to increase their consumption now without decreasing it later and without violating their No-Ponzi-Game condition.

We have to instead imagine that the "neoclassical" households who own the bubble asset, hold it against future sale. This could on the face of it seem rational enough if there were some probability that not only would the bubble continue to exist, but it would also grow so that the return would be at least as high as that yielded on an alternative investment. Owners holding the asset in the expectation of a capital gain, will thus plan to sell at some later point in time. Let t_i be the point in time where household

i wishes to sell and let

$$T = \max[t_1, t_2, ..., t_N].$$

Then nobody will plan to hold the asset after T. The household speculator, i, having $t_i = T$ will thus not have anyone to sell to (other than people who will only pay p_T^*). Anticipating this, no-one would buy or hold the asset the period before, and so on. So no-one will want to buy or hold the asset in the first place.

The conclusion is that $p_t > p_t^*$ cannot be a rational expectations equilibrium in a setup with a finite number of "neoclassical" households.

The same line of reasoning does not, however, go through in an overlapping generations model where *new* households – that is, new traders – enter the economy every period.

(f) An economy with interest rate above the output growth rate In an overlapping generations (OLG) model with an infinite sequence of new decision makers, rational bubbles are under certain conditions theoretically possible. The argument is that with $N \to \infty$, T as defined above is not bounded. Although this unboundedness is a necessary condition for rational bubbles, it is not sufficient, however.

To see why, let us return to the arbitrage examples 1, 2, and 3 where we have $a^{-1} = 1 + r$ so that a hypothetical rational bubble has the form $b_{t+1} = (1 + r)b_t + u_{t+1}$, where $E_t u_{t+1} = 0$. So in expected value the hypothetical bubble is growing at a rate equal to the interest rate, r. If at the same time r is higher than the long-run output growth rate, the value of the expanding bubble asset would sooner or later be larger than GDP and aggregate saving would not suffice to back its continued growth. Agents with rational expectations anticipate this and so the bubble never gets started.

This point is valid when the interest rate in the OLG economy is higher than the growth rate of the economy – which is normally considered the realistic case. Yet, the opposite case *is* possible and in that situation it is less easy to rule out rational asset price bubbles. This is also the case in situations with imperfect credit markets. It turns out that the presence of segmented financial markets or externalities that create a wedge between private and social returns on productive investment may increase the scope for rational bubbles (Blanchard, 2008).

4.5 Conclusion

The empirical evidence concerning asset price bubbles in general and rational asset price bubbles in particular seems inconclusive. It is very difficult to statistically distinguish between bubbles and mis-specified fundamentals. Rational bubbles can also have more complicated forms than the bursting bubble in Example 3 above. For example Evans (1991) and Hall et al. (1999) study "regime-switching" rational bubbles.

Whatever the possible limits to the plausibility of rational bubbles in asset prices, it is useful to be aware of their logical structure and the variety of forms they can take as logical possibilities. Rational bubbles may serve as a benchmark for a variety of "behavioral asset price bubbles", i.e., bubbles arising through particular psychological mechanisms. This would take us to *behavioral finance* theory. The reader is referred to, e.g., Shiller (2003).

For surveys on the theory of rational bubbles and econometric bubble tests, see Salge (1997) and Gürkaynak (2008). For discussions of famous historical bubble episodes, see the symposium in *Journal of Economic Perspectives* 4, No. 2, 1990, and Shiller (2005).

5 Appendix

A. The log-linear specification

In many macroeconomic models with rational expectations the equations are specified as log-linear, that is, as being linear in the logarithms of the variables. If Y, X, and Z are the original positive stochastic variables, defining $y = \ln Y$, $x = \ln X$, and $z = \ln Z$, a log-linear relationship between Y, X, and Z is a relation of the form

$$y = \alpha + \beta x + \gamma z, \tag{43}$$

where α, β, and γ are constants. The motivation for assuming log-linearity can be:

(a) Linearity is convenient because of the simple rule for the expected value of a sum: $E(\alpha+\beta x+\gamma z) = \alpha+\beta E(x)+\gamma E(z)$, where E is the expectation operator. Indeed, for a non-linear function, $f(x, z)$, we generally have $E(f(x, z)) \neq f(E(x), E(z))$.

(b) Linearity in logs may often seem a more realistic assumption than linearity in anything else.

(c) In time series models a logarithmic transformation of the variables followed by formation of first differences can be the road to eliminating a trend in the mean and variance.

As to point (b) we state the following:

CLAIM To assume linearity in logs is equivalent to assuming constant elasticities.

Proof Let the positive variables Y, X and Z be related by $Y = F(X,\ Z)$, where F is a continuous function with continuous partial derivatives. Taking the differential on both sides of $\ln\ Y = \ln F(X, Z)$, we get

$$\begin{aligned} d\ln Y &= \frac{1}{F(X,Z)}\frac{\partial F}{\partial X}dX + \frac{1}{F(X,Z)}\frac{\partial F}{\partial Z}dZ \\ &= \frac{X}{Y}\frac{\partial Y}{\partial X}\frac{dX}{X} + \frac{Z}{Y}\frac{\partial Y}{\partial Z}\frac{dZ}{Z} = \eta_{YX}\frac{dX}{X} + \eta_{YZ}\frac{dZ}{Z} = \eta_{YX}d\ln X + \eta_{YZ}d\ln Z, \end{aligned} \tag{44}$$

where η_{YX} and η_{YZ} are the partial elasticities of Y w.r.t. X and Z, respectively. Thus, defining $y = \ln Y$, $x = \ln X$, and $z = \ln Z$, gives

$$dy = \eta_{YX}dx + \eta_{YZ}dz. \tag{45}$$

Assuming constant elasticities amounts to putting $\eta_{YX} = \beta$ and $\eta_{YZ} = \gamma$, where β and γ are constants. Then we can write (45) as $dy = \beta dx + \gamma dz$. By integration, we get (43) where α is now an arbitrary integration constant. Hereby we have shown that constant elasticities imply a log-linear relationship between the variables.

Now, let us instead start by assuming the log-linear relationship (43). Then,

$$\frac{\partial y}{\partial x} = \beta, \frac{\partial y}{\partial z} = \gamma. \tag{46}$$

But (43), together with the definitions of y, x and z, implies that

$$Y = e^{\alpha+\beta x+\gamma z} = e^{\alpha+\beta\ln X+\gamma\ln Z},$$

from which follows that

$$\frac{\partial Y}{\partial X} = Y\beta\frac{1}{X} \text{ so that } \eta_{YX} \equiv \frac{X}{Y}\frac{\partial Y}{\partial X} = \beta,$$

and

$$\frac{\partial Y}{\partial Z} = Y\gamma\frac{1}{Z} \text{ so that } \eta_{YZ} \equiv \frac{Z}{Y}\frac{\partial Y}{\partial Z} = \gamma.$$

That is, the partial elasticities are constant. □

So, when the variables are in logs, then the coefficients in the linear expressions are the elasticities. Note, however, that the interest rate is normally an exception. It is often regarded as more realistic to let the interest rate itself and not its logarithm enter linearly. Then the associated coefficient indicates the *semi-elasticity* with respect to the interest rate.

B. Conditional expectations and the law of iterated expectations

The mathematical conditional expectation is a weighted sum of the possible values of the stochastic variable with weights equal to the corresponding conditional probabilities.

Let Y and X be two *discrete* stochastic variables with joint probability function $j(y, x)$ and marginal probability functions $f(y)$ and $g(x)$, respectively. If the conditional probability function for Y given $X = x_0$ is denoted $h(y\,|x_0)$, we have $h(y\,|x_0) = j(y, x_0)/g(x_0)$, assuming $g(x_0) > 0$. The conditional expectation of Y given $X = x_0$, denoted $E(Y|X = x_0)$, is then

$$E(Y|X = x_0) = \sum_y y\frac{j(y, x_0)}{g(x_0)}, \tag{47}$$

where the summation is over all the possible values of y.

This conditional expectation is a function of x_0. Since x_0 is just one possible value of the stochastic variable X, we interpret the conditional expectation itself as a stochastic variable and write it as $E(Y|X)$. Generally, for a function of the discrete stochastic variable X, say $k(X)$, the expected value is

$$E(k(X)) = \sum_x k(x)g(x).$$

When we here let the conditional expectation $E(Y|X)$ play the role of $k(X)$ and sum over all x for which $g(x) > 0$, we get

$$\begin{aligned} E(E(Y|X)) &= \sum_x E(Y|x)g(x) = \sum_x \left(\sum_y y\frac{j(y,x)}{g(x)}\right) g(x) \qquad \text{(by (47))} \\ &= \sum_y y\left(\sum_x j(y,x)\right) = \sum_y yf(y) = E(Y). \end{aligned}$$

This result is a manifestation of the *law of iterated expectations:* the unconditional expectation of the conditional expectation of Y is given by the unconditional expectation of Y.

Now consider the case where Y and X are *continuous* stochastic variables with joint probability *density* function $j(y, x)$ and marginal density functions $f(y)$ and $g(x)$, respectively. If the conditional density function for Y given $X = x_0$ is denoted $h(y\,|x_0)$, we have

$h(y\,|x_0) = j(y, x_0)/g(x_0)$, assuming $g(x_0) > 0$. The conditional expectation of Y given $X = x_0$ is

$$E(Y|X = x_0) = \int_{-\infty}^{\infty} y \frac{j(y, x_0)}{g(x_0)} dy, \tag{48}$$

where we have assumed that the range of Y is $(-\infty, \infty)$. Again, we may view the conditional expectation itself as a stochastic variable and write it as $E(Y|X)$. Generally, for a function of the continuous stochastic variable X, say $k(X)$, the expected value is

$$E(k(X)) = \int_R k(x)g(x)dx,$$

where R stands for the range of X. When we let the conditional expectation $E(Y|X)$ play the role of $k(X)$, we get

$$\begin{aligned} E(E(Y|X)) &= \int_R E(Y|x)g(x)dx = \int_R \left(\int_{-\infty}^{\infty} y \frac{j(y, x)}{g(x)} dy \right) g(x)dx \text{ (by (48))} \\ &= \int_{-\infty}^{\infty} y \left(\int_R j(y, x)dx \right) dy = \int_{-\infty}^{\infty} y f(y)dy = E(Y). \end{aligned} \tag{49}$$

This shows us the *law of iterated expectations* in action for continuous stochastic variables*:* the unconditional expectation of the conditional expectation of Y is given by the unconditional expectation of Y.

EXAMPLE Let the two stochastic variables, X and Y, follow a two-dimensional normal distribution. Then, from mathematical statistics we know that the conditional expectation of Y given X satisfies

$$E(Y|X) = E(Y) + \frac{\text{Cov}(Y, X)}{\text{Var}(X)}(X - E(X)).$$

Taking expectations on both sides gives

$$E(E(Y|X)) = E(Y) + \frac{\text{Cov}(Y, X)}{\text{Var}(X)}(E(X) - E(X)) = E(Y). \qquad \square$$

We may also express the law of iterated expectations in terms of subsets of the original outcome space for a stochastic variable. Let the event $\mathcal{A}$ be a subset of the outcome space for Y and let $\mathcal{B}$ be a subset of $\mathcal{A}$. Then the law of iterated expectations takes the form

$$E(E(Y|\mathcal{B})|\mathcal{A}) = E(Y|\mathcal{A}). \tag{50}$$

That is, when $\mathcal{B} \subseteq \mathcal{A}$, the expectation, conditional on $\mathcal{A}$, of the expectation of Y, conditional on $\mathcal{B}$, is the same as the expectation, conditional on $\mathcal{A}$, of Y.

Often we consider a dynamic context where expectations are conditional on dated information I_{t-i} $(i = 1, 2, ...)$. By a, so far, "informal analogy" with (49) we then write the law of iterated expectations this way:

$$E(E(Y_t|I_{t-i})) = E(Y_t), \qquad \text{for } i = 1, 2, ... \tag{51}$$

In words: the unconditional expectation of the conditional expectation of Y_t, given the information up to time $t - i$ equals the unconditional expectation of Y_t. Similarly, by a, so far, "informal analogy" with (50) we may write

$$E(E(Y_{t+2}|I_{t+1})|I_t) = E(Y_{t+2}|I_t). \tag{52}$$

That is, the expectation today of the expectation tomorrow, when more may be known, of a variable the day after tomorrow is the same as the expectation today of the variable the day after tomorrow. Intuitively: you ask a stockbroker in which direction she expects to revise her expectations upon the arrival of more information. If the broker answers "upward", say, then another broker is recommended.

The notation used in the transition from (50) to (52) might seem problematic, though. That is why we talk of "informal analogy". The sets $\mathcal{A}$ and $\mathcal{B}$ are subsets of the outcome space and $\mathcal{B} \subseteq \mathcal{A}$. In contrast, the "information" or "information content" represented by our symbol I_t will, for the uninitiated, inevitably be understood in a meaning *not* fitting the inclusion $I_{t+1} \subseteq I_t$. Intuitively "information" dictates the opposite inclusion, namely as a set which *expands* over time – more and more "information" (like "knowledge" or "available data") is revealed as time proceeds.

It is possible, however, to interpret the information I_t from another angle so as to make the notation in (52) fully comply with that in (50). Let the outcome space Ω denote the set of ex ante possible[15] sequences $\{(Y_t, X_t)\}_{t=t_0}^{T}$, where Y_t and X_t are vectors of date-t endogenous and exogenous stochastic variables, respectively, and where T is the time horizon, possibly $T = \infty$. For $t \in \{t_0, t_0 + 1, \ldots, T\}$, let the subset $\Omega_t \subseteq \Omega$ be defined as the of time t still possible sequences $\{(Y_s, X_s)\}_{s=t}^{T}$. Now, as time proceeds, more and more realizations occur, that is, more and more of the ex ante random states, (Y_t, X_t), become historical data, (y_t, x_t). Hence, as time proceeds, the subset Ω_t *shrinks* in the sense that $\Omega_{t+1} \subseteq \Omega_t$. The increasing amount of information and the "reduced uncertainty" can thus be seen as two sides of the same thing. Interpreting I_t this way, i.e., as "partial lack of uncertainty", the expression (52) means the same thing as

$$E(E(Y_{t+2}|\Omega_{t+1})|\Omega_t) = E(Y_{t+2}|\Omega_t).$$

[15] By "possible" is meant "feasible according to a given model".

This is in complete harmony with (50).

C. Properties of the model-consistent forecast

As in the text of Section 24.2.2, let e_t denote the model-consistent forecast error $Y_t - E(Y_t|I_{t-1})$. Then, if S_{t-1} represents information contained in I_{t-1},

$$\begin{aligned} E(e_t\,|S_{t-1}) &= E(Y_t - E(Y_t\,|I_{t-1})\,|S_{t-1}) = E(Y_t\,|S_{t-1}) - E(E(Y_t\,|I_{t-1})\,|S_{t-1}) \\ &= E(Y_t\,|S_{t-1}) - E(Y_t\,|S_{t-1}) = 0, \end{aligned} \tag{53}$$

where we have used that $E(E(Y_t\,|I_{t-1})\,|S_{t-1}) = E(Y_t\,|S_{t-1})$, by the law of iterated expectations. With $S_{t-1} = I_{t-1}$ we have, as a special case,

$$\begin{aligned} E(e_t\,|I_{t-1}) &= 0, \quad \text{as well as} \\ E(e_t) &= E(Y_t - E(Y_t\,|I_{t-1})) = E(Y_t) - E(E(Y_t\,|I_{t-1})) = 0, \end{aligned} \tag{54}$$

in view of (51) with $i = 1$. This proves property (a) in Section 24.2.3.

As to property (b) in Section 24.2.2, for $i = 1, 2, ...$, let s_{t-i} be an arbitrary variable value belonging to the information I_{t-i}. Then, $E(e_t s_{t-i}\,|I_{t-i}) = s_{t-i}E(e_t\,|I_{t-i}) = 0$, by (53) with $S_{t-1} = I_{t-i}$ (since I_{t-i} is contained in I_{t-1}). Thus, by the principle (51),

$$E(e_t s_{t-i}) = E\left(E(e_t s_{t-i}\,|I_{t-i})\right) = E(0) = 0 \qquad \text{for } i = 1, 2, ... \tag{55}$$

This result is known as the *orthogonality property* of model-consistent expectations (two stochastic variables Z and V are said to be *orthogonal* if $E(ZV) = 0$). From the general formula for the (unconditional) covariance follows

$$\text{Cov}(e_t s_{t-i}) = E(e_t s_{t-i}) - E(e_t)E(s_{t-i}) = 0 - 0 = 0, \qquad \text{for } i = 1, 2, ...,$$

by (54) and (55). In particular, with $s_{t-i} = e_{t-i}$, we get $\text{Cov}(e_t e_{t-i}) = 0$. This proves that model-consistent forecast errors exhibit *lack of serial correlation*.

6 Exercises

1. Let $\{X_t\}$ be a stochastic process in discrete time. Suppose $Y_t = X_t + e_t$ and $X_t = X_{t-1} + \varepsilon_t$, where e_t and ε_t are white noise.

a) Is $\{X_t\}$ a random walk? Why or why not?

b) Is $\{Y_t\}$ a random walk? Why or why not?

c) Calculate the rational expectation of X_t conditional on all relevant information up to and including period $t-1$.

d) What is the rational expectation of Y_t conditional on all relevant information up to and including period $t-1$?

e) Compare with the subjective expectation of Y_t based on the adaptive expectations formula with adjustment speed equal to one.

2. Consider a simple Keynesian model of a closed economy with constant wages and prices (behind the scene), abundant capacity, and output determined by demand:

$$
\begin{aligned}
Y_t &= D_t = C_t + \bar{I} + G_t, && (1)\\
C_t &= \alpha + \beta Y^e_{t-1,t}, \qquad \alpha > 0,\ 0 < \beta < 1, && (2)\\
G_t &= (1-\rho)\bar{G} + \rho G_{t-1} + \varepsilon_t, \qquad \bar{G} > 0,\ 0 < \rho < 1, && (3)
\end{aligned}
$$

where the endogenous variables are Y_t = output (= income), D_t = aggregate demand, C_t = consumption, and $Y^e_{t-1,t}$ = expected output (income) in period t as seen from period $t-1$, while G_t, which stands for government spending on goods and services, is considered exogenous as is ε_t, which is white noise. Finally, investment, $\bar{I}$, and the parameters α, β, ρ, and $\bar{G}$ are given positive constants.

Suppose expectations are "static" in the sense that expected income in period t equals actual income in the previous period.

a) Solve for Y_t.

b) Find the income multiplier (partial derivative of Y_t) with respect to a change in G_{t-1} and ε_t, respectively.

Suppose instead that expectations are rational.

c) Explain what this means.

d) Solve for Y_t.

e) Find the income multiplier with respect to a change in G_{t-1} and ε_t, respectively.

f) Compare the result under e) with that under b). Comment.

3. Consider arbitrage between equity shares and a riskless asset paying the constant rate of return $r > 0$. Let p_t denote the price at the beginning of period t of a share that at the end of period t yields the dividend d_t. As seen from period t there is uncertainty about p_{t+i} and d_{t+i} for $i = 1, 2, \ldots$. Suppose agents have rational expectations and care only about expected return (risk neutrality).

a) Write down the no-arbitrage condition.

Suppose dividends follow the process $d_t = \bar{d} + \varepsilon_t$, where $\bar{d}$ is a positive constant and ε_t is white noise, observable in period t, but not known in advance.

b) Find the fundamental solution for p_t and let it be denoted p_t^*. *Hint:* given $y_t = aE_t y_{t+1} + c\, x_t$, the fundamental solution is $y_t = cx_t + c\sum_{i=1}^{\infty} a^i E_t x_{t+i}$.

Suppose someone claims that the share price follows the process

$$p_t = p_t^* + b_t,$$

with a given $b_0 > 0$ and, for $t = 0, 1, 2, \ldots,$

$$b_{t+1} = \begin{cases} \frac{1+r}{q_t} b_t & \text{with probability } q_t, \\ 0 & \text{with probability } 1 - q_t, \end{cases}$$

where $q_t = f(b_t)$, $f' < 0$.

c) What is an asset price bubble and what is a rational asset price bubble?

d) Can the described b_t process be a rational asset price bubble? *Hint:* a bubble component associated with the inhomogenous equation $y_t = aE_t y_{t+1} + c\, x_t$ is a solution, different from zero, to the homogeneous equation, $y_t = aE_t y_{t+1}$.

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Chapter 16

Money in macroeconomics

Money buys goods and goods buy money; but goods do not buy goods.
–Robert W. Clower (1967).

Up to now we have put monetary issues aside. The implicit assumption has been that the exchange of goods and services in the market economy can be carried out without friction as mere intra- or intertemporal barter. This is, of course, not realistic. At best it can provide an acceptable approximation to reality only for a limited set of macroeconomic issues. We now turn to models in which there is a demand for money. We thus turn to *monetary theory*, that is, the study of causes and consequences of the fact that a large part of the exchange of goods and services in the real world is mediated through the use of money.

16.1 What is money?

16.1.1 The concept of money

In economics *money* is defined as an asset (a store of value) which functions as a generally accepted medium of exchange, i.e., it can be used directly to buy *any* good or service offered for sale in the economy. Bitcoins may also be a medium of exchange, but are not *generally* accepted and are therefore not money. A note of IOU (a bill of exchange) may be a medium of exchange, but is not *generally* accepted and is therefore not money. Moreover, the extent to which an IOU is acceptable in exchange depends on the general state in the economy. In contrast, money is characterized by being a *fully liquid asset*. An asset is *fully liquid* if it can be used instantly, unconditionally, and without any extra costs or restrictions to make payments.

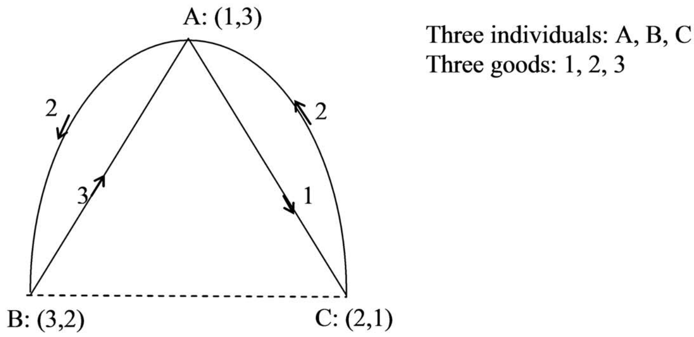

Figure 16.1: No direct exchange possible. A medium of exchange, here good 2, solves the problem (details in text).

Generally, liquidity should be conceived as a matter of degree so that an asset has a higher or lower degree of liquidity depending on the extent to which it can easily be exchanged for money. By "easily" we mean "immediately, conveniently, and cheaply". So an asset's *liquidity* is the *ease* with which the asset can be *converted into money or be used directly for making payments.* Where to draw the line between "money" and "non-money assets" depends on what is appropriate for the problem at hand. In the list below of different monetary aggregates (Section 16.2), M_1 corresponds most closely to the traditional definition of money. Defined as currency in circulation plus demand deposits held by the non-bank public in commercial banks, M_1 embraces all under "normal circumstances" (i.e., beyond financial crises) fully liquid assets in the hands of the non-bank public.[1]

The reason that a market economy uses money is that money facilitates trade enormously, thereby reducing transaction costs. Money helps an economy to avoid the need for a "double coincidence of wants". The classical way of illustrating this is by the *exchange triangle* in Fig. 16.1. The individuals A, B, and C are endowed with one unit of the goods 1, 3, and 2, respectively. But A, B, and C want to consume 3, 2, and 1, respectively. Thus, no direct exchange is possible between two individuals each wanting to consume the other's good. There is a *lack of double coincidence of wants.* The problem can be solved by indirect exchange where A exchanges good 1 for good 2 with C and then, in the next step, uses good 2 in an exchange for good 3 with B. Here good 2 serves as a medium of exchange. If good 2 becomes widely used and accepted as a medium of exchange, it is money. Extending the example to a situation with n goods, we have that exchange without money (i.e., barter) requires $n(n-1)/2$ markets ("trading spots"). Exchange with money, in the form of modern "paper money",

[1] The term "means of payment" is by some used as synonymous with "money", by others as including also media of exchange with slightly lower liquidity.

requires only n markets.

16.1.2 Historical remarks

In the past, ordinary commodities, such as seashells, rice, cocoa, precious metals etc., served as money. That is, commodities that were easily divisible, handy to carry, immutable, and involved low costs of storage and transportation could end up being used as money. This form of money is called *commodity money.* Applying ordinary goods as a medium of exchange is costly, however, because these goods have alternative uses. A more efficient way to trade is by using currency, i.e., coins and notes in circulation with little or no intrinsic value, or pieces of paper, checks, representing claims on such currency. Regulation by a central authority (the state or the central bank) has been of key importance in bringing about this transition into the modern payment system.

Coins, notes, pieces of paper like checks, and electronic signals from smart phones to accounts in a bank have no intrinsic value. Yet they may be generally accepted media of exchange, in which case we refer to them as *paper money.* By having these pieces of paper or electronic signals circulating and the real goods moving only once, from initial producer to final consumer, the trading costs in terms of time and effort are minimized.

In the industrialized countries these paper monies were in the last third of the nineteenth century and until the outbreak of the First World War *backed* through the gold standard. And under the Bretton-Woods agreement, 1947-71, the currencies of the developed Western countries outside the United States were convertible into US dollars at a fixed exchange rate (or rather an exchange rate which is adjustable only under specific circumstances); and US dollar reserves of these countries were (in principle) convertible into gold by the United States at a fixed price (though in practice with some discouragement from the United States).

This indirect gold-exchange standard broke down in 1971-73, and nowadays money in most countries is *unbacked* paper money (including electronic entries in bank accounts). This feature of modern money makes its valuation very different from that of other assets. A piece of paper money in a modern payments system has no worth at all to an individual unless she *expects* other economic agents to value it in the next instant. There is an *inherent circularity* in the acceptance of paper money. Hence the viability of a paper money system is very much dependent on adequate juridical institutions as well as confidence in the ability and willingness of the government and central bank to conduct policies that sustain the purchasing power of the currency. One elementary juridical institution is that of "legal tender", a status which is conferred to certain kinds of money.

An example is the law that a money debt can always be settled by currency and a tax always be paid by currency. A medium of exchange whose market value derives entirely from its legal tender status is called *fiat money* (because the value exists through "fiat", a ruler's declaration). In view of the absence of intrinsic value, maintaining the exchange value of fiat money over time, that is, avoiding high or fluctuating inflation, is one of the central tasks of monetary policy.

16.1.3 The functions of money

The following three functions are sometimes considered to be the definitional characteristics of money:

1. It is a generally accepted medium of exchange.

2. It is a store of value.

3. It serves as a unit of account in which prices are quoted and books kept (the *numeraire*).

On can argue, however, that the last function is on a different footing compared to the two others. Thus, we should make a distinction between the functions that money *necessarily* performs, according to our definition above, and the functions that money *usually* performs. Property 1 and 2 certainly belong to the essential characteristics of money. By its role as a device for making transactions money helps an economy to avoid the need for a double coincidence of wants. In order to perform this role, money *must* be a store of value, i.e., a device that transfers and maintains value over time. The reason that people are willing to exchange their goods for pieces of paper is exactly that these can later be used to purchase other goods. As a store of value, however, money is *dominated* by other stores of value such as bonds and shares that pay a higher rate of return. When nevertheless there is a demand for money, it is due to the *liquidity* of this store of value, that is, its service as a generally accepted medium of exchange.

Property 3, however, is not an indispensable function of money as we have defined it. Though the money unit is usually used as the unit of account in which prices are quoted, this function of money is conceptually distinct from the other two functions and has sometimes been distinct in practice. During times of high inflation, foreign currency has been used as a unit of account, whereas the local money continued to be used as the medium of exchange. During the German hyperinflation of 1922-23 US dollars were the unit of account used in parts of the economy, whereas the mark was the medium of exchange; and during the Russian hyperinflation in the middle of the 1990s again US dollars were often the unit of account, but the rouble was still the medium of exchange.

This is not to say that it is of little importance that money *usually* serves as numeraire. Indeed, this function of money plays an important role for the short-run macroeconomic effects of changes in the money supply. These effects are due to *nominal rigidities*, that is, the fact that prices, usually denominated in money, of most goods and services generally adjust only sluggishly (they are not traded in auction markets).

16.2 The money supply

The money supply is the total amount of money available in an economy at a particular point in time (a stock). As noted above, where to draw the line between assets that should be counted as money and those that should not, depends on the context.

16.2.1 Different measures of the money stock

Usually the money stock in an economy is measured as one of the following alternative *monetary aggregates*:

- M_0, i.e., the *monetary base*, alternatively called *base money, central bank money,* or *high-powered money.* The monetary base is defined as fully liquid claims on the central bank held by the private sector, that is, currency (coins and notes) in circulation plus *bank reserves.* The latter consist of demand deposits held by the commercial banks in the central bank plus currency in the "vaults" of these banks.[2] This monetary aggregate is under the direct control of the central bank and is changed through *open-market operations*, that is, through the central bank trading bonds, usually short-term government bonds, with the private sector. But clearly the monetary base is an imperfect measure of the liquidity in the private sector.

- M_1, defined as *currency in circulation* plus *demand deposits* held by the non-bank general public *in commercial banks.* Currency in circulation is currency held by the general public (households and non-bank firms). The demand deposits are also called *checking accounts* because they are deposits on which checks can be written and payment cards (debit cards) be used. M_1 does not include currency held by commercial banks and demand deposits held by commercial banks in the central bank. But currency in

[2] The commercial banks are usually part of the private sector and by law it is generally only the commercial banks that are allowed to have demand deposits in the central bank – the "banks' bank".

circulation, usually the *major* part of M_0, *is* included in M_1. Most importantly, the commercial banks use a portion of the funds received from depositors to make interest-bearing loans. Through this *bank lending,* M_1 is generally substantially larger than M_0.

The measure M_1 is one measure intended to reflect the quantity of assets serving as media of exchange in the hands of the non-bank part of the private sector. Broader measures of the money stock include:

- $M_2 = M_1$ plus savings accounts and small-denomination time deposits (say below € 100,000) that can easily be converted into a checkable account, although with a penalty. These claims are not instantly liquid, but they are close to.

- $M_3 = M_2$ plus large-denomination time-deposits (say above € 100,000).[3]

As we move down the list, the liquidity of the added assets decreases, while their interest yield increases.[4] Currency is of course fully liquid and earns zero interest. Along with currency, the demand deposits in the commercial banks are normally fully liquid, at least as long as they are guaranteed by a governmental deposit insurance (although normally only up to a certain maximum per account). The interest earned on these demand deposits is usually low or even nil (at least for "small" depositors) and is often ignored in simple theoretical models. When in macroeconomic texts the term "money supply" is used, normally M_1 or M_2 is meant, although part of M_2 is not directly usable as a means of payment and therefore not money in the strict meaning.

A related and theoretically important, simple classification of money types is the following:

1. *Outside money* = money that on net is an asset of the private sector.

2. *Inside money* = money that is not net wealth of the private sector.

Clearly M_0 is *outside money.* Most money in modern economies is *inside money*, however. Deposits at the commercial banks is an example of inside money. These deposits are an asset to their holders, but a liability of the banks.

[3] In casual notation, $M_1 \subset M_2 \subset M_3$, but $M_0 \nsubseteq M_1$ since only a part of M_0 belongs to M_1.

[4] This could be an argument for weighing the different components of a monetary aggregate by their degree of liquidity (see Barnett, 1980, and Spindt, 1985).

Payment cards versus credit cards Does it make sense to include the amounts that people are allowed to charge by using their *credit cards* in the concept of "broad money"? No, this would imply *double counting.* Actually *you* do not pay when you use a credit card at the store. It is the company issuing the credit card that pays to the store (shortly after you made your purchases). You postpone your payment until you receive your monthly bill from the credit card company. That is, the credit card company does the payment for you and gives credit *to you.* It is otherwise with a *payment card* (debet card) where the amount for which you buy is instantly charged your electronic account in the bank.

16.2.2 The money multiplier

Bank lending is the channel through which the monetary base expands to an effective money supply, the "money stock", considerably larger than the monetary base. The excess of the deposits of the non-bank part of the private sector over *bank reserves* ("vault cash" and demand deposits in the central bank) is lent out in the form of bank loans or used to buy government or corporate bonds. The non-bank public then deposits a fraction of these loans on checking accounts. Next, the banks lend out a fraction of these and so on. This process is named the *money multiplier process.* And the ratio of the "money stock", measured as M_1, say, to the monetary base is called the *money multiplier.*

Let

$$\begin{aligned} CUR &= \text{currency in circulation (= held by the non-bank general public)}, \\ DEP &= \text{demand deposits held by the non-bank general public}, \\ \frac{CUR}{DEP} &= cd, \text{ the desired currency-deposit ratio}, \\ RES &= \text{bank reserves } = \text{ currency held by the commercial banks} \\ & \quad \text{("vault cash") plus their demand deposits in the central bank}, \\ \frac{RES}{DEP} &= rd, \text{ the desired reserve-deposit ratio} \geq \text{ the required reserve-deposit ratio.} \end{aligned}$$

Notice that the currency-deposit ratio, cd, is chosen by the non-bank public, whereas the reserve-deposit ratio, rd, refers to the behavior of commercial banks. In many countries there is a minimum reserve-deposit ratio required by law to ensure a minimum liquidity buffer to forestall "bank runs" (situations where many depositors, fearing that their bank will be unable to repay their deposits in full and on time, simultaneously try to withdraw their deposits). On top of the minimum reserve-deposit ratio the banks may hold "excess reserves" depending on their assessment of their lending risks and need for liquidity.

We may express the money multiplier in terms of cd and rd. First, note that

$$M_1 = CUR + DEP = (cd + 1)DEP, \tag{16.1}$$

where DEP is related to the monetary base, M_0, through

$$M_0 = CUR + RES = cdDEP + rdDEP = (cd + rd)DEP.$$

Then, substituting into (16.1) gives

$$M_1 = \frac{cd + 1}{cd + rd} M_0 = mmM_0, \tag{16.2}$$

where $mm = (cd + 1)/(cd + rd)$ is the *money multiplier*.

As a not unrealistic example consider $cd \approx 0.7$ and $rd \approx 0.07$. Then we get $mm \approx 2.2$. When broader measures of money supply are considered, then, of course, a larger money multiplier arises. It should be kept in mind that both cd and rd, and therefore also mm, are neither constant nor exogenous from the point of view of monetary models. They are highly endogenous and depend on many things, including the degree of liquidity, risk, and expected returns on alternative assets − from the banks' perspective as well as the customers'. In the longer run, cd and rd are affected by the evolution of payment technologies.

To some extent it is therefore a matter of simple identities and not particularly informative, when we say that, given M_0 and the currency-deposit ratio, the money supply is smaller, the larger is the reserve-deposit ratio. Similarly, since the latter ratio is usually considerably smaller than one, the money supply is also smaller the larger is the currency-deposit ratio. Nevertheless, the money multiplier turns out to be fairly stable under "normal circumstances". But not always. During 1929-33, in the early part of the Great Depression, the money multiplier in the US fell sharply. Although M_0 increased by 15% during the four-year period, liquidity (M_1) declined by 27%.[5] Depositors became nervous about their bank's health and began to withdraw their deposits (thereby increasing cd) and this forced the banks to hold more reserves (thereby increasing rd). There is general agreement that this banking panic contributed to the depression and the ensuing deflation.

There is another way of interpreting the money multiplier. By definition of cd, we have $CUR = cdDEP$. Let cm denote the non-bank public's desired *currency-money ratio*, i.e., $cm = CUR/M_1$. Suppose cm is a constant. Then

$$CUR = cmM_1 = cm(cd + 1)DEP. \qquad \text{(by (16.1))}$$

[5] Blanchard (2003).

It follows that $cm = cd/(cd+1)$ and $1 - cm = 1/(cd+1)$. Combining this with (16.2) yields

$$M_1 = \frac{1}{\frac{cd}{cd+1} + rd\frac{1}{cd+1}} M_0 = \frac{M_0}{cm + rd(1-cm)} = \frac{1}{1-(1-rd)(1-cm)} M_0 = mmM_0. \tag{16.3}$$

The way the central bank is able to control the monetary base is through *open-market operations*. In its traditional form, *outright open-market operations*, the central bank trades short-term government bonds with the banks. When the central bank buys a government bond from a bank, it takes over a loan from the bank to the government, thereby in effect increasing the monetary base as if lending to the bank. The aim may be to sustain a desired level of M_1 or a desired level of the short-term interest rate or, in an open economy, a desired exchange rate vis-a-vis other currencies. The central bank may alternatively increase the monetary base through a *repurchase agreement*, which is another form of open-market operation, see below.

To obtain a perhaps more intuitive understanding of the money multiplier and the way commercial banks "create money", let us take a *dynamic perspective.* Suppose the central bank increases M_0 by the amount ΔM_0 through purchasing bonds in the market. This is the first round. The seller of the bonds deposits the fraction $1 - cm$ of the proceeds on a checking account in her bank and keeps the rest as cash. The bank keeps the fraction rd of $(1 - cm)\Delta M_0$ as reserves and provides bank loans or buys bonds in the market with the rest. This is the second round. Thus, in the first round money supply is increased by ΔM_0; in the second round it is further increased by $(1 - rd)(1 - cm)\Delta M_0$; in the third round further by $(1 - rd)^2(1 - cm)^2\Delta M_0$, etc.[6] In the end, the total increase in money supply is

$$\begin{aligned}\Delta M_1 &= \Delta M_0 + (1-rd)(1-cm)\Delta M_0 + (1-rd)^2(1-cm)^2\Delta M_0 + ... \\ &= \frac{1}{1-(1-rd)(1-cm)}\Delta M_0 = mm\Delta M_0.\end{aligned}$$

The second last equality comes from the rule for the sum of an infinite geometric series with quotient in absolute value less than one. The conclusion is that the money supply is increased mm times the increase in the monetary base.

16.3 Money demand

Explaining in a precise way how paper money gets purchasing power and how holding money - the "demand for money" in economists' traditional language -

[6] For simplicity, we assume here that cm and rd are constant.

is determined, is a complicated task and not our endeavour here. Suffice it to say that:

- In the presence of sequential trades and the absence of complete information and complete markets, there is a need for a generally accepted medium of exchange − *money.*

- The demand for money, by which we usually mean the quantity of money willingly held by the non-bank public, should be seen as part of a broader *portfolio decision* by which economic agents allocate their financial wealth to different existing assets, including money, and liabilities. The portfolio decision involves a balanced consideration of *after-tax expected return*, *risk*, and *liquidity.*

Money is demanded primarily because of its liquidity service in transactions. Money holding therefore depends on the *amount of transactions* households and firms plan to carry out with money in the near future. Money holding also depends on the *need for flexibility* in spending when there is *uncertainty*: it is appropriate to have ready liquidity in case favorable shopping opportunities should turn up and to have a buffer in case of ill-foreseen adverse events. Keynes (1936, p. 170 ff.) also emphasized the *speculative motive*, i.e., the liquidity demand induced when speculators believe they will know "better than the market" that a fall in the price of bonds, equity shares, or foreign currency will happen very soon.

Generally money earns no interest at all or at least less interest than other assets. Therefore money holding involves a trade-off between the need for liquidity and the wish for interest yield.

The incorporation of a somewhat micro-founded money demand in macromodels is often based on one or another kind of short-cut:

- The *cash-in-advance constraint* (also called the *Clower constraint*).[7] Generally, households' purchases of nondurable consumption goods are in every short period paid for by money held at the beginning of the period. With the cash-in-advance constraint it is simply postulated that to be able to carry out most transactions, you *must* hold money in advance. In continuous time models the household holds a stock of money which is an increasing function of the desired level of consumption per time unit and a decreasing function of the opportunity cost of holding money.

[7] After the American monetary theorist Robert Clower (1967). A better name for the constraint would be "money-in-advance constraint", since by "cash" is usually meant currency.

- The *shopping-costs* approach. Here the liquidity services of money are modelled as reducing shopping time or other kinds of non-pecuniary or pecuniary shopping costs. The shopping time needed to purchase a given level of consumption, c_t, is decreasing in real money holdings and increasing in c_t.
- The *money-in-the-utility function* approach. Here, the indirect utility that money provides through reducing non-pecuniary as well as pecuniary transaction costs is modelled as if the economic agents obtain utility directly from holding money. This will be our approach in the next chapter.
- The *money-in-the-production-function* approach. Here money facilitates the firms' transactions, making the provision of the necessary inputs easier. After all, typically around a third of the aggregate money stock is held by firms.

16.4 What is then the "money market"?

In macroeconomic theory, by the "money market" is usually meant an imaginary market place where at any moment the available aggregate stock of money (supply) "meets" the aggregate desired money holding (demand). Equilibrium in this market is presented by an equation saying that the supply equals the demand in the sense of the amount of money willingly held by the general public. Note that we talk about supply and demand in terms of *stocks* (amounts at a given point in time), not flows. To be specific, let the money supply in focus be money in the sense of M_1 (currency in circulation plus demand deposits in the banks) and let P denote the general price level in the economy (say the GDP deflator). At the demand side, let the aggregate demand for real money balances be represented by the function $L(Y, i)$, where $L_Y > 0$ and $L_i < 0$ ("L" for liquidity demand). The level of aggregate economic activity, Y, enters as an argument because it is an (approximate) indicator of the volume of transactions in the near future for which money is needed. The second argument, i, in the liquidity demand function is some index for the *short-term nominal interest rate* which reflects the opportunity cost of holding money instead of interest-bearing short-term financial claims that are close substitutes to money, i.e., have high relatively high but not full liquidity. We may think of interest-bearing time-deposits that are easily convertible into money, although at a penalty. Or i could be the interest rate on short-term government bonds ("treasury bills") or the interbank rate, see below.[8]

[8] To simplify, in (16.4) we assume that none of the components in the monetary aggregate considered earns interest. In practice demand deposits may earn a small nominal interest. In

So money market equilibrium is present if

$$M_1 = PL(Y, i). \tag{16.4}$$

One of the issues in monetary theory is to account for how this stock equilibrium is brought about. During the history of economic thought there has been different views about which of the variables M_1, P, Y, and i is the equilibrating variable such that the available stock of money becomes willingly held by the agents. Presuming that the central bank somehow controls M_1, classical (pre-Keynesian) monetary theory has P as the equilibrating variable. In Keynes' monetary theory (now mainstream), however, it is i which has this role while the general price level for goods and services is considered sticky in the short run. It will be the bond price, and hence i, which responds.− and establishes the equilibrium (16.4) very fast. Popular specifications of the function L include $L(Y, i) = Y^{\alpha} i^{-\beta}$ (constant elasticity of money demand with respect to i) and $L(Y, i) = Y^{\alpha} e^{-\beta i}$ (constant semi-elasticity of money demand with respect to i), where α and β are positive constants.

One may alternatively think of the "money market" in a more narrow sense, however. We may translate (16.4) into a description of demand and supply for *base money* (currency plus bank reserves in the central bank):

$$M_0 = \frac{P}{mm} L(Y, i), \tag{16.5}$$

where mm is the money multiplier. The right-hand side of this equation reflects that the demand for M_1 via the actions of commercial banks is transformed into a demand for base money.[9] If the general public wants to hold more money, the demand for bank loans rises and when granted, deposits expand. Then the banks try to increase their reserves to maintain the required (or in any case desired) reserve-deposit ratio. A bank that finds it has too little reserves will want to borrow reserves from other banks in what is known as the *interbank market,* often on a day-to-day basis. But the immediate situation is one of excess demand for bank reserves, and if its supply is not increased by the central bank, the interest rate in the interbank market, the *interbank rate,* rises. Then the interest rates on other short-term financial assets (short-term government bonds, time-deposits accounts, commercial paper, etc.) tend to move in the same direction because all these assets compete with each other. Assets offering higher-than-average rate of return will attract funds from assets offering lower-than-average rate of return, thereby roughly averaging out.

this case, i would indicate the excess of the short-term interest rate over this rate.

[9] Although the money multiplier tends to depend positively on i as well as other interest rates, this aspect is unimportant for the discussion below and is ignored in the notation in (16.5).

The "narrow" money market considered in (16.5) is a compact description of what is in the financial market statistics, and by the practitioners, called the "money market". This is the collective name for markets where trade in short-term debt-instruments (time to maturity less than one year). The agents trading in these markets include the central bank, the commercial banks, mortgage credit institutions, and other financial institutions. From a logical point of view a more appropriate collective name than "money market" would be "short-term bond market" or "near-money market". This would be in line with the usual way we use the term "market", namely as a "place" where a certain type of goods or assets are traded *for* money. Moreover, speaking of a "short-term bond market" corresponds well to the standard collective name for the markets for financial assets with maturity of *more* than one year, namely the "capital market" (where "capital" is synonymous with longer-term bonds and equity).

Anyway, in this book we maintain the usual term "money market" for the abstract market place where the aggregate supply of money "meets" the aggregate demand for money. As to what kind of money is in focus, "narrow" or "broad", further specification is always to be added.

Monetary policy and open-market operations

In recent decades the short-term interest rate has become the *main* monetary policy tool of the leading central banks in the world. In recent decades leading central banks, for instance both the Federal Reserve System of the US, the Fed, and the European Central Bank, the ECB, has increasingly focused on the short-term nominal interest rate as their policy tool. These central banks *announce* a *target level* for the chosen *policy interest rate* and then adjusts the monetary base through open-market operations such that the policy interest rate ends up very close to the announced target.

To understand the mechanism let us first imagine that the policy interest rate is the annualized interest rate, i_g, on one-month government bonds. Suppose the payoff is 1 euro at the *maturity date* and that there is no payoff between the *issue date* and the maturity date. Let p be the market price (in euros) of the bond at the issue date. The implicit monthly interest rate, x, is then the solution to the equation $v = (1 + x)^{-1}$, i.e.,

$$x = p^{-1} - 1.$$

Translated into an annual interest rate, this amounts to $i_g = (1+x)^{12} - 1 = p^{-12} - 1$ per year. With $p = 0.9975$, we get $i_g = 0.03049$ per year.[10]

[10] With continuous compounding we have $p = e^{-i_g/12}$ so that $i_g = 12 \ln p^{-1} = 0.03004$ when $p = 0.9975$.

Now, suppose the central bank finds that i_g is too high and therefore enters the market to buy a substantial amount of these bonds from the private sector. To find sellers a higher price of the bonds must be offered. The bond price, p, is thus driven up, and the rate i_g thereby lowered – until the available stocks of bonds and money in their new proportion are willingly held. In practice this adjustment of p, and hence i_g, to a new equilibrium level takes place fast. If the resulting i_g is not as low as the target value, the central bank continues its buying bonds for base money until it is.

In fact, for both the Fed and the ECB the chosen policy interest rate for which they announce a target value is not short-term government bonds but the interbank rate. Even so, the procedure to obtain that rate is still some form of open-market operations where government bonds are traded with the private sector. Because of the competition between different short-term assets in the financial markets, other interest rates, *including the interbank rate*, say i, will also be affected in a downward direction. The central bank continues its trading until the injection of base money has brought the interbank rate down to the announced target level. The interbank market is in the US called the Federal Funds market, and the interest rate.in this market is called the *Federal Funds Rate.*[11] Similarly, the ECB announces a certain target value for its quite similar policy rate called EONIA (Euro Overnight Index Average).

The aim of controlling the policy rate may be to stimulate or dampen the general level of economic activity, and the purpose of this may be controlling inflation, unemployment, or the foreign exchange rate. In this context what really matters is the interest rate households and firms have to pay when they borrow, the "bank lending rate"or the "corporate bond rate". Because of the higher risk involved, these rates tend to exceed the interbank rate by a substantial amount, known as the *interest spread.* In a financial crisis this spread may soar.

Repurchase agreement and repo rate Nowadays a lot of open market operations are carried out in the form of *repurchase agreements, repos* in brief. The central bank announces a specific interest rate at which it is willing to buy short-term government bonds from a commercial bank with the agreement that the bank buys back the bonds after a week, say, at a pre-agreed price such that the implied annualized interest rate on this loan equals the announced interest rate, the *repo rate.* This government bond serves as a collateral in the sense that if the bank defaults, the central bank has the bond.

In a *reverse repo* the buy and buy back roles of the two parties are reversed.

[11] In spite of its name, the Federal Funds Rate is not an interest rate charged by the U.S. central bank but a weighted average of the short-term interest rates commercial banks in the U.S. charge each other on overnight loans on an uncollateral basis.

The zero lower bound on the nominal interest rate In recessions, when the central bank attempts to stimulate aggregate demand by lowering the policy rate, i, it may reach a point where no further lowering is possible no matter how much money supply is increased. This point is attained when $i = 0$, the "zero lower bound". Agents would prefer holding money at zero interest rather than short-term bonds at negative interest. That is, the "=" in the equilibrium condition (16.4), or its equivalent, (16.5), should be replaced by "$\geq$" or, equivalently, $L(Y, i)$ should at $i = 0$ be interpreted as a "set-valued function". Strictly speaking the lower bound is slightly below zero because the alternative to holding bonds is holding money which gives zero interest but involves costs of storing, insuring, and transporting.

Monetary policy and the implications of the zero lower bound (or the slightly less than zero lower bound) are explored later in this book.

16.5 Key questions in monetary theory and policy

Some of the central questions in monetary theory and policy are:

1. How, and through what channels, do changes in the *level* of the money supply (in the M_0 sense, say), or the *growth rate* of the money supply affect (a) the real variables in the economy (resource allocation), and (b) the price level and the rate of inflation?

2. How do the effects of money supply movements depend on whether they occur through open-market operations or through the financing of budget deficits?

3. How do the effects depend on the state of the economy with respect to capacity utilization?

4. How can monetary policy be designed to stabilize the purchasing power of money and optimize the liquidity services to the inhabitants?

5. How can monetary policy be designed to stabilize the economy and "smooth" business cycle fluctuations?

6. Do rational expectations rule out persistent real effects of changes in the money supply?

7. Is hyperinflation always the result of an immense growth in the money supply or can hyperinflation be generated by self-fulfilling expectations?

8. What kind of regulation of commercial banks is conducive to a smooth functioning of the credit system and reduced risk of a financial crisis?

As an approach to some of these issues, we will in the next chapter consider a neoclassical monetary model by Sidrauski (1967). In this model money enters as a separate argument in the utility function. The model has been applied to the study of long-run aspects like the issues 1, 4, and 7 above. The model is less appropriate, however, for short- and medium-run issues such as 3, 5, and 8 in the list. These issues are dealt with in later chapters.

16.6 Literature notes

In the *Arrow-Debreu model*, the basic microeconomic general equilibrium model, there is assumed to exist a *complete set of markets*. That is, there is a market for each "contingent commodity", by which is meant that there are as many markets as there are possible combinations of physical characteristics of goods, dates of delivery, and "states of nature" that may prevail. In such an fictional world any agent knows for sure the consequences of the choices made. All trades can be made once for all and there will thus be no need for any money holding (Arrow and Hahn, 1971).

For the case of incomplete markets, Kiyotaki and Wright (JPE, 1989) and Trejos and Wright (JMCB, 1993) develop a microeconomic theory of how intrinsically valueless notes can obtain the role as a generally accepted means of exchange.

For a detailed account of the different ways of modelling money demand in macroeconomics, the reader is referred to, e.g., Walsh (2003). Concerning "money in the production function", see Mankiw and Summers (1986).

16.7 Exercises

Makroøkonomi 2 Note 3

26.10.2016 Christian Groth

Back to short-run macroeconomics

In this lecture note we shift the focus from long-run macroeconomics to short-run macroeconomics. The *long-run* models concentrated on mechanisms that are important for the economic evolution over a time horizon of at least 10-15 years. With such a horizon it is the development on the supply side (think of capital accumulation, population growth, and technical progress) that is the primary determinant of cumulative changes in output and consumption − the trend. The demand side and monetary factors are important for the fluctuations about the trend. In a long-run perspective these fluctuations have limited quantitative importance. But within a short horizon, say up to four years, the demand-side, monetary factors, nominal rigidities, and expectation errors are quantitatively important. The present note re-introduces these *short-run* factors and aims at suggesting how short-run and long-run theory are linked. This also implies a few remarks about theory dealing with the *medium run,* say 4 to 15 years.[1] The purpose of medium-run theory is to explain the regularities in the fluctuations (business cycles) about the trend and to study what can be accomplished by monetary and fiscal stabilization policy. In that context the *dynamic interaction* between demand and supply factors and the time-consuming adjustment in relative prices play an important role. In this way medium-run theory bridges the gap between the long run and the short run.

1 Stylized facts about the short run

The idea that prices of most goods and services are sticky in the short run rests on the empirical observation that in the short run firms in the manufacturing and service industries typically let output do the adjustment to changes in demand while keeping prices unchanged. In industrialized societies firms are able to do that because under "normal circumstances" there is "abundant production capacity" available in the economy. Three of the most salient short-run features that arise from macroeconomic time series

[1] These number-of-years declarations should not be understood as more than a rough indication. Their appropriateness will depend on the specific historical circumstances and on the problem at hand.

analysis of industrialized market economies are the following (cf. Blanchard and Fischer, 1989, Christiano et al., 1999):

1) Shifts in aggregate demand (induced by sudden changes in the state of confidence, exports, fiscal or monetary policy, or other events) are largely accommodated by changes in quantities rather than changes in nominal prices – *nominal price insensitivity.*

2) Even large movements in quantities are often associated with little or no movement in relative prices – *real price insensitivity.* The real wage, for instance, exhibits such insensitivity in the short run.

3) Nominal prices are *sensitive* to general changes in *input costs.*

These stylized facts pertain to final goods and services. It is not the case that *all* nominal prices in the economy are in the short run insensitive vis-a-vis demand changes. One must distinguish between production of most final goods and services on the one hand and production of primary foodstuff and raw materials on the other. This leads to the associated distinction between "cost-determined" and "demand- determined" prices.

Final goods and services are typically differentiated goods (imperfect substitutes). Their production takes place under conditions of imperfect competition. As a result of existing reserves of production capacity, generally speaking, the *production is elastic w.r.t. demand.* A shift in demand tends to be met by a change in production rather than price. The price changes which do occur are mostly a response to general changes in costs of production. Hence the name "cost-determined" prices.

For *primary foodstuff* and many *raw materials* the situation is different. To increase the supply of most agricultural products requires considerable time. This is also true (though not to the same extent) with respect to mining of raw materials as well as extraction and transport of crude oil. When production is *inelastic* w.r.t. demand in the short run, an increase in demand results in a diminution of stocks and a rise in price. Hence the name "demand-determined prices". The price rise may be enhanced by a speculative element: temporary hoarding in the expectation of further price increases. The price of oil and coffee – two of the most traded commodities in the world market – fluctuate a lot. Through the channel of *costs* the changes in these demand-determined prices spill over to the prices of final goods. Housing construction is time consuming and is also an area where, apart from regulation, demand-determined prices is the rule in the short run.

In industrialized economies manufacturing and services are the main sectors, and the general price level is typically regarded as cost-determined rather than demand determined. Two further aspects are important. First, many wages and prices are set in nominal terms by *price setting agents* like craft unions and firms operating in imperfectly competitive output markets. Second, these wages and prices are in general deliberately kept unchanged for some time even if changes in the environment of the agent occurs; this aspect, possibly due to pecuniary or non-pecuniary costs of changing prices, is known as *nominal price stickiness.* Both aspects have vast consequences for the functioning of the economy as a whole compared with a regime of perfect competition and flexible prices.

Note that *price insensitivity* just refers to the sheer observation of absence of price change in spite of changes in the "environment" − as in the context of facts 1) and 2) above. *Price stickiness* refers to more, namely that prices do not move quickly enough to clear the market in the short run. While price stickiness is in principle a matter of degree, the term includes the limiting case where prices are entirely "fixed" over the period considered − the case of *price rigidity.*

2 A simple short-run model

The simple model presented below is close to what Paul Krugman named the *World's Smallest Macroeconomic Model.*[2] The model is crude but nevertheless useful in at least three ways:

- the model demonstrates the fundamental difference in the *functioning* of an economy with fully flexible prices and one with sticky prices;
- by addressing spillovers across markets, the model is a suitable point of departure for a definition of the Keynesian concept of *effective demand;*
- the model displays the logic behind the Keynesian *refutation* of *Say's law.*

2.1 Elements of the model

We consider a monetary closed economy which produces a consumption good. There are three sectors in the economy, a production sector, a household sector, and a public

[2] Krugman (1999). Krugman tells he learned the model back in 1975 from Robert Hall. As presented here there is an inspiration from Barro and Grossman (1971).

sector with a consolidated government/central bank. Time is discrete. There is a current period, of length a month or a quarter of a year, say, and "the future", compressing the next period and onward. Labor is the only input in production. To simplify notation, the model presents its story as if there is just one representative household and one representative firm owned by the household, but the reader should imagine that there are numerous agents, all alike, of each kind.

The production function has CRS,

$$Y = AN, \qquad A > 0, \tag{1}$$

where Y is aggregate output of a consumption good which is perishable and therefore cannot be stored, A is a technology parameter and N is aggregate employment in the current period. In short- and medium-run macroeconomics the tradition is to use N to denote labor input ("number of hours"), while L is typically used for either money demand ("liquidity demand") or supply of bank loans. We follow this tradition.

The price of the consumption good in terms of money, i.e., the *nominal* price, is P. The wage rate in terms of money, the *nominal* wage, is W. We assume that the representative firm maximizes profit, taking these current prices as given. The nominal profit, possibly nil, is

$$\Pi = PY - WN. \tag{2}$$

There is free exit from the production sector in the sense that the representative firm can decide to produce nothing. Hence, an equilibrium with positive production requires that profits are non-negative.

The household supplies labor inelastically in the amount $\bar{N}$ and receives the profit obtained by the firm, if any. The household demands the consumption good in the amount C^d in the current period (since we want to allow cases of non-market clearing, we distinguish between consumption *demand,* C^d, and realized consumption, C. Current income not consumed is saved for the future. As the output good cannot be stored, the only non-human asset available in the economy is fiat money, which is thus the only asset on hand for saving. There is no private banking sector in the economy. So "money" means the "currency in circulation" (the monetary base) and is on net an asset in the private sector as a whole. Until further notice the money stock is constant.

The preferences of the household are given by the utility function,

$$U = \ln C^d + \beta \ln \frac{\hat{M}}{P^e}, \qquad 0 < \beta < 1, \tag{3}$$

where $\hat{M}$ is the amount of money transferred to "the future", and P^e is the expected future price level. The utility discount factor β (equal to $(1 + \rho)^{-1}$ if ρ is the utility discount rate) reflects "patience".

Consider the household's choice problem. Facing P and W and expecting that the future price level will be P^e, the household chooses C^d and $\hat{M}$ to maximize U s.t.

$$PC^d + \hat{M} = M + WN + \Pi \equiv B, \qquad N \leq N^s = \bar{N}. \tag{4}$$

Here, $M > 0$ is the stock of money held at the beginning of the current period and is predetermined. The actual employment is denoted N and equals the minimum of the amount of employment offered by the firm and the labor supply $\bar{N}$ (the principle of voluntary trade). The sum of initial financial wealth, M, and nominal income, $WN + \Pi$, constitutes the budget, B.[3] Nominal financial wealth at the beginning of the next period is $\hat{M} = M + WN + \Pi - PC^d$, i.e., the sum of initial financial wealth and planned saving where the latter equals $WN + \Pi - PC^d$. The benefit obtained by transferring $\hat{M}$ depends on the expected purchasing power of $\hat{M}$, hence it is $\hat{M}/P^e$ that enters the utility function. Presumably, the household has expectations about labor and profit income also in the future. But there is no feedback from these expectations; current decisions do not affect them. All households are alike and so there is no intertemporal exchange. Ownership rights to the firms' profits, if any, in the future are not tradable. The model is set up such that although there is a future, its role for the present is minimal.

Substituting $\hat{M} = B - PC^d$ into (3), we get the first-order condition

$$\frac{dU}{dC^d} = \frac{1}{C^d} + \beta \frac{P^e}{B - PC^d}(-\frac{P}{P^e}) = 0,$$

which gives

$$PC^d = \frac{1}{1+\beta}B. \tag{5}$$

We see that the marginal (= average) propensity to consume is $(1+\beta)^{-1}$, hence inversely related to the patience parameter β. The planned stock of money to be held at the end of the period is

$$\hat{M} = (1 - \frac{1}{1+\beta})B = \frac{\beta}{1+\beta}B.$$

So, the expected price level, P^e, in the future does not affect the demands, C^d and $\hat{M}$. This is a special feature caused by the additive-logarithmic specification of the utility

[3] As time is discrete, expressions like $M + WN + \Pi$ are legitimate. Although it is meaningless to add a stock and a flow (since they have different denominations), the sum $M + WN + \Pi$ should be interpreted as $M + (WN + \Pi)\Delta t$, where Δt is the period length. With the latter being the time unit, we have Δt = 1.

function in (3). Indeed, with this specification the substitution and income effects of a rise in the expected real gross rate of return, $(1/P^e)/(1/P)$, on saving exactly offset each other. And in this simplistic model, a change in the expected rate of return implies no wealth effect on the current consumption decision.

Inserting (4) and (2) into (5) gives

$$C^d = \frac{B}{P(1+\beta)} = \frac{M + WN + \Pi}{P(1+\beta)} = \frac{\frac{M}{P} + Y}{1+\beta}, \tag{6}$$

In our simple model output demand is the same as the consumption demand C^d. So *clearing* in the output market, in the sense of equality between demand and actual output, requires $C^d = Y$. So, *if* this clearing condition holds, substituting into (6) gives the relationship

$$Y = \frac{M}{\beta P}. \tag{7}$$

This is only a *relationship* between Y and P, not a solution for any of them since both are endogenous variables so far. Moreover, the relationship is *conditional on clearing* in the output market.

We have assumed that agents take prices as given when making their demand and supply decisions. But we have said nothing about whether nominal prices are flexible or rigid as seen from the perspective of the system as a whole.

2.2 The case of fully flexible W and P

What Keynes called "classical economics" is nowadays also often called "Walrasian macroeconomics" (sometime just "pre-Keynesian macroeconomics"). In this theoretical tradition both wages and prices are assumed fully flexible and all markets perfectly competitive.

Firms' ex ante output supply conditional on a hypothetical wage-price pair (W, P) and the corresponding labor demand will be denoted Y^s and N^d, respectively. As we know from microeconomics, the pair (Y^s, N^d) need not be unique, it can easily be a "set-valued function" of (W, P). Moreover, with constant returns to scale in the production function, the range of this function may for certain pairs (W, P) include (∞, ∞).

The distinguishing feature of the Walrasian approach is that wages and prices are assumed fully flexible. Both W and P are thought to adjust immediately so as to clear the labor market and the output market like in a centralized auction market. Clearing in the labor market requires that W and P are adjusted so that actual employment, N, equals labor supply, N^s, which is here inelastic at the given level $\bar{N}$. So

$$N = N^s = \bar{N} = N^d, \tag{8}$$

where the last equality indicates that this employment level is willingly demanded by the firms.

We have assumed a constant-returns-to-scale production function (1). Hence, the clearing condition (8) requires that firms have zero profit. In turn, by (1) and (2), zero profit requires that the real wage equals labor productivity:

$$\frac{W}{P} = A. \tag{9}$$

With clearing in the labor market, output must equal full-employment output,

$$Y = A\bar{N} \equiv Y^f = Y^s, \tag{10}$$

where the superscript "f" stands for "full employment", and where the last equality indicates that this level of output is willingly supplied by the firms. For this level of output to match the demand, C^d, coming from the households, the price level must be

$$P = \frac{M}{\beta Y^f} \equiv P^c, \tag{11}$$

in view of (7) with $Y = Y^f$. This price level is the *classical equilibrium price*, hence the superscript "c". Substituting into (9) gives the *classical equilibrium wage*

$$W = AP^c \equiv W^c. \tag{12}$$

For general equilibrium we also need that the desired money holding at the end of the period equals the available money stock. By *Walras' law* this equality follows automatically from the household's *Walrasian* budget constraint and clearing in the output and labor markets. To see this, note that the *Walrasian* budget constraint is a *special case* of the budget constraint (4), namely the case

$$PC^d + \hat{M} = M + WN^s + \Pi^c, \tag{13}$$

where Π^c is the notional profit associated with the hypothetical production plan (Y^s, N^d), i.e.,

$$\Pi^c \equiv PY^s - WN^d. \tag{14}$$

The Walrasian budget constraint thus *imposes* replacement of the term for *actual* employment, N, with the households' desired labor supply, $N^s(= \bar{N})$. It also *imposes* replacement of the term for *actual* profit, Π, with the hypothetical profit Π^c ("c" for "classical") calculated on the basis of the firms' aggregate production plan (Y^s, N^d).

Now, let the Walrasian auctioneer announce an arbitrary price vector $(W, P, 1)$, with $W > 0$, $P > 0$, and 1 being the price of the numeraire, money. Then the values of excess demands add up to

$$\begin{aligned} & W(N^d - N^s) + P(C^d - Y^s) + \hat{M} - M \\ = \; & WN^d - PY^s + PC^d + \hat{M} - M - WN^s \quad \text{(by rearranging)} \\ = \; & WN^d - PY^s + \Pi^c \quad \text{(by (13))} \\ = \; & WN^d - PY^s + \Pi^c \equiv 0. \qquad \text{(from definition of } \Pi^c \text{ in (14))} \end{aligned}$$

This exemplifies *Walras' law,* saying that with Walrasian budget constraints the aggregate value of excess demands is *identically* zero. Walras' law reflects that when households satisfy their Walrasian budget constraint, then as an arithmetic necessity the economy as a whole has to satisfy an aggregate budget constraint for the period in question. It follows that the equilibrium condition $\hat{M} = M$ is ensured as soon as there is clearing in the output and labor markets. And more generally: if there are n markets and $n - 1$ of these clear, so does the n'th market.

Consequently, when $(W, P) = (W^c, P^c)$, all markets clear in this flexwage-flexprice economy with perfect competition and a representative household with the "endowment"-pair $(M, \bar{N})$. Such a state of affairs is known as a *classical* or *Walrasian equilibrium.*[4] A key feature is expressed by (8) and (10): output and employment are *supply-determined*, i.e., determined by the supply of production factors, here labor.

The intuitive mechanism behind this equilibrium is the following adjustment process. Imagine that in an ultra-short sub-period $W/P - A \neq 0$. In case $W/P - A > 0$ (< 0), there will be excess supply (demand) in the labor market. This drives W down (up). Only when $W/P = A$ and full employment obtains, can the system be at rest. Next imagine that $P - P^c \neq 0$. In case $P - P^c > 0$ (< 0), there is excess supply (demand) in the output market. This drives P down (up). Again, only when $P = P^c$ *and* $W/P = A$ (whereby $W = W^c$), so that the output market clears under full employment, will the system be at rest.

This adjustment process is fictional, however, because outside equilibrium the Walrasian supplies and demands, which supposedly drive the adjustment, are artificial con-

[4]To underline its one-period nature, it may be called a Walrasian *short-run* or a Walrasian *temporary equilibrium.*

structs. Being functions only of initial resources and price signals, the Walrasian supplies and demands are mutually inconsistent outside equilibrium and can therefore not tell what quantities will be traded during an adjustment process. The story needs a considerable refinement unless one is willing to let the mythical "Walrasian auctioneer" enter the scene and bring about adjustment toward the equilibrium prices without allowing trade until these prices are found.

Anyway, assuming that Walrasian equilibrium has been attained, by comparative statics based on (11) and (12) we see that in the classical regime: (a) P and W are proportional to M; and (b) output is at the unchanged full-employment level whatever the level of M. This is the *neutrality of money* result of classical macroeconomics.

The neutrality result also holds in a (quasi-)dynamic context where we consider an actual change in the money stock occurring in historical time. Suppose the government/central bank at the beginning of the period brings about lump-sum transfers to the households in the total amount $\Delta M > 0$. As there is no taxation, this implies a budget deficit which is thus fully financed by money issue.[5] So (4) is replaced by

$$PC^d + \hat{M} = M + \Delta M + W\bar{N} + \Pi^c. \tag{15}$$

If we replace M in the previous formulas by $M' \equiv M + \Delta M$, we see that money neutrality still holds. As *saving* is income minus consumption, there is now positive nominal private saving of size $S^p = \Delta M + W\bar{N} + \Pi^c - PC^d = M' - M = \Delta M$. On the other hand the government dissaves, in that its saving is $S^g = -\Delta M$, where ΔM is the government budget deficit. So national saving is and remains $S \equiv S^p + S^g = 0$ (it *must* be nil as there are no durable produced goods).

2.3 The case of W and P fixed in the short run

In standard Keynesian macroeconomics nominal wages are considered predetermined in the short run, fixed in advance by wage bargaining between workers (or workers' unions) and employers (or employers' unions). Those who end up unemployed in the period do not try to – or are not able to – undercut those employed, at least not in the current period.

Likewise, nominal prices are set in advance by firms facing downward-sloping demand curves. It is understood that there is a large spectrum of differentiated products, and Y

[5] Within the model this is in fact the only way to increase the money stock. As money is the only asset in the economy, a change in the money stock can not be brought about through open market operations where the central bank buys or sells another financial asset.

and C are composites of these. This heterogeneity ought of course be visible in the model − and it will become so in Section 19.3. But at this point the model takes an easy way out and ignores the involved aggregation issue.

Let W in the current period be given at the level $\bar{W}$. Because firms have market power, the profit-maximizing price involves a mark-up on marginal cost, $\bar{W}N/Y = \bar{W}/A$ (which is also the average cost). We assume that the price setting occurs under circumstances where the chosen mark-up becomes a *constant* $\mu > 0$, so that

$$P = (1+\mu)\frac{\bar{W}}{A} \equiv \bar{P}. \tag{16}$$

While $\bar{W}$ is considered exogenous (not determined within the model), $\bar{P}$ is endogenously determined by the given $\bar{W}$, A, and μ. There are barriers to entry in the short run.

Because of the fixed wage and price, the distinction between *ex ante* (also called *planned* or *intended*) demands and supplies and the *ex post* carried out purchases and sales are now even more important than before. This is because the different markets may now also *ex post* feature excess demand or excess supply (to be defined more precisely below). According to the principle that no agent can be forced to trade more than desired, the actual amount traded in a market must equal the minimum of demand and supply. So in the output market and the labor market the actual quantities traded will be

$$Y = \min(Y^d, Y^s) \quad \text{and} \tag{17}$$

$$N = \min(N^d, N^s), \tag{18}$$

respectively, where the superscripts "*d*" and "*s*" are now used for demand and supply in a *new* meaning to be defined below. This principle, that the short side of the market determines the traded quantity, is known as the *short-side rule.* The other side of the market is said to be *quantity rationed* or just *rationed* if there is discrepancy between Y^d and Y^s. In view of the produced good being non-storable, intended inventory investment is ruled out. Hence, the firms try to avoid producing more than can be sold. In (17) we have thus identified the traded quantity with the produced quantity, Y.

But what exactly do we mean by "demand" and "supply" in this context where market clearing is not guaranteed? We mean what is appropriately called the *effective demand* and the *effective supply* ("effective" in the meaning of "operative" in the market, though possibly frustrated in view of the short-side rule). To make these concepts clear, we need first to define an agent's *effective* budget constraint:

DEFINITION 1 An agent's (typically a household's) *effective budget constraint* is the budget constraint conditional on the perceived price and quantity signals from the markets.

It is the last part, "and quantity signals from the markets", which is not included in the concept of a Walrasian budget constraint. The perceived quantity signals are in the present context a) the *actual* employment constraint faced by the household and b) the profit expected to be received from the firms and determined by their *actual* production and sales. So the household's effective budget constraint is given by (4). In contrast, the Walrasian budget constraint is not conditional on quantity signals from the markets but only on the "endowment" $(M, \bar{N})$ and the perceived price signals and profit.

DEFINITION 2 An agent's *effective demand* in a given market is the amount the agent *bids for* in the market, conditional on the perceived price and quantity signals that constrains its bidding. By "bids for" is meant that the agent is both *able* to buy that amount and *wishes* to buy that amount, given the effective budget constraint. Summing over all potential buyers, we get the *aggregate effective demand* in the market.

DEFINITION 3 An agent's *effective supply* in a given market is the amount the agent *offers for sale* in the market, conditional on perceived price and quantity signals that constrains its offering. By "offers for sale" is meant that the agent is both *able* to bring that amount to the market and *wishes* to sell that amount, given the set of opportunities available. Summing over all potential sellers, we get the *aggregate effective supply* in the market.

When $P = \bar{P}$, the aggregate effective output demand, Y^d, is the same as households' consumption demand given by (6) with $P = \bar{P}$, i.e.,

$$Y^d = C^d = \frac{\frac{M}{\bar{P}} + Y}{1 + \beta}. \tag{19}$$

In view of the inelastic labor supply, households' aggregate effective labor supply is simply

$$N^s = \bar{N}.$$

Firms' aggregate effective output supply is

$$Y^s = Y^f \equiv A\bar{N}. \tag{20}$$

Indeed, in the aggregate the firms are *not able* to bring more to the market than full-employment output , Y^f. And every individual firm is not able to bring to the market than what can be produced by "its share" of the labor force. On the other hand, because of the constant marginal costs, every unit sold at the preset price adds to profit. The firms are therefore happy to satisfy any output demand forthcoming – which is in practice testified by a lot of sales promotion.

Firms' aggregate effective demand for labor is constrained by the perceived output demand, Y^d, because the firm would loose by employing more labor. Thus,

$$N^d = \frac{Y^d}{A}. \tag{21}$$

By the short-side rule (17), combined with (20), follows that actual aggregate output (equal to the quantity traded) is

$$Y = \min(Y^d, Y^f) \leqq Y^f.$$

So the following three mutually exclusive cases exhaust the possibilities regarding aggregate output:

$$\begin{aligned} Y &= Y^d < Y^f \quad \text{(the Keynesian regime)}, \\ Y &= Y^f < Y^d \quad \text{(the repressed inflation regime)}, \\ Y &= Y^d = Y^f \quad \text{(the border case)}. \end{aligned}$$

2.3.1 The Keynesian regime: $Y = Y^d < Y^f$.

In this regime we can substitute $Y = Y^d$ into (19) and solve for Y:

$$Y = Y^d = \frac{M}{\beta \bar{P}} \equiv Y^k < Y^f \equiv \frac{M}{\beta P^c} = Y^s. \tag{22}$$

where we have denoted the resulting output Y^k (the superscript "k" for "Keynesian"). The inequality in (**??**) is required by the definition of the Keynesian regime, and the identity comes from (11). Necessary and sufficient for the inequality is that $\bar{P} > P^c \equiv W^c/A$. In view of (16), the economy is thus in the Keynesian regime if and only if

$$\bar{W} > W^c/(1 + \mu). \tag{23}$$

Since $Y < Y^s$ in this regime, we may say there is "excess supply" in the output market or, with a perhaps better term, there is a "buyers' market" situation (sale less than desired). The reservation regarding the term "excess supply" is due to the fact that we should not forget that $Y - Y^s < 0$ is a completely voluntary state of affairs on the part of the price-setting firms.

From (1) and the short-side rule now follows that actual employment will be

$$N = N^d = \frac{Y}{A} = \frac{M}{A\beta\bar{P}} < \bar{N} = N^s. \tag{24}$$

Also the labor market is thus characterized by "excess supply" or a "buyers' market" situation. Profits are $\Pi = \bar{P}Y - \bar{W}N = (1 - \bar{W}/(\bar{P}A))\bar{P}Y = (1 - (1+\mu)^{-1})\beta^{-1}M > 0$, where we have used, first, $Y = AN$, then the price setting rule (16), and finally (**??**).

This solution for (Y, N) is known as a *Keynesian equilibrium* for the current period. It is named an *equilibrium* because the system is "at rest" in the following sense: (a) agents do the best they can given the constraints (which include the preset prices and the quantities offered by the other side of the market); and (b) the chosen actions are *mutually compatible* (purchases and sales match). The term equilibrium is here not used in the Walrasian sense of market clearing through instantaneous price adjustment but in the sense of a *Nash equilibrium* conditional on perceived price and quantity signals. To underline its temporary character, the equilibrium may be called a Keynesian *short-run* (or *temporary*) equilibrium. The flavor of the equilibrium is *Keynesian* in the sense that there is unemployment and at the same time it is aggregate demand in the output market, not the real wage, which is the binding constraint on the employment level. A higher propensity to consume (lower discount factor β) results in higher aggregate demand, Y^d, and thereby a higher equilibrium output, Y^k. In contrast, a lower real wage due to either a higher mark-up, μ, or a lower marginal (= average) labor productivity, A, does *not* result in a higher Y^k. On the contrary, Y^k becomes *lower*, and the causal chain behind this goes via a higher $\bar{P}$, cf. (16) and (**??**). In fact, the given real wage, $\bar{W}/\bar{P} = A/(1+\mu)$, is consistent with unemployment as well as full employment, see below. It is the sticky nominal *price* at an excessive level, caused by a sticky nominal *wage* at an "excessive" level, that makes unemployment prevail through a too low aggregate demand, Y^d. A lower nominal wage would imply a lower $\bar{P}$ and thereby, for a given M, stimulate Y^d and thus raise Y^k.

In brief, the Keynesian regime leads to an equilibrium where output as well as employment are *demand-determined*.

The "Keynesian cross" and effective demand The situation is illustrated by the "Keynesian cross" in the (Y, Y^d) plane shown in Fig. 19.1, where $Y^d = C^d = (1+\beta)^{-1}(M/\bar{P} + Y)$. We see the vicious circle: Output is below the full-employment level because of low consumption demand; and consumption demand is low because of the low employment. The economy is in a *unemployment trap*. Even though at Y^k we have $\Pi > 0$

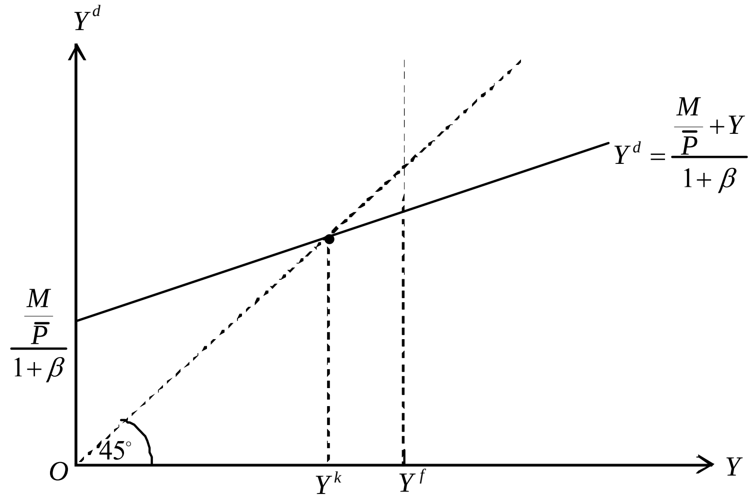

Figure 1: The Keynesian regime ($\bar{W} > W^c/(1+\mu)$; M and Y^f given, $\bar{P}$ fixed).

and there are constant returns to scale, the individual firm has no incentive to increase production because the firm already produces as much as it rightly perceives it can sell at its preferred price. We also see that here money is *not neutral*. For a given $W = \bar{W}$, and thereby a given $P = \bar{P}$, a higher M results in higher output and higher employment.

Although the microeconomic background we have alluded to is a specific "market power story" (one with differentiated goods and downward sloping demand curves), the Keynesian cross in Fig. 19.1 may turn up also for other microeconomic settings. The key point is the fixed $\bar{P} > P^c$ and fixed $\bar{W} < A\bar{P}$.

The fundamental difference between the Walrasian and the present framework is that the latter allows trade outside Walrasian equilibrium. In that situation the households' consumption demand depends *not* on how much labor the households would *prefer* to sell at the going wage, but on how much they are *able* to sell, that is, on a *quantity signal* received from the labor market. Indeed, it is the *actual* employment, N, that enters the operative budget constraint, (4), not the desired employment as in classical or Walrasian theory.

2.3.2 The repressed-inflation regime: $Y = Y^f < Y^d$.

This regime represents the "opposite" case of the Keynesian regime and arises if and only if the opposite of (**??**) holds, namely

$$\bar{W} < W^c/(1+\mu).$$

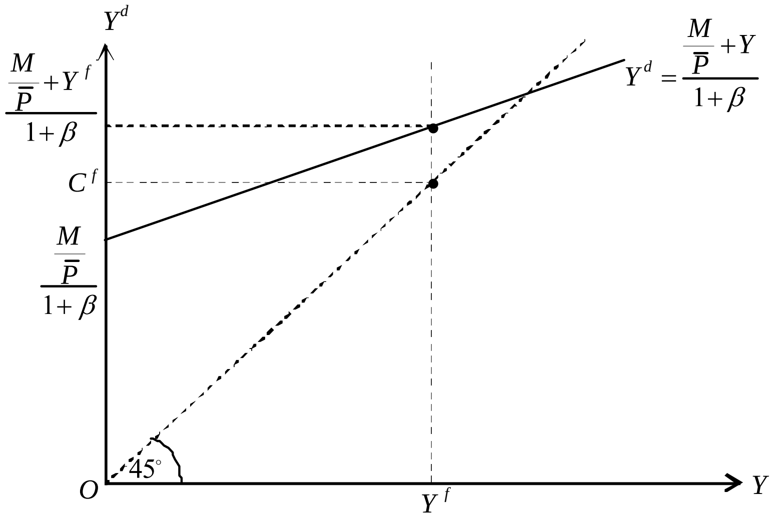

Figure 2: The repressed inflation regime ($\bar{W} < W^c/(1+\mu)$; M and Y^f given, $\bar{P}$ fixed).

In view of (16), this inequality is equivalent to $\bar{P} < W^c/A \equiv P^c$. Hence $M/(\beta\bar{P}) > M/(\beta P^c) = Y^f = A\bar{N}$. In spite of the high output demand, the shortage of labor hinders the firms to produce more than Y^f. With $Y = Y^f$, output demand, which in this model is always the same as consumption demand, C^d, is, from (6),

$$Y^d = \frac{\frac{M}{\bar{P}} + Y^f}{1+\beta} > Y = Y^s = Y^f. \tag{25}$$

As before, effective output supply, Y^s, equals full-employment output, Y^f.

The new element here in that firms perceive a demand level in excess of Y^f. As the real-wage level does not deter profitable production, firms would thus prefer to employ people up to the point where output demand is satisfied. But in view of the short side rule for the labor market, actual employment will be

$$N = N^s = \bar{N} < N^d = \frac{Y^d}{A}.$$

So there is excess demand in both the output market and the labor market. Presumably, these excess demands generate pressure for wage and price increases. By assumption, these potential wage and price increases do not materialize until possibly the next period. So we have a *repressed-inflation equilibrium* $(Y, N) = (Y^f, \bar{N})$, although possibly short-lived.

Fig. 19.2 illustrates the repressed-inflation regime. In the language of the microeconomic theory of quantity rationing, consumers are quantity rationed in the goods market, as realized consumption $= Y = Y^f < Y^d =$ consumption demand. Firms are quantity rationed in the labor market, as $N < N^d$. This is the background for the parlance that in the repressed inflation regime, output and employment are not demand-determined but *supply-determined*. Both the output market and the labor market are *sellers' markets* (purchases less than desired). Presumably, the repressed inflation regime will not last long unless there are wage and price controls imposed by the government, as for instance may be the case for an economy in a war situation.[6]

2.3.3 The border case between the two regimes: $Y = Y^d = Y^f$.

This case arises if and only if $\bar{W} = W^c/(1+\mu)$, which is in turn equivalent to $\bar{P} = (1+\mu)\bar{W}/A = W^c/A \equiv P^c \equiv M/(\beta Y^f)$. No market has quantity rationing and we may speak of both the output market and the labor market as *balanced markets*.

There are two differences compared with the classical equilibrium, however. The first is that due to market power, there is a wedge between the real wage and the marginal productivity of labor. In the present context, though, where labor supply is inelastic, this does not imply inefficiency but only a higher profit/wage-income ratio than under perfect competition (where the profit/wage-income ratio is zero). The second difference compared with the classical equilibrium is that due to price stickiness, the impact of shifts in exogenous variables will be different. For instance a lower M will here result in unemployment, while in the classical model it will just lower P and W and not affect employment.

2.3.4 In terms of effective demands and supplies Walras' law does not hold

As we saw above, with *Walrasian* budget constraints, the aggregate value of excess demands in the given period is zero for any given price vector, $(W, P, 1)$, with $W > 0$ and $P > 0$. In contrast, with *effective* budget constraints, effective demands and supplies, and the short-side rule, this is no longer so. To see this, consider a pair (W, P) where $W < PA$ and $P \neq P^c \equiv M/(\beta Y^f)$. Such a pair leads to either the Keynesian regime or the repressed-inflation regime. The pair *may*, but need not, equal one of the pairs $(\bar{W}, \bar{P})$ considered above in Fig. 19.1 or 19.2 (we say "need not", because the particular μ-markup

[6] As another example of repressed inflation (simultaneous excess demand for consumption goods and labor) we may refer to Eastern Europe before the dissolution of the Soviet Union in 1991. In response to severe and long-lasting rationing in the consumption goods markets, households tended to decrease their labor supply (Kornai, 1979). This example illustrates that if labor supply is elastic, the *effective* labor supply may be less than the Walrasian labor supply due to spillovers from the output market.

relationship between W and P is not needed). We have, *first*, that in both the Keynesian and the repressed-inflation regime, effective output supply equals full-employment output,

$$Y^s = Y^f. \tag{26}$$

The intuition is that in view of $W < PA$, the representative firm *wishes* to satisfy any output demand forthcoming but it is only able to do so up to the point of where the availability of workers becomes a binding constraint.

Second, the aggregate value of excess effective demands is, for the considered price vector $(W, P, 1)$, equal to

$$\begin{aligned} & W(N^d - N^s) + P(C^d - Y^s) + \hat{M} - M \\ = \;& W(N^d - \bar{N}) + PC^d + \hat{M} - M - PY^f \\ = \;& W(N^d - \bar{N}) + WN + \Pi - PY^f \quad \text{(by (4))} \\ = \;& W(N^d - \bar{N}) + PY - PY^f \quad \text{(by (2))} \\ = \;& W(N^d - \bar{N}) + P(Y - Y^f) \begin{cases} < 0 \text{ if } P > M/(\beta Y^f), \text{ and} \\ > 0 \text{ if } P < M/(\beta Y^f) \text{ and } W < PA. \end{cases} \end{aligned} \tag{27}$$

The aggregate value of excess effective demands is thus not identically zero. As expected, it is negative in a Keynesian equilibrium and positive in a repressed-inflation equilibrium.[7] The reason that Walras' law does not apply to effective demands and supplies is that outside Walrasian equilibrium some of these demands and supplies are not realized in the final transactions.

This takes us to Keynes' refutation of Say's law and thereby what Keynes and others regarded as the core of his theory.

2.3.5 Say's law and its refutation

The classical principle "supply creates its own demand" (or "income is automatically spent on products") is named Say's law after the French economist and business man Jean-Baptiste Say (1767-1832). In line with other classical economists like David Ricardo and John Stuart Mill, Say maintained that although mismatch between demand and production can occur, it can only occur in the form of excess production in some industries at the same time as there is excess demand in other industries.[8] General overproduction is impossible. Or, by a classical catchphrase:

[7] At the same time, (27) together with the general equations $N^d = \bar{N}$ and $Y^s = Y^f$, shows that we have $\hat{M} = M$ in a Keynesian equilibrium (where $Y = C^d$) and $\hat{M} < M$ in a repressed-inflation equilibrium (where $Y = Y^f$).

[8] There were two dissidents at this point, Thomas Malthus (1766–1834) and Karl Marx (1818–1883), two classical economists that were otherwise not much aggreeing.

Every offer to sell a good implies a demand for some other good.

By "good" is here meant a produced good rather than just any traded article, including for instance money. Otherwise Say's law would be a platitude (a simple implication of the definition of trade). So, interpreting "good" to mean a produced good, let us evaluate Say's law from the point of view of the result (27). We first subtract $W(N^d - N^s) = W(N^d - \bar{N})$ on both sides of (27), then insert (26) and rearrange to get

$$P(C^d - Y) + \hat{M} - M = 0, \tag{28}$$

for any $P > 0$. Consider the case $W < AP$. In this situation every unit produced and sold is profitable. So any Y in the interval $0 < Y \leq Y^f$ is profitable from the supply side angle. Assume further that $P = \bar{P} > P^c \equiv M/(\beta Y^f)$. This is the case shown in Fig. 19.1. The figure illustrates that aggregate demand *is* rising with aggregate production. So far so well for Say's law. We also see that if aggregate production is in the interval $0 < Y < Y^k$, then C^d $(= Y^d) > Y$. This amounts to excess demand for goods and in effect, by (28), excess supply of money. Still, Say's law is not contradicted. But if instead aggregate production is in the interval $Y^k < Y \leq Y^f$, then C^d $(= Y^d) < Y$; now there is *general overproduction.* Supply no longer creates its own demand. There is a general shortfall of demand. By (28), the other side of the coin is that when $C^d < Y$, then $\hat{M} > M$, which means excess demand for money. People try to hoard money rather than spend on goods. Both the Great Depression in the 1930s and the Great Recession 2008- can be seen in this light.[9]

The refutation of Say's law does not depend on the market power and constant markup aspects we have adhered to above. All that is needed for the argument is that the agents are price takers within the period. Moreover, the refutation does not hinge on *money* being the asset available for transferring purchasing power from one period to the next. We may imagine an economy where M represents *land* available in limited supply. As land is also a non-produced store of value, the above analysis goes through − with one exception, though. This is that ΔM in (15) can no longer be interpreted as a policy choice. Instead, a positive ΔM could be due to discovery of new land.

We conclude that general overproduction is possible, and Say's law is thereby refuted. It might be objected that our "aggregate reply" to Say's law is not to the point since Say had a disaggregate structure with many industries in mind. Considering explicitly a

[9] Paul Krugman stated it this way: "When everyone is trying to accumulate cash at the same time, which is what happened worldwide after the collapse of Lehman Brothers, the result is an end to demand [for output], which produces a severe recession" (Krugman, 2009).

multiplicity of production sectors makes no essential difference, however, as the following example will show.

Many industries* Suppose there is still one labor market, but m industries with production function $y_i = An_i$, where y_i and n_i are output and employment in industry i, respectively, $i = 1, 2, \ldots, m$. Let the preferences of the representative household be given by

$$U = \sum_i \gamma_i \ln c_i + \beta \ln \frac{\hat{M}}{P^e}, \qquad \gamma_i > 0, i = 1, 2, \ldots, m, \quad 0 < \beta < 1.$$

In analogy with (4), the budget constraint is

$$\sum_i P_i c_i + \hat{M} = B \equiv M + W \sum_i n_i + \sum_i \Pi_i = M + \sum_i P_i y_i,$$

where the last equality comes from

$$\Pi_i = P_i y_i - W n_i.$$

Utility maximization gives $P_i c_i = \gamma_i B/(1+\beta)$.

As a special case, consider $\gamma_i = 1/m$ and $P_i = P$, $i = 1, 2, \ldots, m$. Then

$$c_i = \frac{B/m}{(1+\beta)P}, \tag{29}$$

and

$$B = M + P \sum_i y_i \equiv M + PY.$$

Substituting into (29), we thus find demand for consumption good i as

$$c_i = \frac{\frac{M/m}{P} + Y/m}{1+\beta} \equiv y^d, \quad \text{for all } i.$$

Let $P > \min\left[W/A, M/(\beta Y^f)\right]$, where $Y^f \equiv A\bar{N}$. It follows that every unit produced and sold is profitable and that

$$my^d = \frac{\frac{M}{P} + Y}{1+\beta} \leq \frac{\frac{M}{P} + Y^f}{1+\beta} < Y^f,$$

where the weak inequality comes from $Y \leq Y^f$ (always) and the strict inequality from $P > M/(\beta Y^f)$.

Now, suppose good 1 is brought to the market in the amount y_1, where $y^d < y_1 < Y^f/m$. Industry 1 thus experiences a shortfall of demand. Will there in turn necessarily be another industry experiencing excess demand? No. To see this, consider the case $y^d < y_i < Y^f/m$ for all i. All these supplies are profitable from a supply side point of view, and enough labor is available. Indeed, by construction the resource allocation is such that

$$my^d < \sum y_i \equiv Y \leq m\bar{y} < Y^f, \tag{30}$$

where $\bar{y} = \max[y_1, \ldots, y_m] < Y^f/m$. This is a situation where people try to save (hoard money) rather than spend all income on produced goods. It is an example of *general overproduction,* thus falsifying Say's law.

In the special case where all $y_i = Y/m$, the situation for each single industry can be illustrated by a diagram as that in Fig. **??**. Just replace Y^d, Y, Y^k, Y^f, and M in Fig. **??** by y^d, Y/m, $Y^k/m \equiv M/(m\beta P)$, Y^f/m, and M/m, respectively.

2.4 Short-run adjustment dynamics

We now return to the aggregate setup. Apart from the border case of balanced markets, we have considered two kinds of "fix-price equilibria", *repressed inflation* and *Keynesian equilibrium.* Most macroeconomists consider nominal wages and prices to be less sticky upwards than downwards. So a repressed inflation regime is typically regarded as having little durability (unless there are wage and price controls imposed by a government). It is otherwise with the Keynesian equilibrium. A way of thinking about this is the following.

Suppose that up to the current period full-employment equilibrium has applied: $Y = Y^d = M/(\beta\bar{P}) = Y^f$ and $\bar{P} = (1+\mu)\bar{W}/A = W^c/A \equiv P^c \equiv M/(\beta Y^f)$. Then, for some external reason, at the start of the current period a *rise* in the patience parameter occurs, from β to β', so that the new propensity to save is $\beta'/(1+\beta') > \beta/(1+\beta)$. We may interpret this as "precautionary saving" in response to a sudden fall in the general "state of confidence".

Let our "period" be divided into n sub-periods, indexed $i = 0, 1, 2, \ldots, n-1$, of length $1/n$, where n is "large". At least within the first of these sub-periods, the preset $\bar{W}$ and $\bar{P}$ are maintained and firms produce without having yet realized that aggregate demand will be lower than in the previous period. After a while firms realize that sales do not keep track with production.

There are basically two kinds of reaction to this situation. One is that wages and prices are maintained throughout all the sub-periods, while production is gradually scaled down to the Keynesian equilibrium $Y^k = M/(\beta'\bar{P})$. Another is that wages and prices adjust downward so as to soon reestablish full-employment equilibrium. Let us take each case at a time.

Wage and price stay fixed: Sheer quantity adjustment For simplicity we have assumed that the produced goods are perishable. So unsold goods represent a complete loss. If firms fully understand the functioning of the economy and have model-consistent expectations, they will adjust production per time unit down to the level Y^k as fast as possible. Suppose instead that firms have naive adaptive expectations of the form

$$C^e_{i-1,i} = C_{i-1}, \qquad i = 0, 1, 2, ..., n.$$

This means that the "subjective" expectation, formed in sub-period $i-1$, of demand next sub-period is that it will equal the demand in sub-period $i-1$. Let the time-lag between the decision to produce and the observation of the demand correspond to the length of the subperiods. It is profitable to satisfy demand, hence actual output in sub-period i will be

$$Y_i = C^e_{i-1,i} = C^d_{i-1} = \frac{M/\bar{P}}{1+\beta'} + \frac{Y_{i-1}}{1+\beta'},$$

in analogy with (19). This is a linear first-order difference equation in Y_i, with constant coefficients. The solution is (see Math Tools)

$$Y_i = (Y_0 - Y^{*\prime}) \left(\frac{1}{1+\beta'}\right)^i + Y^{*\prime}, \qquad Y^{*\prime} = \frac{M}{\beta'\bar{P}} = Y^k < Y^f. \tag{31}$$

Suppose $\beta' = 0.9$, say. Then actual production, Y_i, converges fast towards the steady-state value Y^k. When $Y = Y^k$, the system is at rest. Fig. 19.x illustrates. Although there is excess supply in the labor market and therefore some downward pressure on wages, the Keynesian presumption is that the workers's side in the labor market generally withstand the pressure.[10]

Fig. 19.x about here (not yet available).

The process (31) also applies "in the opposite direction". Suppose, starting from the Keynesian equilibrium $Y = M/(\beta'\bar{P})$, a *reduction* in the patience parameter β' occurs, such that $M/(\beta'\bar{P})$ increases, but still satisfies $M/(\beta'\bar{P}) < Y^f$. Then the initial condition in (31) is $Y_0 < Y^{*\prime}$, and the greater propensity to consume leads to an upward quantity adjustment.

[10]Possible explanations of downward wage stickiness are discussed in Chapter **??**.

Downward wage and price adjustment* Several of Keynes' contemporaries, among them A. C. Pigou, maintained that the Keynesian state of affairs with $Y = Y^k < Y^f$ could only be very temporary. Pigou's argument was that a fall in the price level would take place and lead to higher purchasing power of M. The implied stimulation of aggregate demand would bring the economy back to full employment. This hypothetically equilibrating mechanism is known as the "real balance effect" or the "Pigou effect" (after Pigou, 1943).

Does the argument go through? To answer this, we imagine that the time interval between different rounds of wage and price setting is as short as our sub-periods. We imagine the time interval between households' decision making to be equally short. Given the fixed markup μ, an initial fall in the preset $\bar{W}$ is needed to trigger a fall in the preset $\bar{P}$. The new *classical* equilibrium price and wage levels will be

$$P^{c\prime} = \frac{M}{\beta' Y^f} \text{ and } W^{c\prime} = AP^{c\prime}.$$

Both will thus be lower than the original ones – by the same factor as the patience parameter has risen, i.e., the factor β'/β. In line with "classical" thinking, assume that soon after the rise in the propensity to save, the incipient unemployment prompts wage setters to reduce $\bar{W}$ and thereby price setters to reduce $\bar{P}$. Let both $\bar{W}$ and $\bar{P}$ after a few rounds be reduced by the factor β'/β. Denoting the resulting wage and price $\bar{W}'$ and $\bar{P}'$, respectively, we then have

$$\bar{W}' = \frac{W^{c\prime}}{1+\mu}, \qquad \bar{P}' = (1+\mu)\frac{\bar{W}'}{A} = \frac{W^{c\prime}}{A} \equiv P^{c\prime} \equiv \frac{M}{\beta' Y^f}.$$

Seemingly, this restores aggregate demand at the full-employment level $Y^d = M/(\beta'\bar{P}') = Y^f$.

While this "classical" adjustment is conceivable in the abstract, Keynesians question its practical relevance for several reasons:

1. Empirically, it seems to be particularly in the downward direction that nominal wages are sticky. And without an initial fall in the nominal wage, the downward wage-price spiral does not get started.

2. If downward wage-price spiral does get started, the implied deflation increases the implicit real interest rate, $(P_t - P_{t+1})/P_{t+1}$. In a more elaborate modeling of consumption and investment, this would tend to dampen aggregate demand rather than the opposite.

3. Additional points, when going a little outside the present simple model, are:

   (a) the monetary base is in reality only a small fraction of financial wealth, and so the real balance effect can not be very powerful unless the fall in the price level is drastic;

   (b) many firms and households have nominal debt, the real value of which would rise, thereby potentially leading to bankruptcies and a worsening of the confidence crisis, thus counteracting a return to full employment.

A clarifying remark. In this context we should be aware that there are two kinds of "price flexibility" to be distinguished: "imperfect" versus "perfect" (or "full") price flexibility. The first kind relates to a *gradual* price process, for instance generated by a wage-price spiral as at item 2 above. The latter kind relates to *instantaneous* and complete price adjustment as with a Walrasian auctioneer. It is the first kind that may be destabilizing rather than the opposite.

2.5 Digging deeper

As it stands the above theoretical framework has many limitations. The remainder of this chapter gives an introduction to how the following three problems have been dealt with in the literature:

(i) Price setting should be explicitly modeled, and in this connection there should be an explanation of price stickiness.

(ii) It should be made clear how to come from the existence of many differentiated goods and markets with imperfect competition to aggregate output and income which in turn constitute the environment conditioning the individual agents' actions.

(iii) The analysis has ignored that capital equipment is in practice an additional factor constraining production.

In subsequent chapters we consider additional problems:

(iv) Also wage setting should be explicitly modeled, and in this connection there should be an explanation of wage stickiness.

(v) At least one additional financial asset, an interest-bearing asset, should enter. This will open up for intertemporal trade and for clarifying the primary function of money as a medium of exchange rather than as a store of value.

(vi) The model should include forward-looking decision making and endogenous expectations.

(vii) The model should be truly dynamic with gradual wage and price changes depending on the market conditions and expectations. This should lead to an explanation why wages and prices do not tend to find their market clearing levels relatively fast.

The next section deals with point (i) and (ii), and Section 19.4 with point (iii).

3 Price adjustment costs

The classical theory of perfectly flexible wages and prices and neutrality of money seems contradicted by overwhelming empirical evidence. At the theoretical level the theory ignores that the dominant market form is not perfect competition. Wages and prices are usually set by agents with market power. And there may be costs associated with changing prices and wages. Here we consider such costs.

The literature has modelled price adjustment costs in two different ways. *Menu costs* refer to the case where there are *fixed costs* of changing price. Another case considered in the literature is the case of *strictly convex adjustment costs*, where the marginal price adjustment cost is increasing in the size of the price change.

The most obvious examples of *menu costs* are of course costs associated with

1. remarking commodities with new price labels,

2. reprinting price lists ("menu cards") and catalogues.

But the term menu costs should be interpreted in a broader sense, including pecuniary as well non-pecuniary costs of:

3. information-gathering,

4. recomputing optimal prices,

5. conveying the new directives to the sales force,

6. the risk of offending customers by frequent and/or large price changes,

7. search for new customers willing to pay a higher price,

8. renegotiating contracts.

Menu costs induce firms to change prices less often than if no such costs were present. And some of the points mentioned in the list above, in particular point 7 and 8, may be relevant also in the different labor markets.

The menu cost theory is one of the microfoundations provided by modern Keynesian economics for the presumption that nominal prices and wages are sticky in the short run. The main theoretical insight of the menu cost theory is the following. There are menu costs associated with changing prices. Even *small* menu costs can be enough to prevent firms from changing their price. This is because the opportunity cost of not changing price is only of second order, i.e., "small"; this is a reflection of the *envelope theorem* (see Appendix). But owing to imperfect competition (price > MC), the effect on aggregate output, employment, and welfare of not changing prices is of first order, i.e., "large".

The menu cost theory provides the more popular explanation of nominal price rigidity. Another explanation rests on the presumption of *strictly convex price adjustment costs*. In this theory the price change cost for firm i is assumed to be $k_{it} = \alpha_i(P_{it} - P_{it-1})^2$, $\alpha_i > 0$. Under this assumption the firm is induced to avoid large price changes, which means that it tends to make frequent, but small price adjustments. This theory is related to the customer market theory. Customers search less frequently than they purchase. A large upward price change may be provocative to customers and lead them to do search in the market, thereby perhaps becoming aware of attractive offers from other stores. The implied "kinked" demand curve can explain that firms are reluctant to suddenly increase their price.

4 Adding interest-bearing assets

To incorporate the key role of financial markets for the performance of the macroeconomy, at least one extra asset should enter in a short-run model, an interest-bearing asset. This gives rise to the IS-LM model that should be familiar from Blanchard, *Macroeconomics*.

An extended IS-LM model is presented in the recent editions of the mentioned text by Blanchard (alone) and in Blanchard et al., *Macroeconomics: A European Perspective*, 2010, Chapter 20. The advantage of the extended version is that the commercial banking sector is introduced more explicitly so that the model incorporates both a centralized bond market and decentralized markets for bank loans.

5 Adding dynamics and a Phillips curve

Adding dynamics, expectations formation, and a Phillips curve leads to a *medium-run model*. An introduction is provided in the first-mentioned Blanchard textbook, chapters 8 and 14. Medium-run models describe fluctuations in production and employment around a trend, often considered related to the "natural rate of unemployment". Adding capital accumulation, technical progress, and growth in the labor force to the model, GDP gets a rising trend.

Roughly speaking, this course, Macroeconomics 2, can be interpreted as dealing with an economy moving along this trend. We have more or less ignored the fluctuations, simply by assuming flexible prices and perfect competition. In a realistic model with imperfect competition and price stickiness in both output and labor markets the natural rate of unemployment is likely to be higher than in an economy with perfect competition. And hump-shaped deviations from trend GDP, that is, business cycles, are likely to arise when the economy is hit by large shocks, for instance a financial crisis.

The third macro course, Macroeconomics 3, deals with short and medium run theory and emphasizes issues related to monetary policy.

6 Appendix

ENVELOPE THEOREM Let $y = f(a, x)$ be a continuously differentiable function of two variables, of which one, a, is conceived as a parameter and the other, x, as a control variable. Let $g(a)$ be a value of x at which $\frac{\partial f}{\partial x}(a, x) = 0$, i.e., $\frac{\partial f}{\partial x}(a, g(a)) = 0$. Let $F(a) \equiv f(a, g(a))$. Provided $F(a)$ is differentiable,

$$F'(a) = \frac{\partial f}{\partial a}(a, g(a)),$$

where $\partial f / \partial a$ denotes the partial derivative of $f(\cdot)$ w.r.t. the first argument.

Proof $F'(a) = \frac{\partial f}{\partial a}(a, g(a)) + \frac{\partial f}{\partial x}(a, g(a))g'(a) = \frac{\partial f}{\partial a}(a, g(a))$, since $\frac{\partial f}{\partial x}(a, g(a)) = 0$ by definition of $g(a)$. $\square$

That is, when calculating the total derivative of a function w.r.t. a parameter and evaluating this derivative at an interior maximum w.r.t. a control variable, the envelope theorem allows us to ignore the terms that arise from the chain rule. This is also the case if we calculate the total derivative at an interior minimum.[11]

[11] For extensions and more rigorous framing of the envelope theorem, see for example Sydsaeter et al.

(2006).

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(2006).