Chapter 12

Overlapping generations in continuous time

12.1 Introduction

In this chapter we return to issues where life-cycle aspects are important. A representative agent framework is therefore not suitable. We shall see how an overlapping generations (OLG) structure can be made compatible with continuous time analysis.

The reason for the transition to continuous time is the following. The two-period OLG models considered in chapters 3-5 have a coarse notion of time. The implicit length of the period is something in the order of 25-30 years. This implies very rough dynamics. And changes within a shorter time horizon can not be studied. Under special conditions three-period OLG models are analytically obedient, but complex. For OLG models with more than three coexisting generations analytical aggregation is close to unmanageable. Empirical OLG models, for specific economies, with a period length of one or a few years, and thereby many coexisting generations, have been developed. Examples include for the U.S. economy the Auerbach-Kotlikoff (1987) model and for the Danish economy the DREAM model (Danish Rational Economic Agents Model). The dynamics and predictions from this kind of models are studied by numerical simulation on a computer. Governments, large organizations and the financial companies use this type of models to assess how changes in economic policy or in external circumstances are likely to affect the economy.

For basic understanding of economic mechanisms, analytical tractability is important, however. With this in mind, a tractable OLG model with a refined notion of time was developed by the French-American economist, Olivier Blanchard, from Massachusetts Institute of Technology. In a paper from 1985 Blanchard simply suggested an OLG model in *continuous* time, in which people have finite, but uncertain lifetime. The model builds on earlier ideas by Yaari (1965) about life-insurance and is sometimes called the Blanchard-Yaari OLG model. For convenience, we stick to the shorter name *Blanchard OLG model*.

The usefulness of the model derives from its close connection to important facts:

- economic interaction takes place between agents belonging to *many* different age groups;
- agents' working life lasts *many* periods; the present discounted value of expected future labor income is thus a key variable in the system; hereby the wealth effect of a change in the interest rate becomes important;
- owing to uncertainty about remaining lifetime and to retirement from the labor market at old age, a large part of saving is channelled to pension arrangements and various kinds of life-insurance;
- taking finite lifetime into account, the model offers a more realistic approach to the study of long-run effects of government budget deficits and government debt than the Ramsey model;
- by including life expectancy among its parameters, the model opens up for studying effects of demographic changes in the industrialized countries such as increased life expectancy due to improved health conditions.

In the next sections we present and discuss Blanchard's OLG model. A simplifying assumption in the model is that expected remaining lifetime for any individual is independent of age. The simplest version of the model assumes in addition that people stay on the labor market until death. This version is known as the *model of perpetual youth* and is presented in Section 12.2. Later in the chapter we extend the model by including retirement at old age, thereby providing a more distinct life-cycle perspective. Among other things this leads to a succinct theory of the interest rate in the long run. In Section 12.5 we apply the Blanchard framework for a study of national wealth and foreign debt in a small open economy. Key variables are listed in Table 12.1.

The model is in continuous time. Chapter 9 gave an introduction to continuous time analysis. In particular we emphasized that flow variables in continuous time should be interpreted as intensities.

Table 12.1. Key variable symbols in the Blanchard OLG model.

Symbol	*Meaning*
$N(t)$	Size of population at time t
b	Birth rate
m	Death rate (mortality rate)
$n \equiv b - m$	Population growth rate
ρ	Pure rate of time preference
$c(v,t)$	Consumption at time t by an individual born at time v
$C(t)$	Aggregate consumption at time t
$a(v,t)$	Financial wealth at time t by an individual born at time v
$A(t)$	Aggregate financial wealth at time t
$w(t)$	Real wage at time t
$r(t)$	Risk-free real interest rate at time t
$L(t)$	Labor force at time t
$h(v,t)$	PV of expected future labor income by an individual
$H(t)$	Aggregate PV of expected future labor income of people alive at time t
δ	Capital depreciation rate
g	Rate of technological progress
λ	Retirement rate

12.2 The model of perpetual youth

We first portray the household sector. We describe its demographic characteristics, preferences, market environment (including a market for life annuities), the resulting behavior by individuals, and the aggregation across the different age groups. The production sector is as in the previous chapters. But in addition to production firms there are now life insurance companies. Finally, general equilibrium and the dynamic evolution at the aggregate level are studied.

The economy is closed. Perfect competition and rational (model consistent) expectations are assumed throughout. Apart from the uncertain lifetime there is no uncertainty.

12.2.1 Households

Demography

We describe a household as consisting of a single adult whose lifetime is uncertain. Let X denote the remaining lifetime (a stochastic variable) of this person. Then

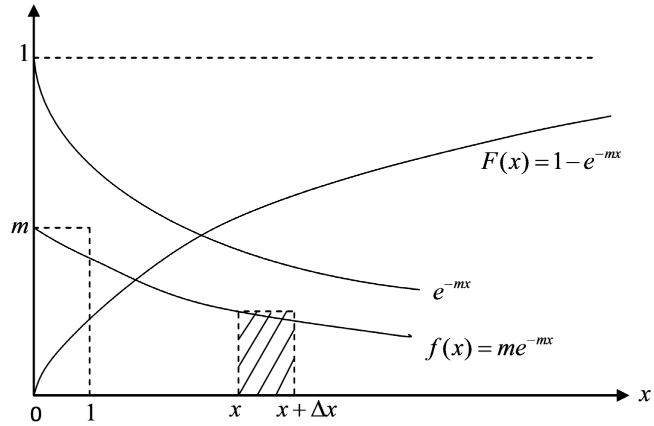

Figure 12.1: The survival probability, e^{-mx}, the exponential cummulative distribution function, $F(x)$, and the associated density function, $f(x)$.

the probability of experiencing X larger than x (an arbitrary positive number) is

$$P(X > x) = e^{-mx}, \tag{12.1}$$

where $m > 0$ is a parameter, reflecting the instantaneous *death rate,* also called *mortality rate.* So (12.1) indicates the *probability of surviving* x more years. The special feature here is that the parameter m is assumed independent of age and the same for all individuals. The reason for introducing this coarse assumption, at least as a first approach, is that it simplifies the analysis a lot by making aggregation easy.

Let us choose one year as our time unit. It then follows from (12.1) that the probability of dying within one year "from now" is approximately equal to m. To see this, note that $P(X \leq x) = 1 - e^{-mx} \equiv F(x)$ is the exponential cumulative distribution function. It follows that the probability density function is $f(x) = F'(x) = me^{-mx}$. We have $P(x < X \leq x + \Delta x) \approx f(x)\Delta x$ for Δx "small". With $x = 0$, this gives $P(0 < X \leq \Delta x) \approx f(0)\Delta x = m\Delta x$. So for a "small" time interval "from now", the probability of dying is approximately proportional to the length of the time interval. And for $\Delta x = 1$, we get $P(0 < X \leq 1) \approx m$, as was to be explained. Fig. 12.1 illustrates.

The expected remaining lifetime is $E(X) = \int_0^\infty x f(x) dx = 1/m$ and is thus the same whatever the current age. This reflects that the exponential distribution is "memory-less". A related unwelcome implication of the assumption (12.1) is that there is no upper bound on *possible* lifetime. Although according to the exponential distribution, the probability of becoming for instance 200 years old is extremely small for values of m consistent with a realistic life expectancy, it is certainly larger than in reality.

Let $N(t)$ be the size of the adult population at time t. We ignore integer problems and consider $N(t)$ as a smooth function of time, t. We assume the events of death are independent across individuals and that the population is "large". Then, by the law of large numbers the actual number of deaths per year at time t is indistinguishable from the expected number, $N(t)m$.[1] Let $b > 0$ denote the *birth rate,* referring to the inflow into the adult population. Like m, b is assumed *constant* over time. Again, appealing to stochastic independence and the law of large numbers, the actual number of births per year at time t is indistinguishable from the expected number, $N(t)b$. So at the aggregate level frequencies and probabilities coincide. By implication, $N(t)$ is growing according to $N(t) = N(0)e^{nt}$, where $n \equiv b - m$ is the population growth rate, a constant. Thus m and b correspond to what demographers call the crude mortality rate and the crude birth rate, respectively.

Let $N(v, t)$ denote the number of people from the birth cohort of the time interval $(v, v + 1)$ still alive at time t (they belong to "vintage" v). Thereby $N(v, t)$ is also the number of people of age $t - v$ at time t, which we perceive as "current time". We have

$$N(v, t) \approx N(v)bP(X > t - v) = N(0)e^{nv}be^{-m(t-v)}. \tag{12.2}$$

Provided parameters have been constant for a long time back in history, from this formula the age composition of the population at time t can be calculated. The number of newborn (age below 1 year) around time t is $N(t, t) \approx N(t)b = N(0)e^{nt}b$. The number of people of age j at time t is approximately

$$N(t - j, t) \approx N(0)e^{n(t-j)}be^{-mj} = N(0)e^{nt}be^{-bj} = N(t)be^{-bj}, \tag{12.3}$$

since $b = n + m$.

Fig. 12.2 shows this age distribution and compares with a stylized empirical age distribution (the hatched curve). The general concavity of the empirical curve and in particular its concentrated "curvature" around 70-80 years' age is not well captured by the theoretical model. Yet the model at least reflects that cohorts of increasing age tend to be smaller and smaller.

By summing over all times of birth we get the total population:

$$\begin{aligned}\int_{-\infty}^{t} N(v, t)dv &= \int_{-\infty}^{t} N(0)e^{nv}be^{-m(t-v)}dv \\ &= N(0)be^{-mt}\int_{-\infty}^{t} e^{(n+m)v}dv = N(0)be^{-mt}\left[\frac{e^{(n+m)v}}{n+m}\right]_{-\infty}^{t} \\ &= N(0)be^{-mt}\frac{e^{(n+m)t} - 0}{b} = N(0)e^{nt} = N(t). \qquad (12.4)\end{aligned}$$

[1] If $\hat{m}$ denotes the frequency of deaths (relative to population), the law of large numbers in this context says that for every $\varepsilon > 0$, $P(|\hat{m} - m| \leq \varepsilon\,|N(t)) \to 1$ as $N(t) \to \infty$.

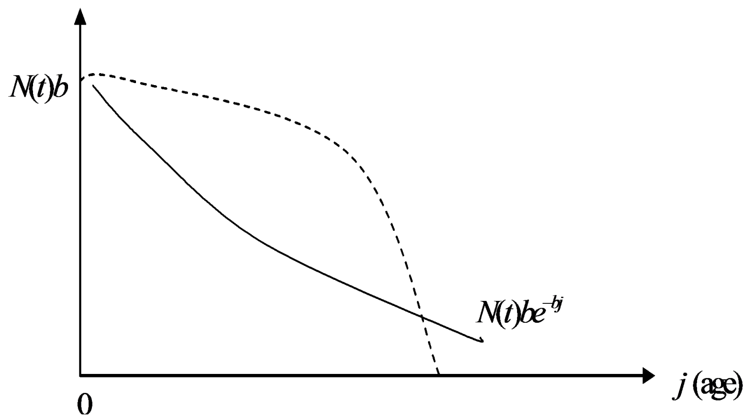

Figure 12.2: Age distribution of the population at time t (the hatched curve depicts a stylized empirical curve).

Preferences

We consider an individual born at time $v \leq t$ and still alive at time t. The consumption flow at time t of the individual is denoted $c(v, t)$. For $s > t$, we interpret $c(v, s)$ as the planned consumption flow at time s in the future. The individual maximizes expected lifetime utility, where the instantaneous utility function is $u(c)$, $u' > 0, u'' < 0$, and the pure rate of time preference (impatience) is a constant $\rho \geq 0$. There is no bequest motive. Expected lifetime utility, as seen from time t, is

$$U_t = E_t \left(\int_t^{t+X} u\left(c(v,s)\right) e^{-\rho(s-t)} ds \right), \tag{12.5}$$

where E_t is the expectation operator conditional on information available at time t. This formula for expected discounted utility is valid for all alive at time t whatever the cohort $v \leq t$ to which they belong; this is due to their common expected remaining lifetime. Hence we can do with only one time index, t, on the symbol U.

There is a convenient way of rewriting the objective function, U_t. Given $s > t$, let $Z(s)$ denote a stochastic variable with two different possible outcomes:

$$Z(s) = \begin{cases} u\left(c(v,s)\right), & \text{if } X > s - t \text{ (i.e., the person is still alive at time } s) \\ 0, & \text{if } X \leq s - t \text{ (i.e., the person is dead at time } s). \end{cases}$$

Then

$$U_t = E_t \left(\int_t^{\infty} Z(s) e^{-\rho(s-t)} ds \right) = \int_t^{\infty} E_t \left(Z(s) e^{-\rho(s-t)} \right) ds.$$

Note that in this context the integration operator $\int_t^\infty(\cdot)ds$ acts like a discrete-time summation operator $\sum_0^\infty$. Hence,

$$\begin{aligned} U_t &= \int_t^\infty e^{-\rho(s-t)}E_t(Z(s))ds \\ &= \int_t^\infty e^{-\rho(s-t)}\left(u\left(c(v,s)\right)P(X>s-t)+0\cdot P(X\leq s-t)\right)ds \\ &= \int_t^\infty e^{-(\rho+m)(s-t)}u\left(c(v,s)\right)ds. \end{aligned} \tag{12.6}$$

We see that the expected discounted utility can be written in a way similar to the intertemporal utility function in the deterministic Ramsey model. The only difference is that the pure rate of time preference, ρ, is replaced by an effective rate of time preference, $\rho + m$. This rate is higher, the higher is the death rate m. This reflects that the likelihood of being alive at time s in the future is a decreasing function of the death rate.

For analytical convenience, we let $u(c) = \ln c$.

The market environment

Since every individual faces an uncertain length of lifetime and there is no bequest motive, there will be a demand for assets that pay a high return as long as the investor is alive, but on the other hand is nullified at death. Assets with this property are called *life annuities*. They will be demanded because they make it possible to ensure a high return until the uncertain time of death and to convert potential wealth after death to higher consumption while still alive.

So we assume there is a market for life annuities (also called "reverse" or "negative life insurance") issued by life insurance or pension companies. Consider a depositor who at some point in time buys a life annuity contract for one unit of account. In return the depositor receives $r + \hat{m}$ units of account per year paid continuously until death. Here r is the risk-free interest rate (for simplicity assumed time-independent) and $\hat{m}$ is an *actuarial compensation* over and above that rate. It is a "compensation" for granting the insurance company ownership of the deposit in the event the depositor dies.

How is the actuarial compensation determined in equilibrium? Well, since the economy is large and deaths are assumed stochastically independent, the insurance companies face no aggregate uncertainty. We further assume the insurance companies have negligible administration costs and that there is free entry and exit. We claim that in this case, $\hat{m}$ must in equilibrium equal the mortality rate m. To see this, let the aggregate deposit in the form of life annuity contracts be A units of account and let the number of depositors be N (N "large"). The aggregate revenue to the insurance company on these contracts is then $rA + NmA/N$

per year. The first term is due to A being invested by the insurance company in manufacturing firms, paying the risk-free interest rate r in return (by assumption there is no risk associated with production). The second term is due to Nm of the depositors dying per year. For each depositor who dies there is a transfer, on average A/N, of wealth to the insurance company sector. This is because the deposits are taken over by the insurance company at death (the company's liabilities to those who die are cancelled).

In the absence of administration costs the total costs faced by the insurance company amount to the payout $(r + \hat{m})A$ per year. So total profit is

$$\Pi = rA + NmA/N - (r + \hat{m})A.$$

Under free entry and exit, equilibrium requires $\Pi = 0$. It follows that $\hat{m} = m$. That is, the actuarial compensation equals the mortality rate.

The *conditional* rate of return, $r + m$, obtained by the depositor as long as alive is called the *actuarial rate of interest*. The actuarial rate of interest is called "conditional" because it is conditional upon survival. In contrast, the *expected unconditional rate* of return on holding a life annuity equals r when $\hat{m} = m$ (see Appendix A). A life annuity is said to be *actuarially fair* if it offers the customer the same expected unconditional rate of return as a safe bond. So in this model the life annuities are actuarially fair.[2]

In return for a high conditional rate of return, $r+m$, the estate of the deceased person looses the deposit at the time of death. In this way individuals dying earlier will support those living longer. The market for life annuities is thus a market for *longevity insurance*.

Given r, the actuarial rate of interest will be higher the higher is the mortality rate, m. The intuition is that a higher m implies lower expected remaining lifetime, $1/m$. The expected duration of the life annuity to be paid is therefore shorter. With an unchanged actuarial compensation, this would make issuing these life annuity contracts more attractive to the life insurance companies and competition among them will drive the compensation $\hat{m}$ up until $\hat{m} = m$ again.

As we shall see, in this model, in equilibrium, all financial wealth will be placed in life annuities and earn the conditional rate of return equal to $r + m$ as long as the customer is alive. This is illustrated in Fig. 12.3 where S^N is aggregate net saving and A is aggregate financial wealth. The flows in the diagram are in real terms with the output good as the unit of account.

Whatever name is in practice used for the real world's pension arrangements, many of them have life annuity ingredients and can in a macroeconomic perspective be considered as belonging to the insurance company box in the diagram. Typical Danish "labor market pension" schemes are an example. The stream of

[2] Appendix A considers the case of an age-dependent mortality rate.

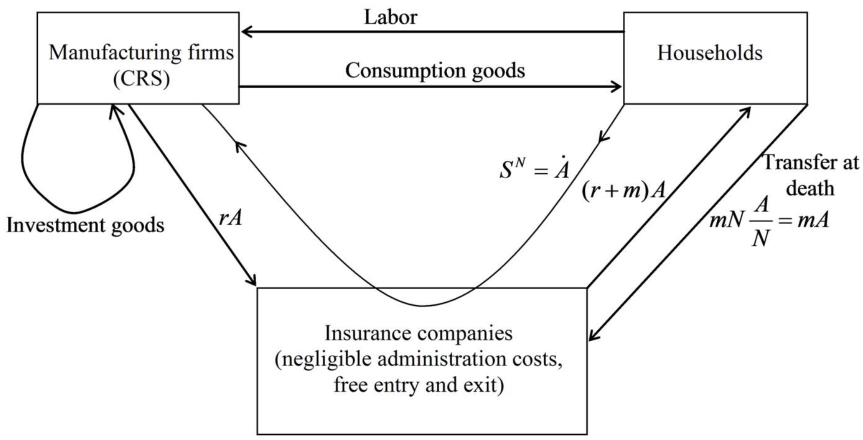

Figure 12.3: Overview of the economy. A is aggregate private financial wealth, S^N is aggregate private net saving.

payouts from such pension arrangements to the customer usually does not start until the customer retires from the labor market, however. This seems in contrast to the model where the flow of dividends to the depositor starts already from the date of purchase of the contract (this is the case of "immediate annuities"). But with perfect credit and annuity markets as simplifying assumed in the model, this difference is immaterial.

What about existence of a market for "ordinary" or "positive" life insurance? In such a market an individual contracts to pay the life insurance company a continuous flow of $\tilde{m}$ units of account per year until death and at death, in return the estate of the deceased person receives one unit of account. Provided the market is active, in equilibrium with free entry and no administration costs, we would have $\tilde{m} = m$ (see Appendix A). In the real world the primary motivation for positive life insurance is care for surviving relatives. But the Blanchard model ignores this motive. Indeed, altruism is absent in the preferences specified in (12.5). Hence there will be no demand for positive life insurance.

The consumption/saving problem

Recall that there is no utility from leisure and, in the present version of the model, no retirement. Hence, labor supply of the individual is inelastic and constant over time. We normalize it to be one unit of labor per year until death.

Let $t = 0$ be the current time and let s denote an arbitrary future point in time. The decision problem for an arbitrary individual born at time $v \leq 0$ is to choose a plan, $(c(v, s))_{s=0}^{\infty}$, so as to maximize expected lifetime utility, U_0, subject to a dynamic budget constraint. The plan is, of course, conditional in the sense of only going to be operative as long as the individual is alive. Letting $u(c) = \ln c$, the decision problem is:

$$\begin{aligned}
\max U_0 &= \int_0^\infty \ln\left(c\left(v,s\right)\right) e^{-(\rho+m)s} ds \quad \text{s.t.} \\
c\left(v,s\right) &\geq 0, \\
\dot{a}(v,s) &\equiv \frac{\partial a(v,s)}{\partial s} = (r\left(s\right)+m)a\left(v,s\right)+w\left(s\right)-c\left(v,s\right), \qquad (12.7) \\
& \text{where } a(v,0) \text{ is given,} \\
\lim_{s\to\infty} a\left(v,s\right) e^{-\int_0^s (r(\tau)+m)d\tau} &\geq 0. \qquad \text{(NPG)}
\end{aligned}$$

Labor income per time unit at time s is $w(s)\cdot 1$, where $w(s)$ is the real wage. The variable $a(v,s)$ appearing in the dynamic budget identity (12.7) is real financial wealth at time s and $a\left(v,0\right)$ is the historically given initial financial wealth. Implicit in the way (12.7) and the solvency condition, (NPG), are written is the assumption that the individual *can* procure debt ($a(v,s) < 0$) at the actuarial rate of interest $r(s)+m$. Nobody will offer loans to individuals at the going risk-free interest rate r. There would be a risk that the borrower dies before having paid off the debt including compound interest. But insurance companies will be willing to offer loans at the actuarial rate of interest, $r(s)+m$. As long as the debt is not paid off, the borrower pays the interest rate $r(s)+m$ per time unit. In case the borrower dies before the debt is paid off, the estate is held free of any obligation. In return for this risk the lender receives the actuarial compensation, m, on top of r until the loan is paid off or the borrower dies.

Owing to heterogeneity in an actual population regarding survival probabilities, asymmetric information, and related credit market imperfections, in real world situations this kind of individual loan contracts are rare.[3] This is ignored by the model. But this simplification is not intolerable since, in the context of the model, it turns out that at least in a neighborhood of the steady state, *all* individuals *will save continuously*, that is, *buy* actuarial notes issued by the insurance companies.

All things considered, we end up with a decision problem similar to that in the Ramsey model, namely with an infinite time horizon and a No-Ponzi-Game condition. The only difference is that ρ has been replaced by $\rho+m$ and r by $r+m$. The constraint implied by the NPG condition is that an eventual debt, $-a(v,s)$, is not allowed in the long run to grow at a rate higher than or equal

[3] And to the extent such loans exist, they tend to be associated with an interest cost over and above the sum of the "actuarially fair" rate (the sum of the risk-free rate and the instantaneous mortality rate). Think of what the interest rate on student loans would be in the absence of government support.

to the effective rate of interest $r(s) + m$. This precludes *permanent* financing of interest payments by new loans.

The consumption-saving problem has the same *form* as in the Ramsey model. We can therefore apply the result from Chapter 9 saying that an interior optimal solution must satisfy a set of first-order conditions leading to the Keynes-Ramsey rule. In the present log utility case the latter takes the form

$$\frac{\dot{c}(v,t)}{c(v,t)} \equiv \frac{\partial c(v,t)/\partial t}{c(v,t)} = r(t) + m - (\rho + m) = r(t) - \rho. \tag{12.8}$$

Moreover, the transversality condition,

$$\lim_{t\to\infty} a\,(v,t)\, e^{-\int_0^t (r(\tau)+m)d\tau} = 0, \tag{12.9}$$

must be satisfied by an optimal solution. These conditions are also sufficient for an optimal solution.

The individual consumption function

The Keynes-Ramsey rule itself is only a rule for the rate of change of consumption. We can, however, determine the *level* of consumption in the following way.

We may construct the intertemporal budget constraint that corresponds to the dynamic budget identity (12.7) combined with (NPG). This amounts to a constraint saying that the present value (PV) of the planned consumption stream can not exceed total initial wealth:

$$\int_0^\infty c\,(v,s)\, e^{-\int_0^s (r(\tau)+m)d\tau} ds \leq a(v,0) + h(v,0), \tag{IBC}$$

where $h(v,0)$ is the initial human wealth of the individual. Human wealth is the PV of the expected future labor income and can here, in analogy with (12.6), be written[4]

$$h(v,0) = \int_0^\infty w\,(s)\, e^{-\int_0^s (r(\tau)+m)d\tau} ds = \frac{H(0)}{N(0)} \equiv \bar{h}(0), \tag{12.10}$$

for all $v \leq 0$. Here $H(0)$ is total human wealth at time 0 and $\bar{h}(0)$ is *average* human wealth in the economy. In this version of the Blanchard model there is no retirement and everybody works the same per year until death. In view of the age-independent death probability, expected remaining participation in the labor market is thus the same for all alive. Hence $h(v,0)$ is independent of v and equal to average human wealth.

[4] For details, see Appendix B.

From Proposition 1 of Chapter 9 we know that, given the relevant dynamic budget identity, here (12.7), (NPG) holds if and only if (IBC) holds. Moreover, there is strict equality in (NPG) if and only if there is strict equality in (IBC).

Considering the Keynes-Ramsey rule as a linear differential equation for c as a function of t, the solution formula for consumption is

$$c(v,t) = c(v,0)e^{\int_0^t (r(\tau)-\rho)d\tau}.$$

But so far we do not know $c(v,0)$. Here the transversality condition (12.9) is of help. From Chapter 9 we know that the transversality condition is equivalent to requiring that the NPG condition is not "over-satisfied" which in turn requires strict equality in (IBC). Substituting our formula for $c(v,t)$ into (IBC) with strict equality yields

$$c(v,0)\int_0^\infty e^{\int_0^t (r(\tau)-\rho)d\tau}e^{-\int_0^t (r(\tau)+m)d\tau}dt = a(v,0) + h(v,0),$$

which reduces to $c(v,0) = (\rho + m)\left[a(v,0) + h(v,0)\right].$

Since initial time is arbitrary and the "effective" time horizon is infinite, we therefore have for any $t \geq 0$ the consumption function

$$c(v,t) = (\rho + m)\left[a(v,t) + h(v,t)\right], \tag{12.11}$$

where $h(v,t)$, in analogy with (12.10), is the PV of the individual's expected future labor income, as seen from time t :

$$h(v,t) = \int_t^\infty w(s)\, e^{-\int_t^s (r(\tau)+m)d\tau}ds = \frac{H(t)}{N(t)} \equiv \bar{h}(t). \tag{12.12}$$

That is, with logarithmic utility the optimal level of consumption is simply proportional to *total* wealth, including human wealth.[5] The factor of proportionality equals the effective rate of time preference, $\rho+m$, and indicates the marginal (and average) propensity to consume out of wealth. The higher is the death rate, m, the shorter is expected remaining lifetime, $1/m$, thus implying a larger marginal propensity to consume (in order to reap the fruits while still alive).

12.2.2 Aggregation

We will now aggregate over the different cohorts, that is, over the different times of birth. Summing consumption over all times of birth, we get aggregate consumption at time t,

$$C(t) = \int_{-\infty}^t c(v,t)N(v,t)dv, \tag{12.13}$$

[5] With a *general* CRRA utility function the marginal propensity to consume out of wealth depends on current and expected future interest rates, as shown in Chapter 9.

where $N(v,t)$ equals $N(v)be^{-m(t-v)}$, cf. (12.2). Similarly, aggregate financial wealth can be written

$$A(t) = \int_{-\infty}^{t} a(v,t)N(v,t)dv, \tag{12.14}$$

and aggregate human wealth is

$$H(t) \equiv N(t)\bar{h}(t) = N(0)e^{nt} \int_{t}^{\infty} w\left(s\right) e^{-\int_t^s (r(\tau)+m)d\tau} ds. \tag{12.15}$$

Since the propensity to consume out of wealth is the same for all individuals, i.e., independent of age, aggregate consumption becomes

$$C(t) = (\rho + m)\left[A(t) + H(t)\right]. \tag{12.16}$$

The dynamics of household aggregates

There are two basic dynamic relations for the household aggregates.[6] The first relation is

$$\dot{A}(t) = r(t)A(t) + w(t)L(t) - C(t). \tag{12.17}$$

Note that the rate of return here is $r(t)$ and thereby differs from the conditional rate of return for the individual during lifetime, namely $r(t)+m$. The difference derives from the fact that for the household sector as a whole, $r(t)+m$ is only a *gross* rate of return. The actuarial compensation m is paid by the household sector itself − via the life-insurance companies. There is a transfer of wealth when people die, in that the liabilities of the insurance companies are cancelled. First, $N(t)m$ individuals die per time unit and their average wealth is $A(t)/N(t)$. The implied transfer is in total $N(t)mA(t)/N(t)$ per time unit from those who die. This is what finances the actuarial compensation m to those who are still alive and have placed their savings in life annuity contracts issued by the insurance sector. Hence, the average *net* rate of return on financial wealth for the household sector as a whole is

$$(r(t)+m)A(t) - N(t)m\frac{A(t)}{N(t)} = r(t)A(t),$$

in conformity with (12.17). In short: the reason that (12.17) does not contain the actuarial compensation is that this compensation is only a transfer from those who die to those who are still alive. The unconditional rate of return in the economy is just $r(t)$.

[6] Here we only describe the intuition behind these relations. Their formal derivation is given in Appendix C.

The second important dynamic relation for the household sector as a whole is

$$\dot{C}(t) = [r(t) - \rho + n]C(t) - b(\rho + m)A(t). \tag{12.18}$$

To interpret this, note that three effects are in play:

1. The dynamics of consumption at the individual level follows the Keynes-Ramsey rule

$$\dot{c}(v,t) \equiv \frac{\partial c(v,t)}{\partial t} = \left(r\left(t\right) - \rho\right) c\left(v,t\right).$$

This explains the term $[r(t) - \rho + n]C(t)$ in (12.18), except for $nC(t)$.

2. The appearance of $nC(t)$ is a trivial population growth effect: defining $C \equiv cN$, we have

$$\dot{C} = (\dot{c}N + c\dot{N}) = (\frac{\dot{c}}{c} + \frac{\dot{N}}{N})cN \equiv (\frac{\dot{c}}{c} + n)C.$$

3. The subtraction of the term $b(\rho + m)A(t)$ in (12.18) is more challenging. This term is due to a *generation replacement effect.* In every short instant some people die and some people are born. The first group has financial wealth, the last group not. The inflow of newborns is $N(t)b$ per time unit and since they have no financial wealth, the replacement of dying people by these young people lowers aggregate consumption. To see by how much, note that the average financial wealth in the population is $A(t)/N(t)$ and the consumption effect of this is $(\rho + m)A(t)/N(t)$, cf. (12.16). This implies, *ceteris paribus,* that the turnover of generations reduces aggregate consumption per time unit by

$$N(t)b(\rho + m)\frac{A(t)}{N(t)} = b(\rho + m)A(t)$$

per time unit. This explains the last term in (12.18). The generation replacement effect makes the growth rate of aggregate consumption smaller than what the Keynes-Ramsey rule suggests.

Whereas the Keynes-Ramsey rule describes individual consumption dynamics, we see that the *aggregate* consumption dynamics do not follow the Keynes-Ramsey rule. The reason is the generation replacement effect. This "compositional effect" is a characteristic feature of overlapping generations models. It distinguishes these models from representative agent models, like the Ramsey model.

12.2.3 The representative firm

As description of the technology, the firms, and the factor markets we apply the simple neoclassical competitive one-sector setup that we have used in previous chapters. The technology of the representative firm in the manufacturing sector is given by

$$Y(t) = F(K(t), T(t)L(t)), \tag{12.19}$$

where F is a neoclassical production function with CRS, and $Y(t), K(t)$, and $L(t)$ are output, capital input, and labor input, respectively, per time unit. The technology level $T(t)$ grows at a constant rate $g \geq 0$, that is, $T(t) = T(0)e^{gt}$, where $T(0) > 0$. Ignoring for the time being the explicit dating of the variables, maximization of profit $(= F_1(K, TL) - (r+\delta)K - wL)$ under perfect competition leads to

$$\frac{\partial Y}{\partial K} = F_1(K, TL) = f'(\tilde{k}^d) = r + \delta, \tag{12.20}$$

$$\frac{\partial Y}{\partial L} = F_2(K, TL)T = \left[f(\tilde{k}^d) - \tilde{k}^d f'(\tilde{k}^d)\right] T \equiv \tilde{w}(\tilde{k}^d)T = w, \tag{12.21}$$

where $\delta > 0$ is the constant rate of capital depreciation, $\tilde{k}^d \equiv K^d/(TL^d)$ is the desired effective capital-labor ratio, and f is defined by $f(\tilde{k}^d) \equiv F(\tilde{k}^d, 1)$. We have $f' > 0$, $f'' < 0$, and $f(0) = 0$ (the latter condition in view of the upper Inada condition for the marginal productivity of labor, cf. Appendix C to Chapter 2).

Alternatively, we may imagine that the production firms own the capital stock they use and finance their gross investment by issuing bonds, as illustrated in Fig. 12.3. It still holds that total costs per unit of capital is the sum of the interest rate and the capital depreciation rate. The insurance companies use their deposits to buy the bonds issued by the manufacturing firms.

12.2.4 Dynamic general equilibrium (closed economy)

Clearing in the labor market entails $L^d = N$, where N is aggregate labor supply which equals the size of the population. Clearing in the market for capital goods entails $K^d = K$, where K is the aggregate capital stock available in the economy. Hence, in equilibrium $\tilde{k}^d = \tilde{k} \equiv K/(TN)$, which is predetermined at any point in time. The equilibrium factor prices at time t are thus given as

$$r(t) = f'(\tilde{k}(t)) - \delta, \quad \text{and} \tag{12.22}$$

$$w(t) = \tilde{w}(\tilde{k}(t))T(t). \tag{12.23}$$

Deriving the dynamic system

We will now derive a dynamic system in terms of $\tilde{k}$ and $\tilde{c} \equiv C/(TN)$. In a closed economy where natural resources (land etc.) are ignored, aggregate financial wealth equals, by definition, the market value of the capital stock, which is $1 \cdot K$.[7] Thus

$$A = K \text{ for all } t.$$

From (12.17) therefore follows:

$$\begin{aligned}
\dot{K} &= \dot{A} = rK + wL - C \\
&= [F_1(K,TL) - \delta]\,K + F_2(K,TL)TL - C \quad \text{(by (12.20) and (12.21))} \\
&= F_1(K,TL)K + F_2(K,TL)TL - \delta K - C \\
&= F(K,TL) - \delta K - C \quad \text{(by Euler's theorem)} \\
&= Y - \delta K - C. \qquad (12.24)
\end{aligned}$$

So, not surprisingly, we end up with a standard national product accounting relation for a closed economy. In fact we could directly have written down the result (12.24). Its formal derivation here only serves as a check that our product and income accounting is consistent.

To find the law of motion of $\tilde{k} \equiv K/(TN)$, we log-differentiate w.r.t. time (take logs on both sides and differentiate w.r.t. time) so as to get

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{K}}{K} - \frac{\dot{T}}{T} - \frac{\dot{N}}{N} = \frac{F(K,TN) - C - \delta K}{K} - (g+n),$$

from (12.24). Multiplying through by $\tilde{k} \equiv K/(TN)$ gives

$$\dot{\tilde{k}} = \frac{F(K,TN) - C}{TN} - (\delta + g + n)\tilde{k} = f(\tilde{k}) - \tilde{c} - (\delta + g + n)\tilde{k},$$

since $\tilde{c} \equiv C/(TN)$. Any path $(\tilde{k}_t, \tilde{c}_t)_{t=0}^{\infty}$ satisfying this equation and starting from the historically given initial value, $\tilde{k}_0$, is a technically feasible path. Which of these paths become realized depends on households' effective utility discount rate, $\rho + m$, and the market form, which is here perfect competition.

To find the law of motion of $\tilde{c}$, we first insert (12.22) and $A = K$ into (12.18) to get

$$\dot{C} = \left[f'(\tilde{k}) - \delta - \rho + n\right] C - b(\rho + m)K. \qquad (12.25)$$

Log-differentiating $C/(TN)$ w.r.t. time yields

$$\begin{aligned}
\frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{\dot{C}}{C} - \frac{\dot{T}}{T} - \frac{\dot{N}}{N} = f'(\tilde{k}) - \delta - \rho + n - b(\rho+m)\frac{K}{C} - g - n \text{ (from (12.25))} \\
&= f'(\tilde{k}) - \delta - \rho - b(\rho+m)\frac{\tilde{k}}{\tilde{c}} - g,
\end{aligned}$$

[7] There are no capital installation costs and so the value of a unit of installed capital equals the replacement cost per unit before installation. This replacement cost is one.

By rearranging:

$$\dot{\tilde{c}} = \left[f'(\tilde{k}) - \delta - \rho - g\right]\tilde{c} - b(\rho + m)\tilde{k}.$$

Our two coupled differential equations in $\tilde{k}$ and $\tilde{c}$ constitute the dynamic system of the Blanchard model. Since the parameters n, b, and m are connected through $n \equiv b - m$, one of them should be eliminated to avoid confusion. It is natural to have b and m as the basic parameters and then consider $n \equiv b - m$ as a derived one. Consequently we write the system as

$$\dot{\tilde{k}} = f(\tilde{k}) - \tilde{c} - (\delta + g + b - m)\tilde{k}, \tag{12.26}$$

$$\dot{\tilde{c}} = \left[f'(\tilde{k}) - \delta - \rho - g\right]\tilde{c} - b(\rho + m)\tilde{k}. \tag{12.27}$$

Observe that initial $\tilde{k}$ equals a predetermined value, $\tilde{k}_0$, while initial $\tilde{c}$ is a forward-looking variable, an endogenous jump variable. Therefore we need more information to pin down the dynamic evolution of the economy. Fortunately, for each individual household we have a transversality condition like that in (12.9). Indeed, for any fixed pair (v, t_0), where $t_0 \geq 0$ and $v \leq t_0$, the transversality condition takes the form

$$\lim_{t\to\infty} a(v,t)e^{-\int_{t_0}^{t}(r(\tau)+m)d\tau} = 0. \tag{12.28}$$

In comparison, note that the transversality condition (12.9) was seen from the special perspective of $(v, t_0) = (v, 0)$, which is only of relevance for those alive already at time 0.

Phase diagram

To get an overview of the dynamics, we draw a phase diagram. There are two reference values of $\tilde{k}$, namely the golden rule value, $\tilde{k}_{GR} > 0$, and a certain benchmark value, $\bar{\tilde{k}} > 0$. These are given by

$$f'\left(\tilde{k}_{GR}\right) - \delta = g + b - m = g + n, \qquad \text{and} \qquad f'(\bar{\tilde{k}}) - \delta = \rho + g, \tag{12.29}$$

respectively. Let us for simplicity assume that f satisfies the Inada conditions. Given $b \geq m$, both $\tilde{k}_{GR}$ and $\bar{\tilde{k}}$ exist,[8] and they are unique in view of $f'' < 0$. We have $\bar{\tilde{k}} \lesseqgtr \tilde{k}_{GR}$ for $\rho \gtreqless b - m$, respectively. The reason that $\bar{\tilde{k}}$ is an important benchmark value will be apparent in a moment.

[8]Here we use that when $b \geq m$ and $\delta > 0$, $g \geq 0$, and $\rho \geq 0$, then $\delta + g + b - m > 0$ and $\delta + \rho + g > 0$. Then the Inada conditions ensure that the two equations in $\tilde{k}_{GR}$ and $\bar{\tilde{k}}$, respectively, given by (12.29), have a solution.

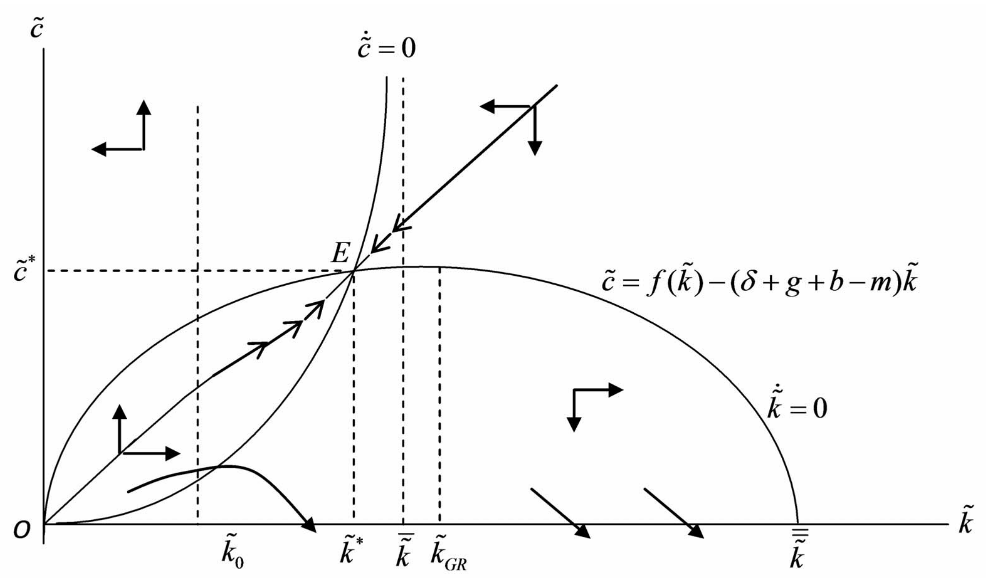

Figure 12.4: Phase diagram of the model of perpetual youth.

Equation (12.26) shows that

$$\dot{\tilde{k}} = 0 \text{ for } \tilde{c} = f(\tilde{k}) - (\delta + g + b - m)\tilde{k}. \tag{12.30}$$

The locus $\dot{\tilde{k}} = 0$ is shown in Fig. 12.4. It starts at the origin O, reaches its maximum at the golden rule capital-labor ratio, and crosses the horizontal axis at the capital-labor ratio $\bar{\bar{\tilde{k}}} > \tilde{k}_{GR}$, satisfying $f(\bar{\bar{\tilde{k}}}) = (\delta + g + b - m)\bar{\bar{\tilde{k}}}$. The existence of a $\bar{\bar{\tilde{k}}}$ with this property is guaranteed by the upper Inada condition for the marginal productivity of capital.

The horizontal arrows in the diagram are explained the following way. Imagine that for some fixed value $\tilde{k}_1 < \bar{\bar{\tilde{k}}}$ we draw the vertical line $\tilde{k} = \tilde{k}_1$ in the positive quadrant. We then consider greater and greater values of $\tilde{c}$ along this line. To begin with $\tilde{c}$ is small and therefore, by (12.26), $\dot{\tilde{k}}$ is positive. At the point where the imagined vertical line crosses the $\dot{\tilde{k}} = 0$ locus, we have $\dot{\tilde{k}} = 0$. And above this point we have $\dot{\tilde{k}} < 0$ due to the now large consumption level. In brief, by (12.26) follows

$$\dot{\tilde{k}} \gtreqless 0 \text{ for } \tilde{c} \lesseqgtr f(\tilde{k}) - (\delta + g + b - m)\tilde{k},$$

respectively. Or even briefer: (12.26) implies $\partial\dot{\tilde{k}}/\partial\tilde{c} = -1$. The horizontal arrows thus indicate the *direction* of movement of $\tilde{k}$ in the different regions of the phase diagram as determined by the differential equation (12.26).

Equation (12.27) shows that

$$\dot{\tilde{c}} = 0 \text{ for } \tilde{c} = \frac{b(\rho + m)\tilde{k}}{f'(\tilde{k}) - \delta - \rho - g}. \tag{12.31}$$

Hence,

$$\text{along the } \dot{\tilde{c}} = 0 \text{ locus, } \lim_{\tilde{k} \to \bar{\tilde{k}}} \tilde{c} = \infty,$$

so that the $\dot{\tilde{c}} = 0$ locus is asymptotic to the vertical line $\tilde{k} = \bar{\tilde{k}}$. Moving along the $\dot{\tilde{c}} = 0$ locus in the other direction, we see from (12.31) that $\lim_{\tilde{k} \to 0} \tilde{c} = 0$, as illustrated in Fig. 12.4.[9]

To explain the vertical arrows in the figure, for some fixed value $\tilde{c}_1$ (not too large) we imagine the corresponding horizontal line $\tilde{c} = \tilde{c}_1$ in the positive quadrant has been drawn. We then consider greater and greater values of $\tilde{k}$ along this line. To begin with, $\tilde{k}$ is small and therefore $f'(\tilde{k})$ is large so that, by (12.27), $\dot{\tilde{c}}$ is positive. At the point where the imagined horizontal line crosses the $\dot{\tilde{c}} = 0$ locus, we have $\dot{\tilde{c}} = 0$. And to the right, we have $\dot{\tilde{c}} < 0$ because $\tilde{k}$ is now large and $f'(\tilde{k})$ therefore small. The vertical arrows thus indicate the *direction* of movement of $\tilde{c}$ in the different regions of the phase diagram as determined by the differential equation (12.27). In brief: by (12.27) follows $\partial\dot{\tilde{c}}/\partial\tilde{k} = f''(\tilde{k})\tilde{c} - b(\rho + m) < 0$.

Steady state Fig. 12.4 also shows the steady-state point E, where the $\dot{\tilde{c}} = 0$ locus crosses the $\dot{\tilde{k}} = 0$ locus. The corresponding effective capital-labor ratio is $\tilde{k}^*$, to which is associated the (technology-corrected) consumption level $\tilde{c}^*$. Given our assumptions, including the Inada conditions, there exists one and only one steady state with positive effective capital-labor ratio. To see this, notice that in steady state the right-hand sides of (12.30) and (12.31) are equal to each other.

[9] The $\dot{\tilde{c}} = 0$ locus is positively sloped everywhere since, by (12.31),

$$\frac{d\tilde{c}}{d\tilde{k}}\Big|_{\dot{\tilde{c}}=0} = b(\rho + m)\frac{f'(\tilde{k}) - \delta - \rho - g - \tilde{k}f''(\tilde{k})}{\left(f'(\tilde{k}) - \delta - \rho - g\right)^2} > 0, \text{ whenever } f'(\tilde{k}) - \delta > \rho + g.$$

The latter inequality holds whenever $\tilde{k} < \bar{\tilde{k}}$.

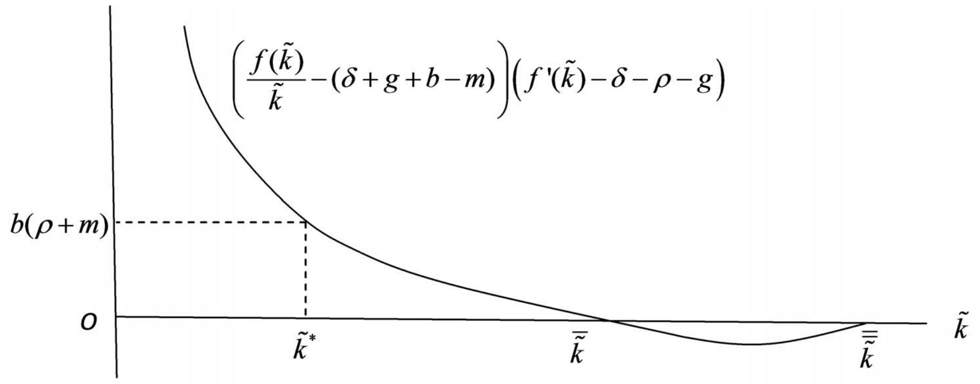

Figure 12.5: Existence of a unique $\tilde{k}^*$.

After ordering this implies

$$\left(\frac{f(\tilde{k})}{\tilde{k}} - (\delta + g + b - m)\right) \left[f'(\tilde{k}) - \delta - \rho - g\right] = b\,(\rho + m)\,. \qquad (12.32)$$

The left-hand side of this equation is depicted in Fig. 12.5. Since both the average and marginal productivities of capital are decreasing in $\tilde{k}$, the value of $\tilde{k}$ satisfying the equation is unique, given the requirement $\tilde{k} < \bar{\tilde{k}}$. And such a value exists due to the Inada conditions.[10]

The equilibrium path By equilibrium path is meant the solution (if any) to the model. More precisely, given the model, a path $(\tilde{k}(t),\, \tilde{c}(t))_{t=0}^{\infty}$ is an *equilibrium path* (with perfect foresight), if: (a) the path is *technically feasible,* that is, it satisfies the accounting equation (12.26) and has $\tilde{k}(t) = \tilde{k}_0$, where $\tilde{k}_0$ is the historically given initial effective capital-labor ratio; (b) the path is consistent with *market clearing* for all $t \geq 0$, given firms' profit maximization and households' utility maximization conditional on their expectations and budget constraints; and (c) along the path the evolution of the pair $(w(t),\, r(t))$, where $w(t) = \tilde{w}(\tilde{k}(t))T(t)$ and $r(t) = f'(\tilde{k}(t)) - \delta$, is as expected by the households – *expectations are fulfilled.*

Let us first ask the question: can the steady-state point E, that is, a path with $(\tilde{k}(t), \tilde{c}(t)) = (\tilde{k}^*, \tilde{c}^*)$ for all $t \geq 0$, be an equilibrium path? The answer is "yes", *if* the historically given initial value, $\tilde{k}_0$, happens to equal the steady-state value, $\tilde{k}^*$. Then the first requirement, (a), is fulfilled because the steady state satisfies the accounting equation (12.26) and starts out with $\tilde{k}(0) = \tilde{k}_0$ (which

[10]There is also a *trivial* steady state, namely the origin, which will never be realised as long as initial $\tilde{k}$ is positive.

in this case happens to equal to $\tilde{k}^*$). The second requirement, (b), is fulfilled because, by satisfying both the differential equations, (12.26) and (12.27), *and* the transversality conditions (12.28), the steady state is consistent with market clearing for all $t \geq 0$, given firms' profit maximization and households' utility maximization conditional on their expectations and budget constraints. The third requirement, (c), is fulfilled because expectations are fulfilled in the steady state

Here we claimed, without proof, that the steady state satisfies the transversality condition (12.28) for every $t_0 \geq 0$ and every $v \leq t_0$. A proof is given in Appendix D.

But generally, $\tilde{k}_0$ will differ from $\tilde{k}^*$, for instance we may have $0 < \tilde{k}_0 < \tilde{k}^*$ as in Fig. 12.4. So the equilibrium path cannot coincide with the steady state, at least not to begin with. To find a candidate for the equilibrium path in this case, we consider the dynamics in a neighborhood of the steady state. In Fig. 12.4 the horizontal and vertical arrows in this neighborhood indicate that the steady-state point E is a *saddle point*.[11] Graphically, this means that from the left and the right, respectively, in the phase diagram there is one and only one trajectory − the saddle path − converging toward the steady state, as indicated also in Fig. 12.4. As a construction line we have in the phase diagram drawn the vertical (stippled) line along which $\tilde{k} = \tilde{k}_0 > 0$. Consider the point where this vertical line crosses the saddle path. We claim that the section of the saddle path stretching from this point to E is, by construction, an equilibrium path of the economy. By arguments similar to those we applied for the steady state, we find that this converging path fulfils the requirements (a), (b), and (c) for an equilibrium path. This includes satisfaction of the transversality condition (12.28) for every $t_0 \geq 0$ and every $v \leq t_0$. Truly, in view of continuity, when a transversality condition holds for the steady state, it also holds for any path converging to the steady state.

Thereby we have found a solution to the model, given $0 < \tilde{k}_0 < \tilde{k}^*$. And the ordinate to the point where the vertical line $\tilde{k}_0 = \tilde{k}^*$ crosses the saddle path is the associated equilibrium value of initial consumption, $\tilde{c}(0)$. If instead $\tilde{k}_0 > \tilde{k}^*$, the corresponding section of the upper saddle path makes up the equilibrium path for that case.

Might there exist *other* equilibrium paths? No, the other paths in the phase diagram in Fig. 12.4 violate either the transversality conditions of the households (paths that in the long run point South-East) or their NPG conditions, and therefore also their transversality conditions, (paths that in the long run point North-West).

The conclusion is that there *exists* a solution to the model, and it is *unique*.

[11] To supplement this graphical inspection, a formal proof is given in Appendix E.

Moreover, the solution implies convergence to the steady state. In the language of differential equations this convergence has the form of *conditional* asymptotic stability of the steady state. The conditionality is due to the convergence being dependent on the initial value of the jump variable $\tilde{c}(0)$, cf. the phase diagram. But by assuming perfect foresight into the indefinite future, the model imposes that the market mechanism is able to induce people to choose their initial consumption levels in exact accordance with their transversality conditions, thereby making the technology-corrected average individual consumption level at time 0, $\tilde{c}(0)$, exactly equal to the ordinate of the point where the vertical line $\tilde{k}_0 = \tilde{k}^*$ crosses the saddle path. This is both necessary and sufficient for the convergence.

Saddle-point stability An equivalent way to characterize the dynamics of the model is to say that the unique non-trivial steady state E is *saddle-point stable*.[12] Indeed, the point E satisfies the four definitional conditions for saddle-point stability in a two-dimensional dynamic system with a steady state: (1) the steady state should be a saddle point; (2) one of the two endogenous variables should be predetermined, while the other should be a jump variable; (3) the saddle path should not be parallel to the jump-variable axis; and (4) there is a boundary condition on the system such that the diverging paths are ruled out as solutions.

These four requirements are satisfied by the present model. As noted above, requirement (1) holds. So does (2), since $\tilde{k}$ is a predetermined variable, and $\tilde{c}$ is a jump variable. The saddle path is not parallel to the $\tilde{c}$ axis, and it is the only path that satisfies *all* the conditions of dynamic general equilibrium, thereby ensuring the requirements (3) and (4).

Comments on the solution

Over time the economy moves along the saddle path toward the steady-state point E. If $\tilde{k}_0 < \tilde{k}^* \leq \tilde{k}_{GR}$, as illustrated in Fig. 12.4, then both $\tilde{k}$ and $\tilde{c}$ grow over time until the steady state is "reached". This is just one example, however. We could alternatively have $\tilde{k}_0 > \tilde{k}^*$ and then $\tilde{k}$ would be falling during the adjustment process.

An implication of $\tilde{c}(t) \equiv c(t)/T(t) \to \tilde{c}^*$ for $t \to \infty$ is that per capita consumption grow in the long run at the same rate as technology, the rate g. More precisely, for $t \to \infty$,

$$c(t) \text{ weakly approaches the steady-state path } c^*(t) = \tilde{c}^* T(0) e^{gt}. \qquad (12.33)$$

[12] The formal argument, which is more intricate than for the Ramsey model, is given in Appendix D, where also the arrows indicating paths that cross the $\tilde{k}$-axis are explained.

The qualification "weakly" is added to indicate that we claim no more than that the *ratio* $c(t)/c^*(t)$ $(= \tilde{c}(t)/c^*) \rightarrow 1$ for $t \rightarrow \infty$. This need not imply that the *difference* between $c(t)$ and $c^*(t)$ converges to nil. The difference can be written $c(t) - c^*(t) \equiv \tilde{c}(t)T(t) - \tilde{c}^*T(t) = (\tilde{c}(t) - \tilde{c}^*)T(t)$, where the first factor in the last expression goes to zero while the second factor goes to $+\infty$ for $t \rightarrow \infty$. Which of the factors moves faster depends on circumstances.

Similarly, an implication of $\tilde{k}(t) \rightarrow \tilde{k}^*$ for $t \rightarrow \infty$ is that

$$\tilde{w}(\tilde{k}(t)) \equiv \frac{w(t)}{T(t)} \rightarrow \tilde{w}(\tilde{k}^*) \text{ for } \quad t \rightarrow \infty, \tag{12.34}$$

where the wage function $\tilde{w}$ is defined in (12.21). This implies that the real wage in the long run grows at the rate g, and, more precisely, that

$w(t)$ weakly approaches the steady-state path $w^*(t) = \tilde{w}(\tilde{k}^*)T(0)e^{gt}$.

These results are similar to those of the Ramsey model. More interesting are the following two observations:

1. *The long-run real interest rate will be higher than in the "corresponding" Ramsey long-run equilibrium when the latter exists.* For $t \rightarrow \infty$,

$$r(t) = f'(\tilde{k}(t)) - \delta \rightarrow f'(\tilde{k}^*) - \delta = r^* > f'(\bar{\tilde{k}}) - \delta = \rho + g, \tag{12.35}$$

where the inequality follows from $\tilde{k}^* < \bar{\tilde{k}}$. Suppose the Ramsey economy described in Chapter 10 has the same f, ρ and g and has $u(c) = \ln c$, i.e., $\theta = 1$. Then, if and only if $\rho - n > 0$ (cf. the assumption (A1) of Chapter 10), does a long-run equilibrium exist in that Ramsey economy, and its long-run interest rate will be $r_R^* = \rho + g$. Owing to finite lifetime $(m > 0)$, the present version of the Blanchard OLG model, with the same ρ and n, unambiguously predicts a higher long-run interest rate than this "corresponding" Ramsey economy has. The positive probability of not being alive at a certain time in the future leads to less saving and therefore less capital accumulation. So the economy ends up with a lower effective capital-labor ratio and thereby a higher real interest rate.[13]

2. *The Perpetual Youth model does not rule out dynamic inefficiency.* From the definition of $\tilde{k}_{GR}$ and $\bar{\tilde{k}}$ in (12.29) follows that $\bar{\tilde{k}} \lesseqqgtr \tilde{k}_{GR}$ for $\rho \gtreqqless n$, respectively, where $n \equiv b - m$. Suppose $0 \leq \rho < n$. Then $\bar{\tilde{k}} > \tilde{k}_{GR}$ and we *can* have $\tilde{k}_{GR} < \tilde{k}^* < \bar{\tilde{k}}$. In contrast, a steady state of a Ramsey economy will always be situated to the left of $\tilde{k}_{GR}$ and dynamic efficiency thereby be ensured.

[13] When retirement at old age is added to the model, this, however, no longer necessarily holds, cf. Section 12.3 below.

How comes this difference? It comes from a parameter restriction imposed in the Ramsey model because it is needed to ensure existence of general equilibrium in that model. This parameter restriction is named (A1) in Chapter 10 and requires that $\rho - n > (1-\theta)g$. The restriction reduces to $\rho - n > 0$ in the case $\theta = 1$, which is the relevant case for comparison between the two models. When $\theta = 1$ in the Ramsey model, its steady-state value of $\tilde{k}$, denoted $\tilde{k}^*_R$, say, satisfies

$$f'(\tilde{k}^*_R) - \delta = \rho + g > n + g = f'(\tilde{k}_{GR}) - \delta,$$

where the inequality comes from the parameter restriction $\rho - n > 0$. *If* we impose $\rho - n > 0$ in the Blanchard OLG model, its steady state will also be to the left, in fact further to the left, of $\tilde{k}_{GR}$, since we get

$$f'(\tilde{k}^*) - \delta > f'(\bar{\tilde{k}}) - \delta = \rho + g > n + g = f'(\tilde{k}_{GR}) - \delta.$$

It is only in the opposite case, namely the case $\rho - n \leq 0$, that the possibility of $\bar{\tilde{k}} \geq \tilde{k}_{GR}$ arises. And in the OLG model, the weak inequality $\rho - n \leq 0$ is *not* in conflict with existence of dynamic general equilibrium. But in the Ramsey model it is. Certainly, in the Ramsey model the weak inequality $\rho - n \leq 0$ leads to $r^* \leq g + n$. The right-hand side of this inequality will equal the growth rate of wL in a hypothetical steady state. So the present value of future labor income of the representative household would be *infinite* and consumption demand thus unbounded. This rules out equilibrium. And it is the reason that the Ramsey model from the beginning only allows $\rho - n > 0$, or more generally $\rho - n > (1-\theta)g$. Hence a Ramsey economy always has steady state to the left of the golden rule and therefore, in contrast to an OLG model, never ends up in dynamic inefficiency.

We can relax the parameter restrictions $\delta > 0$, $\rho \geq 0$, and $b \geq m$ that have hitherto been assumed in the Blanchard OLG model for ease of exposition. To ensure existence of a solution to the household's decision problem, we need that the effective utility discount rate is positive, i.e., $\rho + m > 0$.[14] Further, from the definition of $\tilde{k}_{GR}$ and $\bar{\tilde{k}}$ in (12.29) we need $\delta + g + b - m > 0$ and $\delta + \rho + g > 0$ where, by definition, $m > 0$, $\delta \geq 0$, and $g \geq 0$. Hence, as long as $\delta + g > 0$, we can allow $n \equiv b - m$ and/or ρ to be negative (not "too" negative, though) without interfering with the existence of general equilibrium.

Remark on a seeming paradox It might seem like a paradox that the economy can be in steady state and at the same time have $r^* - \rho > g$. By the

[14] Of course we also need that the present discounted value of future labor income is well-defined (i.e., not infinite) and this requires $r^* + m > g$. In view of (12.35), however, this is automatically satisfied when $\rho \geq 0$ and $m > 0$.

Keynes-Ramsey rule, when $r^* - \rho > g$, individual consumption is growing faster than productivity, which grows at the rate g. How can such an evolution be sustained? The answer lies in the fact that we are *not* considering an economy with a representative agent, but an economy with a composition effect in the form of the turnover of the generations. Whereas individual consumption can grow at a relatively high rate, this consumption only exists as long as the individual is alive. Hence per capita consumption, $c \equiv C/N$, behaves differently. From (12.18) we have in steady state

$$\frac{\dot{c}}{c} = r^* - \rho - b(\rho + m)\frac{a}{c} < r^* - \rho,$$

where $a \equiv A/N$ (average financial wealth). The consumption by those who die is replaced by that of the newborn who have less financial wealth, hence lower consumption than the average citizen. To take advantage of $r^* - \rho > g \geq 0$, the young (and in fact everybody) save, thereby becoming gradually richer and improving their standard of living relatively fast. Owing to the generation replacement effect, however, per capita consumption grows at a lower rate. In steady state *this* rate equals g, as indicated by (12.33).

Is this consistent with the aggregate consumption function? The answer is affirmative since by dividing through by $N(t)$ in (12.16) we end up with

$$c(t) = (\rho+m)(k(t)+\bar{h}(t)) \equiv (\rho+m)(\tilde{k}(t)+\tilde{h}(t))T(t) = (\rho+m)(\tilde{k}^*+\tilde{h}^*)T(0)e^{gt}, \tag{12.36}$$

in steady state, where

$$\begin{aligned}\tilde{h}^* &\equiv \left(\frac{\bar{h}(t)}{T(t)}\right)^* = \frac{\int_t^\infty w(s)e^{-(r^*+m)(s-t)}ds}{T(t)} = \frac{\int_t^\infty \tilde{w}(\tilde{k}^*)T(s)e^{-(r^*+m)(s-t)}ds}{T(t)} \\ &= \tilde{w}(\tilde{k}^*)\int_t^\infty e^{g(s-t)}e^{-(r^*+m)(s-t)}ds = \frac{\tilde{w}(\tilde{k}^*)}{r^*+m-g}. \end{aligned} \tag{12.37}$$

The last equality in the first row comes from (12.34); in view of $r^* > \rho + g$, the numerator, $r^* + m - g$, in the second row is a positive constant. Hence, both $\tilde{k}^*$ and $\tilde{h}^*$ are constants. In this way the consumption function in (12.36) confirms the conclusion that per capita consumption in steady state grows at the rate g.

The demographic transition and the long-run interest rate

In the last more than one hundred years the industrialized countries have experienced a gradual decline in the three demographic parameters m, b, and n. Clearly m has gone down, thus increasing life expectancy, $1/m$. But also $n \equiv b - m$ has gone down, hence b has gone down even more than m. What effect on r^* should we expect? A "rough answer" can be based on the Blanchard model.

It is here convenient to consider n and m as the basic parameters and $b \equiv n+m$ as a derived one. So in (12.26) and (12.27) we substitute $b \equiv n+m$. Then there is only one demographic parameter affecting the position of the $\dot{\tilde{k}} = 0$ locus, namely n. Three effects are in play:

a. *Labor-force growth effect.* The lower n results in an upward shift of the $\dot{\tilde{k}} = 0$ locus in Fig. 12.4, hence a tendency to expansion of $\tilde{k}$. This "capital deepening" is due to the fact that slower growth in the labor force implies less capital "dilution".

b. *Life-cycle effect.* Given n, the lower m results in a clockwise turn of the $\dot{\tilde{c}} = 0$ locus in Fig. 12.4. This enforces the tendency to expansion of $\tilde{k}$. The explanation is that the higher life expectancy, $1/m$, increases the incentive to save and thus reduces consumption $C = (\rho+m)(A+H)$. Thereby, capital accumulation is promoted.

c. *Generation replacement effect.* Given m, the lower $b = n+m$ results in lower n, hence a further clockwise turn of the $\dot{\tilde{c}} = 0$ locus in Fig. 12.4. This additional capital deepening is explained by a composition effect. Lower b implies a smaller proportion of young people (with the same human wealth as others, but less financial wealth) in the population, thus leading to smaller H/A, hence smaller $[C/(\rho+m)]/A = (A+H)/A$, by the consumption function. As C/A is thus smaller, $S/A \equiv (Y-C)/A$ will be larger, resulting again in more capital accumulation.

Thus all three effects on the effective capital-labor ratio are positive. Consequently, we should expect a lower marginal productivity of capital and a lower real interest rate in the long run. There are a few empirical long-run studies pointing in this direction (see, e.g., Doménil and Lévy, 1990).

We called our answer to this demographic question a "rough answer". Being based on a *comparative* method, the analysis has its limitations. The comparative method compares the evolution of two distinct economies having the same structure and parameter values except with respect to the parameter the role of which we want to study.

A more appropriate approach would consider dynamic effects of a parameter changing in *historical time* in a given economy. This would be a truly dynamic approach but is much more complex, requiring an extended model with demographic dynamics. In contrast the Blanchard OLG model presupposes a stationary age distribution in the population. That is, the model depicts a situation where m, b, and n have stayed at their current values for a long time and are not changing. A time-dependent n, for example, would require expressions like $N(t) = N(0)e^{\int_0^t n_s ds}$, which gives rise to a much more complicated model.

12.3 Adding retirement

So far the model has assumed that everybody works full-time until death. This is clearly a weakness of a model that is intended to reflect life-cycle aspects of economic behavior. We therefore extend the model by incorporating gradual (but exogenous) retirement from the labor market. Following Blanchard (1985), we assume retirement is *exponential* (thereby still allowing simple analytical aggregation across cohorts).

Gradual retirement and aggregate labor supply

Suppose labor supply, ℓ, per year at time t for an individual born at time v depends only on age, $t - v$, according to

$$\ell(t-v) = e^{-\lambda(t-v)}, \tag{12.38}$$

where $\lambda > 0$ is the *retirement rate*. That is, higher age implies lower labor supply.[15] The graph of (12.38) in $(t - v, \ell)$ space looks like the solid curve in Fig. 12.2 above. Though somewhat coarse, this gives at least a flavour of retirement: old persons don't supply much labor. Consequently an incentive to save for retirement emerges.

Aggregate labor supply now is

$$\begin{aligned} L(t) &= \int_{-\infty}^{t} \ell(t-v)N(v,t)dv \\ &= \int_{-\infty}^{t} e^{-\lambda(t-v)}N(0)e^{nv}be^{-m(t-v)}dv \qquad \text{(from (12.38) and (12.4))} \\ &= bN(0)e^{-(\lambda+m)t}\int_{-\infty}^{t} e^{(\lambda+b)v}dv = bN(0)e^{-(\lambda+m)t}\frac{e^{(\lambda+b)t}-0}{\lambda+b} \\ &= bN(0)e^{nt}\frac{1}{\lambda+b} = \frac{b}{\lambda+b}N(t). \end{aligned} \tag{12.39}$$

For given population size $N(t)$, earlier retirement (larger λ) implies lower aggregate labor supply. Similarly, given $N(t)$, a higher birth rate, b, entails a larger aggregate labor supply. This is because a higher b amounts to a larger fraction of

[15] An alternative interpretation of (12.38) would be that labor *productivity* is a decreasing function of age (as in Barro and Sala-i-Martin, 2004, pp. 185-86).

young people in the population and the young have a larger than average labor supply. Moreover, as long as the birth rate and the retirement rate are constant, aggregate labor supply grows at the same rate as population.

By the specification (12.38) the labor supply per year of a newborn is one unit of labor. In (12.39) we thus measure the labor force in units equivalent to the labor supply per year of one newborn.

The essence of retirement is that the aggregate labor supply depends on the age distribution in the population. The formula (12.39) presupposes that the age distribution has been constant for a long time. Indeed, the derivation of (12.39) assumes that the parameters b, m, and λ took their current values a long time ago so that there has been enough time for the age distribution to reach its steady state.

Human wealth

The present value at time t of expected future labor income for an individual born at time v is

$$\begin{aligned} h(v,t) &= \int_t^{\infty} w(s)\,\ell(s-v)e^{-\int_t^s (r(\tau)+m)d\tau}ds \\ &= \int_t^{\infty} w(s)\,e^{-\lambda(s-v)}e^{-\int_t^s (r(\tau)+m)d\tau}ds \\ &= e^{-\lambda(t-v)}\int_t^{\infty} w(s)\,e^{-\lambda(s-t)}e^{-\int_t^s (r(\tau)+m)d\tau}ds \\ &= e^{-\lambda(t-v)}\int_t^{\infty} w(s)\,e^{-\int_t^s (r(\tau)+\lambda+m)d\tau}ds = e^{-\lambda(t-v)}h(t,t) \quad (12.40) \end{aligned}$$

where

$$h(t,t) = \int_t^{\infty} w(s)\,e^{-\int_t^s (r(\tau)+\lambda+m)d\tau}ds, \quad (12.41)$$

which is the human wealth of a newborn at time t (in (12.40) set $v = t$). Hence, aggregate human wealth for those alive at time t is

$$\begin{aligned} H(t) &= \int_{-\infty}^{t} h(v,t)N(v,t)dv = h(t,t)\int_{-\infty}^{t} e^{-\lambda(t-v)}N(v,t)dv \\ &= h(t,t)\int_{-\infty}^{t} e^{-\lambda(t-v)}N(0)e^{nv}be^{-m(t-v)}dv \\ &= h(t,t)bN(0)e^{-(\lambda+m)t}\int_{-\infty}^{t} e^{(\lambda+b)v}dv \\ &= h(t,t)bN(0)e^{-(\lambda+m)t}\frac{e^{(\lambda+b)t}-0}{\lambda+b} \\ &= h(t,t)N(0)e^{nt}\frac{b}{\lambda+b} = h(t,t)N(t)\frac{b}{\lambda+b} = h(t,t)L(t), \quad (12.42) \end{aligned}$$

by substitution of (12.39). That is, aggregate human wealth at time t is the same as the human wealth of a newborn at time t times the size of the labor force at time t. This result is due to the labor force being measured in units equivalent to the labor supply of one newborn.

Combining (12.41) and (12.42) gives

$$H(t) = \frac{b}{\lambda+b}N(t)\int_{t}^{\infty} w(s)\, e^{-\int_t^s (r(\tau)+\lambda+m)d\tau}ds. \quad (12.43)$$

If $\lambda = 0$, this reduces to the formula (12.15) for aggregate human wealth in the simple Blanchard model. We see from (12.43) that the future wage level $w(\tau)$ is effectively discounted by the sum of the interest rate, the retirement rate, and the death rate. This is not surprising. The sooner you retire and the sooner you are likely to die, the less important to you are the wage levels in the future.

Since the propensity to consume out of wealth is still the same for all individuals, aggregate consumption is, as before,

$$C(t) = (\rho+m)\left[A(t)+H(t)\right]. \quad (12.44)$$

Dynamics of household aggregates

The increase in aggregate financial wealth per time unit is

$$\dot{A}(t) = r(t)A(t) + w(t)L(t) - C(t). \quad (12.45)$$

The only difference compared to the simple Blanchard model is that it is now aggregate employment, $L(t)$, rather than population, $N(t)$, that enters the term for aggregate labor income.

The second important dynamic relation for the household sector is the one describing the increase in aggregate consumption per time unit. Instead of $\dot{C}(t) = [r(t) - \rho + n]C(t) - b(\rho + m)A(t)$ from the simple Blanchard model, we now get

$$\dot{C}(t) = [r(t) - \rho + \lambda + n]C(t) - (\lambda + b)(\rho + m)A(t). \tag{12.46}$$

We see the retirement rate λ enters in two ways. This is because the generation replacement effect now has two sides. On the one hand, as before, the young that replace the old enter the economy with *no financial wealth*. On the other hand now they arrive with *more human wealth* than the average citizen. Through this channel the replacement of generations implies an increase per time unit in human wealth equal to λH, *ceteris paribus*. Indeed, the "rejuvenation effect" on individual labor supply is proportional to labor supply: $\partial\ell(t-v)/\partial v = \lambda\ell(t-v)$, from (12.38). In analogy, with a slight abuse of notation we can express the *ceteris paribus effect* on aggregate consumption as

$$\frac{\partial C}{\partial t} = (\rho + m)\frac{\partial H}{\partial t} = (\rho + m)\lambda H = \lambda(C - (\rho + m)A),$$

where the first and the last equality come from (12.44). This explicates the difference between the new equation (12.46) and the corresponding one from the simple model.[16]

The equilibrium path

With $r = f'(\tilde{k}) - \delta$ and $A = K$, (12.46) can be written

$$\dot{C} = \left[f'(\tilde{k}) - \delta - \rho + \lambda + n\right] C - (\lambda + b)(\rho + m)K. \tag{12.47}$$

Once more, the dynamics of general equilibrium can be summarized in two differential equations in $\tilde{k} \equiv K/(TL) \equiv k/T$ and $\tilde{c} \equiv C/(TN) \equiv c/T$. The differential equation in $\tilde{k}$ can be based on the national product identity for a closed economy: $Y = C + \dot{K} + \delta K$. Isolating $\dot{K}$ and using the definition of $\tilde{k}$, we get

$$\dot{\tilde{k}} = f(\tilde{k}) - \frac{C}{TL} - (\delta + g + n)\tilde{k} = f(\tilde{k}) - \frac{\lambda + b}{b}\tilde{c} - (\delta + g + b - m)\tilde{k}, \tag{12.48}$$

since $C/(TL) \equiv cN/(TL) = (N/L)c/T = (\lambda + b)\tilde{c}/b$ from (12.39).

As to the other differential equation, log-differentiating $\tilde{c}$ w.r.t. time yields

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{\dot{C}}{C} - \frac{\dot{T}}{T} - \frac{\dot{N}}{N} = f'(\tilde{k}) - \delta - \rho + \lambda + n - (\lambda + b)(\rho + m)\frac{K}{C} - g - n,$$

[16] This explanation of (12.46) is only intuitive. A formal derivation can be made by using a method analogous to that applied in Appendix C.

from (12.47). Hence,

$$\begin{aligned}\dot{\tilde{c}} &= \left[f'(\tilde{k}) - \delta - \rho + \lambda - g\right]\tilde{c} - (\lambda + b)(\rho + m)\frac{K}{C}\tilde{c} \\ &= \left[f'(\tilde{k}) - \delta - \rho + \lambda - g\right]\tilde{c} - (\lambda + b)(\rho + m)\frac{K}{TL}\cdot\frac{L}{N},\end{aligned}$$

implying, in view of (12.39),

$$\dot{\tilde{c}} = \left[f'(\tilde{k}) - \delta - \rho + \lambda - g\right]\tilde{c} - b(\rho + m)\tilde{k}. \tag{12.49}$$

The transversality conditions of the households are still given by (12.28).

Phase diagram The equation describing the $\dot{\tilde{k}} = 0$ locus is

$$\tilde{c} = \frac{b}{\lambda + b}\left[f(\tilde{k}) - (\delta + g + b - m)\tilde{k}\right]. \tag{12.50}$$

The equation describing the $\dot{\tilde{c}} = 0$ locus is

$$\tilde{c} = \frac{b\,(\rho + m)\,\tilde{k}}{f'(\tilde{k}) - \delta - \rho + \lambda - g}. \tag{12.51}$$

Let the value of $\tilde{k}$ such that the denominator of (12.51) vanishes be denoted $\bar{\tilde{k}}$, that is,

$$f'(\bar{\tilde{k}}) = \delta + \rho - \lambda + g. \tag{12.52}$$

Such a value exists if, in addition to the Inada conditions, the inequality

$$\lambda < \delta + \rho + g$$

holds, saying that the retirement rate is not "too large". We assume this to be true. The $\dot{\tilde{k}} = 0$ and $\dot{\tilde{c}} = 0$ loci are illustrated in Fig. 12.6. The $\dot{\tilde{c}} = 0$ locus is everywhere to the left of the line $\tilde{k} = \bar{\tilde{k}}$ and is asymptotic to this line.

As in the simple Blanchard model, the steady state $(\tilde{k}^*, \tilde{c}^*)$ is saddle-point stable. The economy moves along the saddle path towards the steady state for $t \to \infty$. Hence, for $t \to \infty$,

$$r_t = f'(\tilde{k}_t) - \delta \to f'(\tilde{k}^*) - \delta \equiv r^* > \rho + g - \lambda, \tag{12.53}$$

where the inequality follows from $\tilde{k}^* < \bar{\tilde{k}}$. Again we may compare with the Ramsey model which, with $u(c) = \ln c$, has long-run interest rate equal to $r_R^* = \rho + g$.

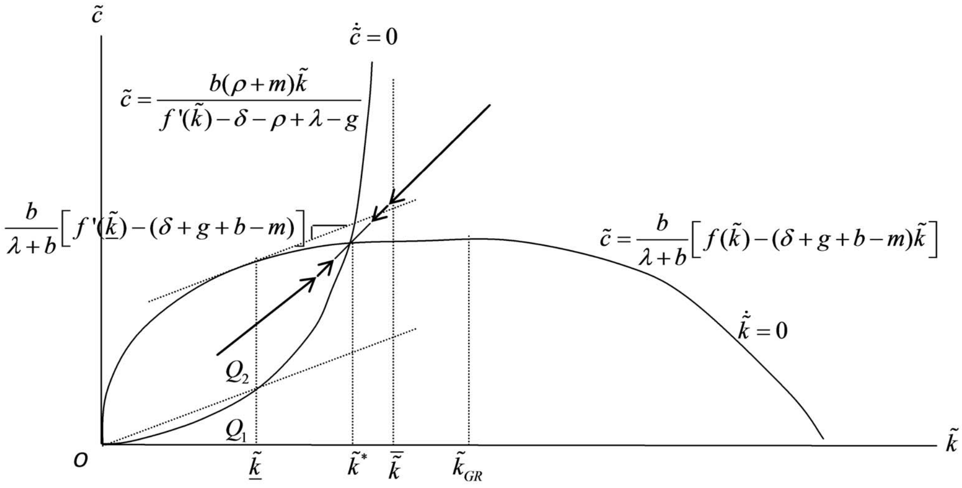

Figure 12.6: Phase diagram of the Blanchard model with retirement.

In the Blanchard OLG model extended with retirement, the long-run interest rate may differ from this value because of two effects of life-cycle behavior, that go in opposite directions. On the one hand, as mentioned earlier, finite lifetime ($m > 0$) leads to a higher effective utility discount rate, hence *less* saving and therefore a tendency for $r^* > \rho + g$. On the other hand, retirement entails an incentive to save *more* (for the late period in life with low labor income). This results in a tendency for $r^* < \rho + g$, everything else equal.

The presence of retirement implies that a new kind of apparent paradox may arise: the growth rate of individual consumption, $c(v, t)$, may be *lower* than that of per capita consumption, $c(t) \equiv \tilde{c}(t)T(t)$. In steady state $c(v, t)$ grows at the rate $r^* - \rho$ as long as the individual is alive, while per capita consumption, $c(t)$, grows at the rate g. In view of (12.53) and $g \geq 0$, the greatest lower bound for the former growth rate is $g - \lambda$ and there is scope for the inequality $g - \lambda < r^* - \rho < g$ to hold. So every individual alive may have a growth rate of consumption below the per capita consumption growth rate, g. The former may even be negative (namely if $g - \lambda < r^* - \rho < 0$) in spite of $g > 0$.[17] How can this be possible? Again the answer lies in the generation replacement effect. On the one hand, $r^* - \rho < 0$ induces every individual to have declining consumption until death (and, outside the model, may be also needs as old *are* smaller). On the other hand, if at the same time $g > 0$, every new generation starts adult life with higher initial consumption than the previous one. This is because technical progress endows new generations with higher initial human wealth than that with which the previous generations entered the economy.

[17] See Exercise 12.??

Another new phenomenon due to retirement is the theoretical possibility of dynamic inefficiency which was absent before. Recall that the golden rule capital-labor ratio, $\tilde{k}_{GR}$, is characterized by

$$f'(\tilde{k}_{GR}) - \delta = g + n,$$

where in the present case $n = b - m$. There are two cases to consider:

Case 1: $\lambda \leq \rho - n$. Then $\rho - \lambda \geq n$, so that $\bar{\tilde{k}} \leq \tilde{k}_{GR}$, implying that $\tilde{k}^*$ is below the golden rule value. In this case the long-run interest rate, r^*, satisfies $r^* = f'(\tilde{k}^*) - \delta > g + n$, that is, the economy is dynamically efficient.

Case 2: $\lambda > \rho - n$. We now have $\rho - \lambda < n$, so that $\bar{\tilde{k}} > \tilde{k}_{GR}$. Hence, it is possible that $\tilde{k}^* > \tilde{k}_{GR}$, implying $r^* = f'(\tilde{k}^*) - \delta < g + n$, so that there is sustained over-saving and the economy is dynamically inefficient. Owing to the retirement, this can arise even when $\rho > n$. A situation with $r^* \leq g + n$ has theoretically interesting implications for solvency and sustainability of fiscal policy, a theme we considered in Chapter 6. On the other hand, as argued in Section 4.2 of Chapter 4, the empirics point to dynamic *efficiency* as the most plausible case.

The reason that a high retirement rate λ (early retirement) *may*, theoretically, lead to over-saving is that early retirement implies a longer span of the period as almost fully retired. Hence there is a need to do more saving for retirement. In general equilibrium, however, there is an effect in the opposite direction. This is that early retirement reduces the labor force relative to population. Thereby the drag on capital accumulation coming from a given level of *per capita* consumption is enlarged, as revealed by the term $((\lambda + b)/b)\tilde{c}$ in (12.48).

12.4 The rate of return in the long run

Blanchard's OLG model provides a succinct and yet multi-facetted theory of the level of the interest rate in the long run. Of course, in the real world there are many different types of uncertainty which simple macro models like the present one ignore. Yet we may interpret the real interest rate of these models as reflecting the general level around which the different interest rates of an economy fluctuate, i.e., a kind of average rate of return.

In this perspective Blanchard's theory of the rate of return differs from the modified golden rule theory from Ramsey's and Barro's models by allowing a role for demographic parameters. The Blanchard model predicts a long-run interest rate in the interval

$$\rho + g - \lambda < r^* < \rho + g + b. \tag{12.54}$$

The left-hand inequality, which reflects the role of retirement, was proved above (see (12.53)). In the right-hand inequality appears the positive birth rate b which allows the interest rate to exceed the level $\rho + g$, i.e., the level of the interest rate in the corresponding Ramsey model. An algebraic proof of this upper bound for r^* is provided in Appendix F. Below is a graphical argument, which is more intuitive.

Proof (sketch) of the upper inequality in (12.54)*

Let $\underline{\tilde{k}} > 0$ be some value of $\tilde{k}$ less than $\tilde{k}^*$. The vertical line $\tilde{k} = \underline{\tilde{k}}$ in Fig. 12.6 crosses the horizontal axis and the $\dot{\tilde{c}} = 0$ locus at the points Q_1 and Q_2, respectively. Adjust the choice of $\underline{\tilde{k}}$ such that the ray OQ_2 is parallel to the tangent to the $\dot{\tilde{k}} = 0$ locus at $\tilde{k} = \underline{\tilde{k}}$ (evidently this can always be done). We then have

$$\text{slope of } OQ_2 = \frac{|Q_1Q_2|}{|OQ_1|} = \frac{b}{\lambda + b}\left[f'(\underline{\tilde{k}}) - (\delta + g + n)\right].$$

By construction we also have

$$\text{slope of } OQ_2 = \frac{b(\rho + m)\underline{\tilde{k}}}{f'(\underline{\tilde{k}}) - \delta - \rho - g + \lambda} \cdot \frac{1}{\underline{\tilde{k}}},$$

where $\tilde{k}$ cancels out. Equating the two right-hand sides and ordering gives

$$\begin{aligned} \frac{b(\rho + m)}{f'(\underline{\tilde{k}}) - \delta - \rho - g + \lambda} &= \frac{b}{\lambda + b}\left[f'(\underline{\tilde{k}}) - (\delta + g + n)\right] \Rightarrow \\ \frac{(\lambda + b)(\rho + m)}{f'(\underline{\tilde{k}}) - \delta - \rho - g + \lambda} &= f'(\underline{\tilde{k}}) - (\delta + g + n). \qquad (12.55) \end{aligned}$$

This implies a quadratic equation in $f'(\underline{\tilde{k}})$ with the positive solution

$$f'(\underline{\tilde{k}}) = \delta + \rho + g + b. \qquad (12.56)$$

Indeed, with (12.56) we have:

$$\begin{aligned} \text{left-hand side of (12.55)} &= \frac{(\lambda + b)(\rho + m)}{\delta + \rho + g + b - \delta - \rho - g + \lambda} \\ &= \frac{(\lambda + b)(\rho + m)}{b + \lambda} = \rho + m, \qquad \text{and} \\ \text{right-hand side of (12.55)} &= \rho + m, \end{aligned}$$

so that (12.55) holds. Now, from $\tilde{k}^* > \underline{\tilde{k}}$ and $f'' < 0$ follows that $f'(\tilde{k}^*) < f'(\underline{\tilde{k}})$. Hence,

$$r^* = f'(\tilde{k}^*) - \delta < f'(\underline{\tilde{k}}) - \delta = \rho + g + b,$$

where the last equality follows from (12.56). This confirms the right-hand inequality in (12.54). □

How r^* depends on parameters

Let us first consider a numerical example.

EXAMPLE 1 Using one year as our time unit, a rough estimate of the rate of technological progress g for the Western countries since World War II is $g = 0.02$. To get an assessment of the birth rate b, we may coarsely estimate $n = b - m$ to be 0.005. An expected lifetime (as adult) around 55 years, equal to $1/m$ in the model, suggests that $m = 1/55 \approx 0.018$. Hence $b = n + m \approx 0.023$. What about the retirement rate λ? An estimate of the labor force participation rate is $L/N = 0.75$, equal to $b/(b + \lambda)$ in the model, so that, from (12.39), $\lambda = b(1 - L/N)/(L/N) \approx 0.008$. Now, guessing $\rho = 0.02$, (12.54) gives $0.032 < r^* < 0.063$. □

As this example illustrates, the interval (12.54) gives only a very rough idea about the level of r^*. And such an interval is of limited help for assessing how r^* depends on the model's parameters.

To dig deeper, given the production function f, let us see if we can determine r^* as an implicit function of the parameters. In steady state, $\tilde{k} = \tilde{k}^*$ and the right-hand sides of (12.50) and (12.51) are equal to each other. After ordering we have

$$\left(\frac{f(\tilde{k}^*)}{\tilde{k}^*} - (\delta + g + b - m)\right)\left[f'(\tilde{k}^*) - \delta - \rho + \lambda - g\right] = (\lambda + b)(\rho + m). \quad (12.57)$$

A diagram showing the left-hand side and right-hand side of this equation will look qualitatively like Fig. 12.5 above. The equation defines $\tilde{k}^*$ as an implicit function φ of the parameters g, δ, n, m, ρ, and λ, i.e., $\tilde{k}^* = \varphi(g, b, m, \rho, \lambda, \delta)$. The partial derivatives of φ have the sign structure $\{-, -, ?, -, ?, -\}$ (to see this, use implicit differentiation or simply curve shifting in a graph like Fig. 12.5). Then, from $r^* = f'(\tilde{k}^*) - \delta$ follows $\partial r^*/\partial x = (\partial r^*/\partial \tilde{k}^*)\partial\varphi/\partial x = f''(\tilde{k}^*)\partial\varphi/\partial x$ for $x \in \{g, b, m, \rho, \lambda, \delta\}$. These partial derivatives have the sign structure $\{+, +, ?, +, ?, ?\}$. This tells us how the long-run interest rate qualitatively depends on these parameters.

For example, a higher rate of technological progress results in a higher rate of return, r^*. The higher g is, the greater is the expected future wage income and the associated consumption possibilities even without any current saving. This discourages saving, and thereby capital accumulation, and a lower effective capital-labor ratio in steady state is the result, hence a higher long-run interest rate. In turn this is what is needed to sustain a higher long-run per capita growth rate equal to g. A higher mortality rate has an ambiguous effect on the rate of return in the long run. On the one hand a higher m shifts the $\dot{\tilde{k}} = 0$ curve in Fig. 9.6 upward because of the implied lower labor force growth rate. For given aggregate saving this entails more capital deepening in the economy. On the other hand, a higher m also implies less incentive for saving and therefore a counter-clockwise turn of the $\dot{\tilde{c}} = 0$ curve. The net effect of these two forces on $\tilde{k}^*$, hence on r^*, is ambiguous. But as (12.57) indicates, if b is increased along with m so as to keep n unchanged, $\tilde{k}^*$ falls and r^* rises.

Earlier retirement similarly has an ambiguous effect on the rate of return in the long run. On the one hand a higher λ shifts the $\dot{\tilde{k}} = 0$ curve in Fig. 9.6 downward because the lower labor force participation rate reduces per capita output. On the other hand, a higher λ also implies a clockwise turn of the $\dot{\tilde{c}} = 0$ curve. This is because the need to provide for a longer period as retired implies more saving and capital accumulation in the economy. The net effect of these two forces on $\tilde{k}^*$, hence on r^*, is ambiguous.

Also $\partial r^*/\partial\delta$ can not be signed without further specification, because $\partial r^*/\partial\delta = (\partial r^*/\partial\tilde{k}^*)\partial\varphi/\partial\delta = f''(\tilde{k}^*)\partial\varphi/\partial\delta - \partial\delta/\partial\delta = f''(\tilde{k}^*)\partial\varphi/\partial\delta - 1$, where we cannot apriori tell whether the first term exceeds 1 or not.

Of course, by specifying the production function and assigning numbers to the parameters, numerical solutions can be studied.

Further perspectives

A theoretically important factor for consumption-saving behavior − and thereby r^* − is missing in the version of the Blanchard model considered here. This factor is the desire for consumption smoothing or its inverse, the intertemporal elasticity of substitution in consumption. Since our version of the model is based on the special case $u(c) = \ln c$, the intertemporal elasticity of substitution in consumption is fixed to be 1. Now assume, more generally, that $u(c) = c^{1-\theta}/(1-\theta)$, where $\theta > 0$ and $1/\theta$ is the intertemporal elasticity of substitution in consumption. Then the dynamic system becomes three-dimensional and in that way more complicated. Nevertheless it can be shown that a higher θ implies a higher real interest rate in the long run.[18] The intuition is that in an economy with sustained productivity growth, a higher θ means less willingness to offer current consumption for more future consumption and this implies less saving. Thus, $\tilde{k}^*$ becomes lower and r^* higher. Also public debt tends to affect r^* positively in a closed economy, as we will see in the next chapter.

[18] See Blanchard (1985).

We end this section with some general reflections. Economic theory is a set of propositions that are organized in a hierarchic way and have an economic interpretation. A theory of the real interest rate should say something about the factors and mechanisms that determine the level of the interest rate. In fact, in a more realistic setup with uncertainty, it is the level of interest *rates,* including the risk-free rate, that should be determined. We would like the theory to explain both the short-run level of interest rates and the long-run level, that is, the average level over several decades. The Blanchard model can be one part of such a theory as far as long-run interest rates is concerned. Abstracting from monetary factors, nominal price rigidities, and short-run fluctuations in aggregate demand, the model is certainly less reliable as a description of the short run.

Note that the interest rate considered so far is the short-term interest rate. What is important for consumption and, in particular, investment is rather the long-term interest rate (internal rate of return on long-term bonds). With perfect foresight, the long-term rate is just a weighted average of expected future short-term rates.[19] In steady state these rates will be the same and so the present theory applies. In a world with uncertainty, however, the link between the long-term rate and the expected future short-term rates is more difficult to discern, affected as it may be by changing risk and liquidity premia.

12.5 National wealth and foreign debt

We will embed the Blanchard setup in a small open economy (henceforth SOE). The purpose is to study how national wealth and foreign debt in the long run are determined, when technological change is exogenous. Our SOE is characterized by:

(a) Perfect mobility across borders of goods and financial capital.

(b) Domestic and foreign financial claims are perfect substitutes.

(c) The need for means of payment is ignored and so is the need for a foreign exchange market.

(d) No mobility of labor across borders.

(e) Labor supply is inelastic, but age-dependent.

The assumptions (a) and (b) imply interest rate equality (see Section 5.3 in Chapter 5). That is, the interest rate in our SOE must equal the interest rate

[19] See Chapter 22.

in the world market for financial capital. This interest rate is exogenous to our SOE. We denote it r and assume r is positive and constant over time.

Apart from this, households, firms, and market structure are as in the Blanchard model for the closed economy with gradual retirement. We maintain the assumptions of perfect competition, no government sector, and no uncertainty except with respect to individual life lifetime.

Elements of the model

Firms choose effective capital-labor ratio $\tilde{k}(t)$ so that $f'(\tilde{k}(t)) = r + \delta$. The unique solution to this equation is denoted $\tilde{k}^*$. Thus,

$$f'(\tilde{k}^*) = r + \delta. \tag{12.58}$$

How does $\tilde{k}^*$ depend on r? To find out, we interpret (12.58) as implicitly defining $\tilde{k}^*$ as a function of r, $\tilde{k}^* = \varphi(r)$. Taking the total derivative w.r.t. r on both sides of (12.58), then gives $f''(\tilde{k}^*)\varphi'(r) = 1$, from which follows

$$\frac{d\tilde{k}^*}{dr} = \varphi'(r) = \frac{1}{f''(\tilde{k}^*)} < 0. \tag{12.59}$$

With continuous clearing in the labor market, employment equals labor supply, which, as in (12.39), is

$$L(t) = \frac{b}{\lambda + b} N(t), \qquad \text{for all } t,$$

where $N(t)$ is population, $b \equiv n + m$ is the birth rate, and $\lambda \geq 0$ is the retirement rate. We have $\tilde{k}(t) = \tilde{k}^*$, so that

$$K(t) = \tilde{k}^* T(t) L(t) = \tilde{k}^* T(t) \frac{b}{\lambda + b} N(t), \tag{12.60}$$

for all $t \geq 0$. This gives the endogenous stock of physical capital in the SOE at any point in time. If r shifts to a higher level, $\tilde{k}^*$ shifts to a lower level and the capital stock immediately adjusts, as shown by (12.59) and (12.60), respectively. This instantaneous adjustment is a counter-factual prediction of the model; it is a signal that the model ought to be modified so that adjustment of the capital stock takes place more gradually. We come back to this in Chapter 14 in connection with the theory of convex capital adjustment costs. For the time being we ignore adjustment costs and proceed as if (12.60) holds for all $t \geq 0$. This simplification would make short-run results very inaccurate, but is less problematic in long-run analysis.

In equilibrium firms' profit maximization implies the real wage

$$w(t) = \frac{\partial Y(t)}{\partial L(t)} = \left[f(\tilde{k}^*) - \tilde{k}^* f'(\tilde{k}^*) \right] T(t) \equiv \tilde{w}(\tilde{k}^*)T(t) = \tilde{w}^* T(t), \qquad (12.61)$$

where $\tilde{w}^*$ is the real wage per unit of effective labor. It is constant as long as r and δ are constant. So the real wage, w, per unit of "natural" labor grows over time at the same rate as technology, the rate $g \geq 0$. Notice that $\tilde{w}^*$ depends negatively on r in that

$$\frac{d\tilde{w}^*}{dr} = \frac{d\tilde{w}(\tilde{k}^*)}{d\tilde{k}^*}\frac{d\tilde{k}^*}{dr} = -\tilde{k}^* f''(\tilde{k}^*)\frac{1}{f''(\tilde{k}^*)} = -\tilde{k}^* < 0, \qquad (12.62)$$

where we have used (12.59).

From now we suppress the explicit dating of the variables when not needed for clarity. As usual A denotes aggregate private financial wealth. Since the government sector is ignored, A is the same as *national wealth* of the SOE. And since land is ignored, we have

$$A \equiv K - D,$$

where D denotes net foreign debt, that is, financial claims on the SOE from the rest of the world. Then $A_f \equiv -D$ is net holding of foreign assets. Net national income of the SOE is $rA + wL$ and aggregate net saving is $S^N = rA + wL - C$, where C is aggregate consumption. Hence,

$$\dot{A} = S^N = rA + wL - C. \qquad (12.63)$$

So far essentially everything is as it would be in a Ramsey (representative agent) model for a small open economy.[20] When we consider the change over time in *aggregate* consumption, however, an important difference emerges. In the Ramsey model the change in aggregate consumption is given simply as an aggregate Keynes-Ramsey rule. But the life-cycle feature arising from the finite horizons leads to something quite different in the Blanchard model. As we saw in Section 12.3 above,

$$\dot{C} = (r - \rho + \lambda + n)C - (\lambda + b)(\rho + m)A, \qquad (12.64)$$

where the last term is the generation replacement effect.

[20]The fact that labor supply, L, deviates from population, N, if the retirement rate, λ, is positive, is a minor difference compared with the Ramsey model. As long as λ and b are constant, L is still proportional to N.

The law of motion

All parameters are non-negative and in addition we will throughout, not unrealistically, assume that

$$r > g - m. \tag{A1}$$

This assumption ensures that the model has a solution even for $\lambda = 0$ (see (12.66) below). To obtain a dynamic system capable of being in a steady state, we introduce growth-corrected variables, $\tilde{a} \equiv A/(TN) \equiv a/T$ and $\tilde{c} \equiv C/(TN) \equiv c/T$. Log-differentiating $\tilde{a}$ w.r.t. t gives

$$\begin{aligned} \frac{\dot{\tilde{a}}}{\tilde{a}} &= \frac{\dot{A}}{A} - \frac{\dot{T}}{T} - \frac{\dot{N}}{N} = \frac{rA + wL - C}{A} - (g+n) \qquad \text{or} \\ \dot{\tilde{a}}(t) &= (r - g - n)\tilde{a}(t) + \tilde{w}^* \frac{b}{\lambda + b} - \tilde{c}(t), \end{aligned} \tag{12.65}$$

where $\tilde{w}^*$ is given in (12.61) and we have used $L(t)/N(t) = b/(\lambda + b)$ from (12.39). We might proceed by using (12.64) to get a differential equation for $\tilde{c}(t)$ in terms of $\tilde{a}(t)$ and $\tilde{c}(t)$ (analogous to what we did for the closed economy). The interest rate is now a constant, however, and then a more direct approach to the determination of $\tilde{c}(t)$ in (12.65) is convenient.

Consider the aggregate consumption function $C = (\rho + m)(A + H)$. Substituting (12.61) into (12.43) gives

$$H(t) = \frac{b}{\lambda + b} N(t)\tilde{w}^* T(t) \int_t^\infty e^{-(r+\lambda+m-g)(\tau - t)} d\tau = \frac{bN(t)\tilde{w}^* T(t)}{\lambda + b} \frac{1}{r + \lambda + m - g}, \tag{12.66}$$

where we have used that (A1) ensures $r + \lambda + m - g > 0$. It follows that

$$\frac{H(t)}{T(t)N(t)} = \frac{b\tilde{w}^*}{(\lambda + b)(r + \lambda + m - g)} \equiv \tilde{h}^*,$$

where $\tilde{h}^* > 0$ by (A1). Growth-corrected consumption can now be written

$$\tilde{c}(t) = (\rho + m)\left(\frac{A(t)}{T(t)N(t)} + \frac{H(t)}{T(t)N(t)}\right) = (\rho + m)(\tilde{a}(t) + \tilde{h}^*). \tag{12.67}$$

Substituting for $\tilde{c}$ into (12.65), inserting $b \equiv n + m$, and ordering gives the law of motion of the economy:

$$\dot{\tilde{a}}(t) = (r - \rho - g - b)\tilde{a}(t) + \frac{(r + \lambda - \rho - g)b\tilde{w}^*}{(r + \lambda + m - g)(\lambda + b)}. \tag{12.68}$$

The dynamics are thus reduced to *one* differential equation in growth-corrected national wealth; moreover, the equation is linear and even has constant coefficients. If we want it, we can therefore find an explicit solution. Given $\tilde{a}(0) = \tilde{a}_0$ and $r \neq \rho + g + b$, the solution is

$$\tilde{a}(t) = (\tilde{a}_0 - \tilde{a}^*)e^{-(\rho+g+b-r)t} + \tilde{a}^*, \tag{12.69}$$

where

$$\tilde{a}^* = \frac{(r + \lambda - \rho - g)b\tilde{w}^*}{(r + \lambda + m - g)(\lambda + b)(\rho + g + b - r)}, \tag{12.70}$$

which is the growth-corrected national wealth in steady state. Substitution of (12.69) into (12.67) gives the corresponding time path for growth-corrected consumption, $\tilde{c}(t)$. In steady state growth-corrected consumption is

$$\tilde{c}^* = \frac{(\rho + m)b\tilde{w}^*}{(r + \lambda + m - g)(\rho + g + b - r)}. \tag{12.71}$$

It can be shown that along the paths generated by (12.65), the transversality conditions of the households are satisfied (see Appendix D).

Let us first consider the case of stability. That is, while (A1) is maintained, we assume

$$r < \rho + g + b. \tag{12.72}$$

The phase diagram in $(\tilde{a}, \dot{\tilde{a}})$ space for this case is shown in the upper panel of Fig. 12.7. The lower panel of the figure shows the path followed by the economy in $(\tilde{a}, \tilde{c})$ space, for a given initial $\tilde{a}$ above $\tilde{a}^*$. The equation for the $\dot{\tilde{a}} = 0$ line is

$$\tilde{c} = (r - g - n)\tilde{a} + \tilde{w}^* \frac{b}{\lambda + b},$$

from (12.65). Different scenarios are possible. (Note that all conclusions to follow, and in fact also the above steady-state values, can be derived without reference to the explicit solution (12.69).)

The case of medium impatience In Fig. 12.7, as drawn, it is presupposed that $\tilde{a}^* > 0$, which, given (12.72), requires $r - (g + b) < \rho < r + \lambda - g$. We call this the case of medium impatience. Note that the economy is always at some point on the line $\tilde{c} = (\rho + m)(\tilde{a} + \tilde{h}^*)$, in view of (12.67). If we, as for the closed economy, had based the analysis on *two* differential equations in $\tilde{a}$ and $\tilde{c}$, respectively, then a saddle path would emerge and this would coincide with the $\tilde{c} = (\rho + m)(\tilde{a} + \tilde{h}^*)$ line in Fig. 12.7.[21]

[21] Although the $\dot{\tilde{a}} = 0$ line is drawn with a positive slope, it could alternatively have a negative slope (corresponding to $r < g+n$); stability still holds. Similarly, although growth-corrected per

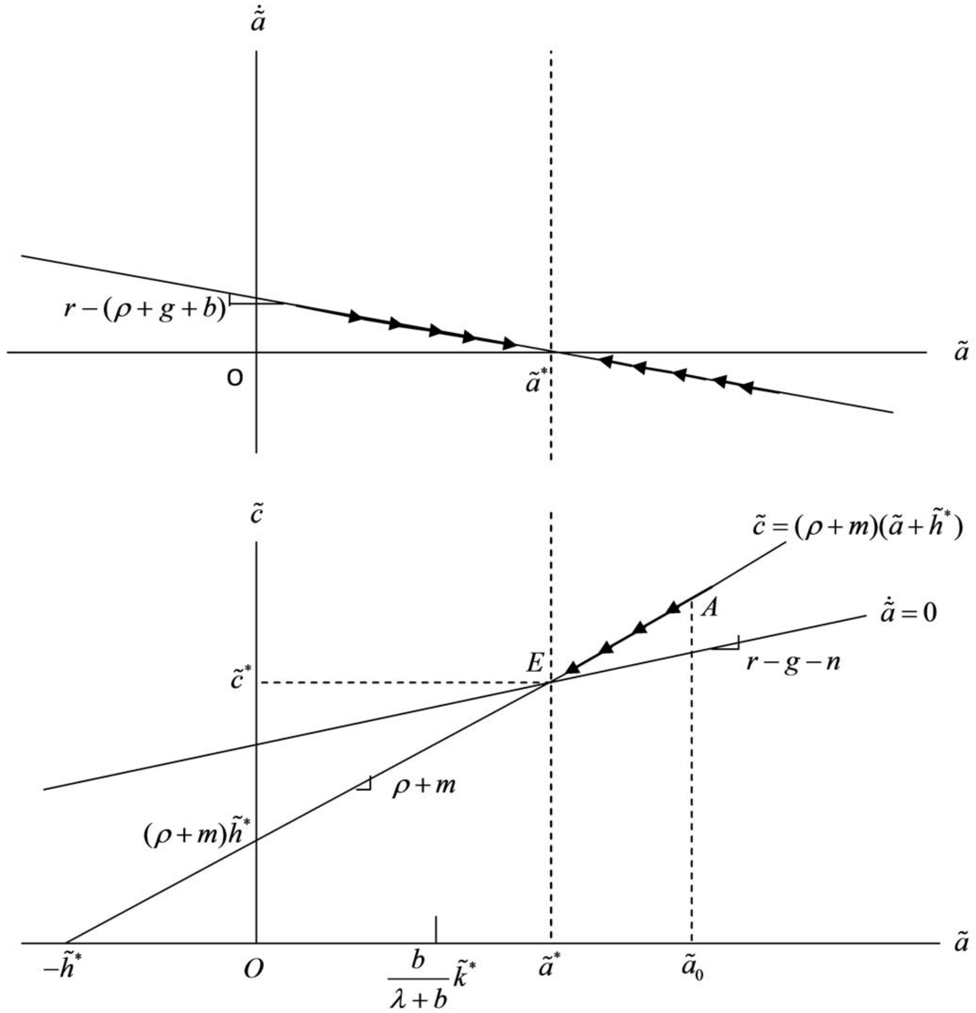

Figure 12.7: Adjustment process for an SOE with *medium impatience*, i.e., $r-(g+b) < \rho < r+\lambda-g$.

The case of high impatience Not surprisingly, a^* in (12.70) is a decreasing function of the impatience parameter ρ. A SOE with $\rho > r + \lambda - g$ (high impatience) has $\tilde{a}^* < 0$. That is, the country ends up with negative national wealth, a scenario which from pure economic logic is definitely possible, if there is a perfect international credit market. One should remember that "national wealth", in its usual definition, also used here, includes only financial wealth. Theoretically it can be negative if at the same time the economy has enough human wealth, H, to make *total* wealth, $A + H$ positive. Since $\tilde{c}^* > 0$, a steady state *must* have $\tilde{h}^* > -\tilde{a}^*$, in view of (12.67).

Negative national wealth of the SOE will reflect that all the physical capital of the SOE and part of its human wealth is in a sense mortgaged. Such an outcome, however, is not likely to be observed in practice. This is so for at least two reasons. First, whereas the analysis assumes a perfect international credit market, in the real world there is limited scope for writing enforceable international financial contracts. Lenders' risk perceptions depend on the level of debt and even within one's own country, access to loans with human wealth as a collateral is limited. Second, long before all physical capital of the impatient SOE is mortgaged or has become directly taken over by foreigners, the government presumably would intervene for *political* reasons.

The case of low impatience Alternatively, if (A1) is strengthened to $r > g+b$, we can have $0 \leq \rho \leq r-(g+b)$. This is the case of low impatience or high patience. Then the stability condition (12.72) is no longer satisfied.

Consider first the generic subcase $0 \leq \rho < r-(g+b)$. In this case the solution formula (12.69) is still valid. The slope of the adjustment path in the upper panel of Fig. 12.7 will now be positive and the $\tilde{c}$ line in the lower panel will be less steep than the $\dot{\tilde{a}} = 0$ line. There is no economic steady state any longer since the $\tilde{c}$ line will no longer cross the $\dot{\tilde{a}} = 0$ line for any positive level of consumption. There is a "fictional" steady-state value, $\tilde{a}^*$, which is negative and unstable. It is only "hypothetical" because it is associated with *negative* consumption, cf. (12.71). With $\tilde{a}_0 > -\tilde{h}^*$, the excess of r over $\rho+g+b$ results in high sustained saving so as to keep $\tilde{a}$ growing *forever*.[22] This means that national wealth, A, grows permanently at a rate higher than $g + n$. The economy grows *large* in the long run. But then, sooner or later, the world market interest rate can no longer be independent of what happens in this economy. The capital deepening resulting from the fast-growing country's capital accumulation will eventually affect the world economy

capita capital, $K/(TN)$ $(L/N)K/(TL) = b\tilde{k}^*/(\lambda + b)$, is in Fig. 12.7 smaller than $\tilde{a}^*$, it could just as well be larger. Both possibilities are consistent with the case of medium impatience.

[22] The reader is invited to draw the phase diagram in $(\tilde{a}, \tilde{c})$ space for this case, cf. Exercise 12.??.

and reduce the gap between r and ρ, so that the incentive to accumulate receives a check $-$ like in a closed economy. Thus, the SOE assumption ceases to fit. Think of China's development since the early 1970s.

The alternative subcase is the knife-edge case $\rho = r - (g + b)$. In this case the solution formula (12.69) is no longer valid. Instead we get

$$\tilde{a}(t) = \tilde{a}(0) + \frac{(r + \lambda - \rho - g)b\tilde{w}^*}{(r + \lambda + m - g)(\lambda + b)}t \to \infty \quad \text{for} \quad t \to \infty.$$

Foreign assets and debt

Returning to the stability case, where (12.72) holds, let us be more explicit about the evolution of net foreign debt. Or rather, in order to visualize by help of Fig. 12.7, we will consider net foreign assets, $A_f \equiv A - K = -D$. How are growth-corrected net foreign assets determined in the long run? We have

$$\tilde{a}_f \equiv \frac{A_f}{TN} \equiv \frac{A - K}{TN} = \tilde{a} - \frac{L}{N}\frac{K}{TL} = \tilde{a} - \frac{b}{\lambda + b}\tilde{k}^* \text{ (by (12.39))}.$$

Thus, by stability of $\tilde{a}$, for $t \to \infty$

$$\tilde{a}_f \to \tilde{a}_f^* = \tilde{a}^* - \frac{b}{\lambda + b}\tilde{k}^*.$$

The country depicted in Fig. 12.7 happens to have $\tilde{a}_0 > \tilde{a}^* > b\tilde{k}^*/(\lambda + b)$. So growth-corrected net foreign assets decline during the adjustment process. Yet, net foreign assets remain positive also in the long run. The interpretation of the positive $\tilde{a}_f$ is that only a part of national wealth is placed in physical capital in the home country, namely up to the point where the net marginal productivity of capital equals the world market rate of return r.[23] The remaining part of national wealth would result in a rate of return below r if invested within the country and is therefore better placed in the world market for financial capital.

Implicit in the described evolution over time of net foreign assets is a unique evolution of the current account surplus. By definition, the current account surplus, CAS, equals the increase per time unit in net foreign assets, i.e.,

$$CAS \equiv \dot{A}_f = \dot{A} - \dot{K}.$$

[23] The term foreign debt, as used here, need not have the contractual form of debt, but can just as well be equity. Although it may be easiest to imagine that capital in the different countries is always owned by the country's own residents, we do not presuppose this. And as long as we ignore uncertainty, the ownership pattern is in fact irrelevant from an economic point of view.

This says that CAS can also be viewed as the difference between net saving and net investment. Taking the time derivative of $\tilde{a}_f$ gives

$$\dot{\tilde{a}}_f = \frac{TN\dot{A}_f - A_f(T\dot{N} + N\dot{T})}{(TN)^2} = \frac{CAS}{TN} - (g+n)\tilde{a}_f.$$

Consequently, the movement of the growth-corrected current account surplus is given by

$$\begin{aligned} \frac{CAS}{TN} &= \dot{\tilde{a}}_f + (g+n)\tilde{a}_f = \dot{\tilde{a}} + (g+n)\tilde{a} - \frac{(g+n)b}{\lambda+b}\tilde{k}^* \\ &= (r-\rho-m)\tilde{a} + \frac{(r+\lambda-\rho-g)b\tilde{w}^*}{(r+\lambda+m-g)(\lambda+b)} - \frac{(g+n)b}{\lambda+b}\tilde{k}^*, \end{aligned}$$

where the second equality follows from the definition of $\tilde{a}_f$ and the third from (12.68). Yet, in our perfect-markets-equilibrium framework there is no bankruptcy-risk and no borrowing difficulties and so the current account is not of particular interest.

Returning to Fig. 12.7, consider a case where the rate of impatience, ρ, is somewhat higher than in the figure, but still satisfying the inequality $\rho < r+\lambda-g$. Then $\tilde{a}^*$, although smaller than before, is still positive. Since $\tilde{k}^*$ is not affected by a rise in ρ, it is $\tilde{a}_f^*$ that adjusts and might now end up negative, tantamount to net foreign debt, $\tilde{d}^* \equiv -\tilde{a}_f^*$, being positive.

Let us take the US economy as an example. Even if it is not really a small economy, the US economy may be small enough compared to the world economy for the SOE model to have something to say.[24] In the middle of the 1980s the US changed its international asset position from being a net creditor to being a net debtor. Since then, the US net foreign debt as a percentage of GDP has been rising, reaching 22 % in 2004.[25] With an output-capital ratio around 50 %, this amounts to a debt-capital ratio $D/K = (D/Y)Y/K = 11$ %.

A different movement has taken place in the Danish economy (which of course fits the notion of an SOE better). After World War II and until recently, Denmark had positive net foreign debt. In the aftermath of the second oil price shock in 1979-80, the debt rose to a maximum of 42 % of GDP in 1983. After 1991 the debt has been declining, reaching 11 % of GDP in 2004 (a development supported by the oil and natural gas extracted from the North Sea). Since 2011 Denmark has had positive net foreign assets.[26]

[24] The share of the US in world GDP was 29 % in 2003, but if calculated in purchasing power-corrected exchange rates only 21 % (World Economic Outlook, April 2004, IMF). The fast economic growth of, in particular, China and India since the early 1980s has produced a downward trend for the US share.

[25] Source: US Department of Commerce.

[26] Source: Statistics Denmark.

The adjustment speed

By *speed of adjustment* of a variable which converges in a monotonic way is meant the proportionate rate of decline per time unit of its distance to its steady-state value. Defining $\psi \equiv \rho + g + b - r$, from (12.69) we find, for $\tilde{a}(t) \neq \tilde{a}^*$,

$$-\frac{d\left|\tilde{a}(t)-\tilde{a}^*\right|/dt}{\left|\tilde{a}(t)-\tilde{a}^*\right|} = -\frac{d(\tilde{a}(t)-\tilde{a}^*)/dt}{\tilde{a}(t)-\tilde{a}^*} = -\frac{(\tilde{a}_0-\tilde{a}^*)e^{-\psi t}(-\psi)}{\tilde{a}(t)-\tilde{a}^*} = \psi.$$

Thus, ψ measures the speed of adjustment of growth-corrected national wealth. We get an estimate of ψ in the following way. With one year as the time unit, let $r = 0.04$ and let the other parameters take values equal to those given in the numerical example in Section 12.3. Then $\psi = 0.023$, telling us that 2.3 percent of the gap, $\tilde{a}(t) - \tilde{a}^*$, is eliminated per year.

We may also calculate the *half-life.* By half-life is meant the time it takes for half the initial gap to be eliminated. Thus, we seek the number τ such that

$$\frac{\tilde{a}(\tau)-\tilde{a}^*}{\tilde{a}_0-\tilde{a}^*} = \frac{1}{2}.$$

From (12.69) follows that $(\tilde{a}(\tau) - \tilde{a}^*)/(\tilde{a}_0 - \tilde{a}^*) = e^{-\psi\tau}$. Hence, $e^{-\psi\tau} = 1/2$, implying that half-life is

$$\tau = \frac{\ln 2}{\psi} \approx \frac{0.69}{0.023} \approx 30 \text{ years.}$$

The conclusion is that adjustment processes involving accumulation of national wealth are slow.

12.6 Concluding remarks

One of the strengths of the Blanchard OLG model compared with the Ramsey model comes to the fore in the analysis of a small open economy. The Ramsey model is a representative agent model so that the Keynes-Ramsey rule holds at both the individual and aggregate level. When applied to a small open economy with exogenous r, the Ramsey model therefore needs the *knife-edge condition* $\rho + \theta g = r$ (where θ is the absolute value of the elasticity of marginal utility of consumption).[27] Indeed, if $\rho + \theta g > r$, the $\tilde{a}$ in a Ramsey economy approaches

[27] A *knife-edge condition* is a condition imposed on a parameter value such that the set of values satisfying this condition has an empty interior in the space of all possible values. For the SOE all four terms entering the Ramsey condition $\rho + \theta g = r$ are parameters. Assuming the condition is satisfied thus amounts to imposing a knife-edge condition, which is unlikely to hold in the real world and which may lead to non-robust results.

a negative number (namely minus the growth-corrected human wealth) and $\tilde{c}$ tends to zero in the long run − an implausible scenario.[28] And if $\rho + \theta g < r$, the economy will tend to grow large relative to the world economy and so, eventually, the SOE framework is no longer appropriate for this economy. It is this lack of robustness which motivates the term "knife-edge" condition. If the parameter values are in a hair's breadth distance from satisfying the condition, qualitatively different behavior of the dynamic system results.

In contrast to the Ramsey model, the Blanchard OLG model deals with life-cycle behavior. Within a "fat" set of parameter values, namely those satisfying the inequalities (A1) and (12.72), it gives robust results for a small open economy.

A further strength of Blanchard's model is that it allows studying effects of alternative age *compositions* of a population. Compared with Diamond's OLG model, the Blanchard model has a less coarse demographic structure and a more refined notion of time. And by taking uncertainty about life-span into account, the model opens up for incorporating markets for life annuities (and similar forms of private pension arrangements). In this way important aspects of reality are included. On the other hand, from an empirical point of view it is a weakness that the propensity to consume out of wealth in the model is the same for a young and an old. In this respect the model lacks a weighty life-cycle feature. This limitation, of cause, comes from the unrealistic premise that the mortality rate is the same for all age groups. Another limitation is that individual asset ownership in the model depends only on age through own accumulated saving. In reality, there is considerable intra-generation differences in asset ownership due to differences in inheritance (Kotlikoff and Summers, 1981; Charles and Hurst, 2003; Danish Economic Council, 2004). Some extensions of the Blanchard OLG model are mentioned in Literature notes.

12.7 Literature notes

Three-period OLG models are under special conditions analytically obedient, see for instance de la Croix and Michel (2002).

Naive econometric studies trying to estimate consumption Euler equations (the discrete time analogue to the Keynes-Ramsey rule) on the basis of aggregate data and a representative agent approach can be seriously misleading. About this, see Attanasio and Weber, RES 1993, 631-469, in particular p. 646.

That Blanchard's OLG model in continuous time becomes three-dimensional if $\theta \neq 1$, is shown in Blanchard (1985). In that article it is also shown that a higher

[28] For a Ramsey-type model with a finite number of infinitely-lived households with different time-preference rates, Uzawa (1968) showed that asymptotically the entire private wealth will be owned by the household with the lowest time-preference rate.

θ implies a higher real interest rate in the long run. That in the perpetual youth model in steady state, individual consumption is growing faster than productivity is due to the young having less financial wealth than the average citizen. We saw that in the Blanchard model with gradual retirement, as in Section 12.3, there is a countervailing effect due to the young having more *human* wealth than the average citizen. Wendner (2010) explores the possibility that the latter effect may dominate and studies how this interferes with the issue of over- or underconsumption created by a "keeping up with the Joneses" externality.

Blanchard (1985) also sketched a more refined life-cycle pattern of the age profile of earned income involving initially rising labor income and then declining labor income with age. This can be captured by assuming that labor supply (or productivity) is the difference between two negative exponentials, $\ell(t-v) = m_1 e^{-\omega_1(t-v)} - m_2 e^{-\omega_2(t-v)}$, where all parameters are positive and $\omega_2/\omega_1 > m_1/m_2 > 1$.

Blanchard's model has been extended in many different directions. Calvo and Obstfeld (1988), Boucekkine et al. (2002), and Heijdra and Romp (2007, 2009) incorporate age-specific mortality. Endogenous education and retirement are included in Boucekkine et al. (2002), Grafenhofer et al. (2005), Sheshinski (2009), and Heijdra and Romp (2009). Matsuyama (1987) includes convex capital adjustment costs. Reinhart (1999) uses the Blanchard framework in a study of endogenous productivity growth. Blanchard (1985), Calvo and Obstfeld (1988), Blanchard and Fischer (1989), and Klundert and Ploeg (1989) apply the framework for studies of fiscal policy and government debt. These last issues will be the topic of the next chapter.

12.8 Appendix

A. Actuarially fair life insurance

Negative life insurance A life annuity contract is defined as *actuarially fair* if it offers the investor the same expected unconditional rate of return as a safe bond. We now check whether the life annuity contracts in equilibrium of the Blanchard model have this property. For simplicity, we assume that the risk-free interest rate is a constant, r.

Buying a life annuity contract at time t means that the depositor invests one unit of account at time t in such a contract. In return the depositor receives a conditional continuous flow of receipts equal to $r+\hat{m}$ per time unit until death. At death the invested unit of account is lost from the point of view of the depositor (or rather the estate of the depositor). The time of death is stochastic, and so the unconditional rate of return, R, is a stochastic variable. Given the constant and age-independent mortality rate m, the expected unconditional return in the short time interval $[t, t+\Delta t)$ is approximately $(r+\hat{m})\Delta t(1-m\Delta t) - 1 \cdot m\Delta t$, where $m\Delta t$ is the approximate probability of dying within the time interval $[t, t+\Delta t)$ and $1-m\Delta t$ is the approximate probability of surviving. The expected unconditional *rate* of return, ER, is the expected return *per time unit* per unit of account invested. Thus,

$$ER \approx \frac{(r+\hat{m})\Delta t(1-m\Delta t) - m\Delta t}{\Delta t} = (r+\hat{m})(1-m\Delta t) - m. \qquad (12.73)$$

In the limit for $\Delta t \to 0$, we get $ER = r + \hat{m} - m$. In equilibrium, as shown in Section 12.2.1, $\hat{m} = m$ and so $ER = r$. This shows that the life annuity contracts in equilibrium are actuarially fair.

Positive life insurance To put negative life insurance in perspective, we also considered positive life insurance. We claimed that the charge of $\tilde{m}$ per time unit until death on a positive life insurance contract must in equilibrium equal the death rate, m. This can be shown in the following way. The contract stipulates that the depositor pays the insurance company a premium of $\tilde{m}$ units of account per time unit until death. In return, at death the estate of the deceased person receives one unit of account from the insurance company. The expected revenue obtained by the insurance company on such a contract in the short time interval $[t, t+\Delta t)$ is approximately $\tilde{m}\Delta t(1-m\Delta t) + 0 \cdot m\Delta t$. In the absence of administration costs the expected cost is approximately $0 \cdot (1-m\Delta t) + 1 \cdot m\Delta t$. We find the expected profit *per time unit* to be

$$E\pi \approx \frac{\tilde{m}\Delta t(1-m\Delta t) - m\Delta t}{\Delta t} = \tilde{m} - \tilde{m}m\Delta t - m.$$

In the limit for $\Delta t \to 0$ we get $E\pi = \tilde{m} - m$. Equilibrium with free entry and exit requires $E\pi = 0$, hence $\tilde{m} = m$, as was to be shown.

Like the negative life insurance contract, the positive life insurance contract is said to be *actuarially fair* if it offers the investor (now the insurance company) the same expected unconditional rate of return as a safe bond. In equilibrium it does so. We see this by replacing $\hat{m}$ by $\tilde{m}$ and applying the argument leading to (12.73) once more, this time from the point of view of the insurance company. At time t the insurance company makes a demand deposit of one unit of account in the financial market (or buys a short-term bond) and at the same time contracts to pay one unit of account to a customer at death in return for a flow of contributions, $\tilde{m}$, per time unit from the customer until death. The payout of one unit of account to the estate of the deceased person is financed by cashing the demand deposit (or stopping reinvesting in short-term bonds). Since in equilibrium $\tilde{m} = m$, the conclusion is that $ER = r$.

Age-dependent mortality rates Let X denote the age at death of an individual. Ex ante X is a stochastic variable. Then the *instantaneous mortality rate* for a person of age x, also called the *hazard rate* of death at age x, is defined as

$$m(x) = \lim_{\Delta x \to 0} \frac{P(x < X \leq x + \Delta x)/\Delta x}{P(X > x)} = \frac{f(x)}{1 - F(x)}, \tag{12.74}$$

where $1 - F(x)$ is the survival function, that is, the unconditional probability of becoming at least x years old. The associated cumulative distribution function is $F(x)$ (the probability of dying before age x). And $f(x) = F'(x)$ is the probability density function. Empirically the *instantaneous mortality rate* is an increasing function of age, x. The classical Gompertz-Makeham formula specifies $m(x)$ as $m(x) = \mu_0 + \mu_1 e^{\mu_2 x}$, where $\mu_0 > 0, \mu_1 > 0$, and $\mu_2 > 0$.

The Blanchard assumption of an age independent mortality rate is the case $\mu_1 = 0$ so that $m(x) = \mu_0 \equiv m$. Indeed, the Blanchard model assumes $1 - F(x) = e^{-mx}$ so that, by (12.74) with $f(x) = me^{-mx}$, we have $m(x) = me^{-mx}/e^{-mx} = m$, a constant.

B. Present value of expected future labor income

Here we show, for the case without retirement ($\lambda = 0$), that the present value, $h(v, t)$, of an individual's expected future labor income can be written as in (12.12) (or as in (12.10) if $t = 0$). For the case with no retirement we have

$$h(v, t) \equiv E_t \int_t^{t+X} w(s) e^{-\int_t^s r(\tau) d\tau} ds, \tag{12.75}$$

where X stands for remaining lifetime, a stochastic variable. The rate of discount for future labor income *conditional on being alive at the moment concerned* is the risk-free interest rate r.

Now, consider labor income of the individual at time $s > t$ as a stochastic variable, $Z(s)$, with two different possible outcomes:

$$Z(s) = \begin{cases} w(s), & \text{if still alive at time } s \\ 0, & \text{if dead at time } s. \end{cases}$$

Then we can rewrite (12.75) as

$$\begin{aligned} h(v, t) &= E_t \int_t^{\infty} Z(s) e^{-\int_t^s r(\tau) d\tau} ds = \int_t^{\infty} E_t(Z(s)) e^{-\int_t^s r(\tau) d\tau} ds \\ &= \int_t^{\infty} w(s) P(X > s - t) + 0 \cdot P(X \leq s - t) e^{-\int_t^s r(\tau) d\tau} ds \\ &= \int_t^{\infty} w(s) e^{-m(s-t)} e^{-\int_t^s r(\tau) d\tau} ds = \int_t^{\infty} w(s) e^{-\int_t^s (r(\tau) + m) d\tau} ds \end{aligned} \tag{12.76}$$

This confirms (12.12). When we discount the *potential* labor income in all future, the relevant discount rate is the actuarial rate of interest, i.e., the risk-free interest rate *plus* the death rate.

C. Aggregate dynamics

C.1. Aggregate dynamics in the perpetual youth model (Section 12.2)

In Section 12.2.2 we gave an intuitive explanation of why aggregate financial wealth and aggregate consumption follow the rules

$$\dot{A}(t) = r(t)A(t) + w(t)L(t) - C(t), \tag{*}$$

and

$$\dot{C}(t) = [r(t) - \rho + n]C(t) - b(\rho + m)A(t), \tag{**}$$

respectively. Here we will prove these equations, appealing to *Leibniz's formula* for differentiating an integral with respect to a parameter appearing both in the integrand and in the limits of integration.

Leibniz's formula Let $f(v,t)$ and $f_t(v,t)$ be continuous. Suppose too that $g(t)$ and $h(t)$ are differentiable. Then

$$\begin{aligned} F(t) &= \int_{g(t)}^{h(t)} f(v,t)dv \Rightarrow \\ F'(t) &= f(t,h(t))h'(t) - f(t,g(t))g'(t) + \int_{g(t)}^{h(t)} \frac{\partial f(v,t)}{\partial t} dv. \end{aligned}$$

For proof, see, e.g., Sydsæter and Hammond (2008). In case $g(t) = -\infty$, one should replace $g'(t)$ by 0. Similarly, if $h(t) = +\infty$, $h'(t)$ should be replaced by 0.

Proof of (*) The intuitive validity of the accounting rule (*) notwithstanding, we cannot be sure that our concepts and book-keeping are consistent, until we have provided a proof.

Aggregate financial wealth is

$$A(t) = \int_{-\infty}^{t} a(v,t)N(0)e^{nv}be^{-m(t-v)}dv, \tag{12.77}$$

where we have inserted (12.2) into (12.14). Using Leibniz's formula with $g'(t) = 0$ and $h'(t) = 1$, we get

$$\begin{aligned}\dot{A}(t) &= a(t,t)N(0)e^{nt}b - 0 + \int_{-\infty}^{t} N(0)b\frac{\partial}{\partial t}[a(v,t)e^{nv}e^{-m(t-v)}]dv \\ &= 0 - 0 + N(0)b\int_{-\infty}^{t} e^{nv}[a(v,t)e^{-m(t-v)}(-m) + \frac{\partial a(v,t)}{\partial t}e^{-m(t-v)}]dv \\ &= -mN(0)b\int_{-\infty}^{t} a(v,t)e^{nv}e^{-m(t-v)}dv + N(0)b\int_{-\infty}^{t}\frac{\partial a(v,t)}{\partial t}e^{nv}e^{-m(t-v)}dv,\end{aligned}$$

where the second equality comes from $a(t,t) = 0$, which is due to the absence of bequests. Inserting (12.77), this can be written

$$\dot{A}(t) = -mA(t) + N(0)b\int_{-\infty}^{t}\frac{\partial a(v,t)}{\partial t}e^{nv}e^{-m(t-v)}dv. \tag{12.78}$$

Thus, the increase per time unit in aggregate financial wealth equals the "intake" (i.e., the increase in financial wealth of those still alive) minus the "discharge" due to death, $mN(t)A(t)/N(t) = mA(t)$.

In (12.78) the term $\partial a(v,t)/\partial t$ stands for the increase per time unit in financial wealth of an individual born at time v and still alive at time t. By definition this is the same as the saving of the individual, hence the same as income minus consumption. Thus, $\partial a(v,t)/\partial t = (r(t)+m)a(v,t) + w(t) - c(v,t)$. Substituting this into (12.78) gives

$$\begin{aligned}\dot{A}(t) &= -mA(t) + N(0)b\left[(r(t)+m)\int_{-\infty}^{t} a(v,t)e^{nv}e^{-m(t-v)}dv\right. \\ &\quad \left. +w(t)\int_{-\infty}^{t} e^{nv}e^{-m(t-v)}dv - \int_{-\infty}^{t} c(v,t)e^{nv}e^{-m(t-v)}dv\right] \\ &= -mA(t) + (r(t)+m)A(t) + w(t)N(t) - C(t), \end{aligned} \tag{12.79}$$

by (12.77), (12.4), and (12.13). Reducing the last expression in (12.79) and noting that $N(t) = L(t)$ gives (*). □

We can prove (**) in a similar way:

Proof of (**) The PV, as seen from time t, of future labor income of any individual is as in (12.76), since labor income is independent of age in this simple version of the Blanchard model. Hence, aggregate human wealth is

$$H(t) = N(t)\bar{h}(t) = N(0)e^{nt}\int_{t}^{\infty} w(s)\, e^{-\int_t^s (r(\tau)+m)d\tau}ds. \tag{12.80}$$

After substituting (12.3) into (12.13), differentiation w.r.t. t (use again Leibniz's formula with $g'(t) = 0$ and $h'(t) = 1$) gives

$$\begin{aligned}
\dot{C}(t) &= c(t,t)N(0)e^{nt}b - 0 + \int_{-\infty}^{t} N(0)b\frac{\partial}{\partial t}[c(v,t)e^{nv}e^{-m(t-v)}]dv \\
&= c(t,t)N(t)b - 0 + N(0)b\int_{-\infty}^{t} e^{nv}[-c(v,t)e^{-m(t-v)}m + \frac{\partial c(v,t)}{\partial t}e^{-m(t-v)}]dv \\
&= (\rho+m)\bar{h}(t)N(t)b - mN(0)b\int_{-\infty}^{t} c(v,t)e^{nv}e^{-m(t-v)}dv \\
&\quad +N(0)b\int_{-\infty}^{t} \frac{\partial c(v,t)}{\partial t}e^{nv}e^{-m(t-v)}dv, \qquad (12.81)
\end{aligned}$$

where the last equality derives from the fact that the consumption function for an individual born at time v is $c(v,t) = (\rho + m)\left[a(v,t) + \bar{h}(t)\right]$, which for $v = t$ takes the form $c(t,t) = (\rho + m)\bar{h}(t)$, since $a(t,t) = 0$. Using (12.80) and (12.13) in (12.81) yields

$$\dot{C}(t) = b(\rho+m)H(t) - mC(t) + N(0)b\int_{-\infty}^{t} \frac{\partial c(v,t)}{\partial t}e^{nv}e^{-m(t-v)}dv. \qquad (12.82)$$

From the Keynes-Ramsey rule we have $\partial c(v,t)/\partial t = (r(t) - \rho)c(v,t)$. Substituting this into (12.82) gives

$$\begin{aligned}
\dot{C}(t) &= b(\rho+m)H(t) - mC(t) + N(0)b(r(t)-\rho)\int_{-\infty}^{t} c(v,t)e^{nv}e^{-m(t-v)}dv \\
&= b(\rho+m)H(t) - mC(t) + (r(t)-\rho)C(t) \\
&= b(\rho+m)H(t) - bC(t) + nC(t) + (r(t)-\rho)C(t) \\
&= b(\rho+m)H(t) - b(\rho+m)(A(t)+H(t)) + (r(t)-\rho+n)C(t) \\
&= (r(t)-\rho+n)C(t) - b(\rho+m)A(t),
\end{aligned}$$

where the second equality comes from (12.13), the third from $n \equiv b - m$, and the fourth from the aggregate consumption function, $C(t) = (\rho + m)(A(t) + H(t))$. Hereby we have derived (**). □

A more direct proof of (**) An alternative and more direct way of proving (**) may be of interest also in other contexts. The aggregate consumption function immediately gives

$$\dot{C}(t) = (\rho+m)(\dot{A}(t) + \dot{H}(t)). \qquad (12.83)$$

Differentiation of (12.80) w.r.t. t (using Leibniz's Formula with $g'(t) = 1$ and $h'(t) = 0$) gives

$$\begin{aligned}
\dot{H}(t) &= \dot{N}(t)\bar{h}(t) + N(t)\left[-w(t) + \int_t^\infty w\left(s\right) e^{-\int_t^s (r(\tau)+m)d\tau}(r(t)+m)ds\right] \\
&= nH(t) - w(t)N(t) + (r(t)+m)N(t)\int_t^\infty w\left(s\right) e^{-\int_t^s (r(\tau)+m)d\tau}ds \\
&= (r(t)+m+n)H(t) - w(t)N(t),
\end{aligned}$$

where the two last equalities follow from (12.80) and (12.76), respectively. Inserting this together with (*) into (12.83) gives

$$\begin{aligned}
\dot{C}(t) &= (\rho+m)\left[r(t)A(t) + w(t)N(t) - C(t) + (r(t)+m+n)H(t) - w(t)N(t)\right] \\
&= (\rho+m)\left[r(t)A(t) - C(t) + (r(t)+m+n)(\frac{C(t)}{\rho+m} - A(t))\right] \\
&= (\rho+m)r(t)A(t) - (\rho+m)C(t) + (r(t)+m+n)C(t) \\
&\quad -(\rho+m)(r(t)+b)A(t) \\
&= (r(t) - \rho + n)C(t) - b(\rho+m)A(t),
\end{aligned}$$

where the second equality comes from the aggregate consumption function and the third from $b \equiv m + n$. □

C.2. Aggregate dynamics in the model with retirement (Section 12.3-4)

(no text currently available)

D. Transversality conditions and why the diverging paths cannot be equilibrium paths

In Section 12.2.4 we claimed that for every $t_0 \geq 0$ and every $v \leq t_0$, the transversality condition (12.28) is satisfied at the steady-state point E in Fig. 12.4. The following lemma is a key step in proving this.

LEMMA D1. At the steady-state point E, for fixed v, $a(v,t)$ ultimately grows at the rate $r^* - \rho$ if lifetime allows.

Proof. We have

$$\begin{aligned}\frac{\partial a(v,t)/\partial t}{a(v,t)} &= \frac{(r(t)+m)a(v,t)+w(t)-c(v,t)}{a(v,t)} \\ &= r(t)+m+\frac{w(t)-(\rho+m)(a(v,t)+\bar{h}(t))}{a(v,t)} \\ &= r(t)-\rho+\frac{w(t)-(\rho+m)\bar{h}(t)}{a(v,t)} \\ &= r(t)-\rho+\frac{\tilde{w}(\tilde{k}(t))-(\rho+m)\tilde{h}(t)}{a(v,t)}T(t). \qquad (12.84)\end{aligned}$$

In a small neighborhood of the steady state, where $\tilde{k}(t) \approx \tilde{k}^*$ and $\tilde{h}(t) \equiv \bar{h}(t)/T(t) \approx \tilde{h}^*$, cf. (12.37), the right-hand side of (12.84) can be approximated by

$$\begin{aligned}& r^* - \rho + \frac{\left[\tilde{w}(\tilde{k}^*) - (\rho+m)\frac{\tilde{w}(\tilde{k}^*)}{r^*+m-g}\right]T(t)}{a(v,t)} \\ = \; & r^* - \rho + \frac{(r^*-\rho-g)\tilde{w}(\tilde{k}^*)T(t)}{(r^*+m-g)a(v,t)} > r^*-\rho > g, \qquad (12.85)\end{aligned}$$

where both inequalities come from (12.35). Thus, at least close to the steady state, $a(v,t)$ grows at a higher rate than technology. It follows that for an imaginary person with infinite lifetime, $T(t)/a(v,t) \to 0$ for $t \to \infty$, so that, by (12.84), $(\partial a(v,t)/\partial t)/a(v,t) \to r^* - \rho$ for $t \to \infty$, as was to be shown. □

At E the discount factor in (12.28) becomes $e^{-(r^*+m)(t-t_0)}$, where $r^* = f'(\tilde{k}^*) - \delta$. In view of $\rho \geq 0$ and $m > 0$, we have $r^* - \rho < r^* + m$. From this, in combination with Lemma D1, follows that all the transversality conditions, (12.28), hold at the steady-state point E and hence also along any path converging to that steady-state point.[29]

Note that, by (12.85), even the limiting value of $\partial a(v,t)/\partial t)/a(v,t)$, $r^* - \rho$, is higher than g. Thus, due to the generation replacement effect, individual financial wealth tends to grow *faster* than average wealth, $A(t)/N(t)$, which for $\lambda = 0$ equals K/L and in the long run grows at the rate g. This explains why transversality conditions can not be checked in the same simple way as in the Ramsey model.

In the text of Section 12.2.4 we also claimed that all the *diverging* paths in the phase diagram of Fig. 12.4 violate the individual transversality conditions (12.28). Let us first explain why in Fig. 12.4 paths which start from below the

[29] The argument can be extended to the case with retirement ($\lambda > 0$) and to the small open economy (even in the case with "low impatience").

stable arm tend to *cross* the $\tilde{k}$-axis. The slope of any path generated by the differential equations for $\tilde{c}$ and $\tilde{k}$, is

$$\frac{d\tilde{c}}{d\tilde{k}} = \frac{d\tilde{c}/dt}{d\tilde{k}/dt} = \frac{\dot{\tilde{c}}}{\dot{\tilde{k}}} = \frac{\left[f'(\tilde{k}) - \delta - \rho - g\right]\tilde{c} - b(\rho + m)\tilde{k}}{f(\tilde{k}) - \tilde{c} - (\delta + g + b - m)\tilde{k}}, \tag{12.86}$$

whenever $\dot{\tilde{k}} \neq 0$. Close to the $\tilde{k}$-axis, this slope is positive for $0 < \tilde{k} < \bar{\bar{\tilde{k}}}$ and negative for $\tilde{k} > \bar{\bar{\tilde{k}}}$ (to see this, put $\tilde{c} \approx 0$ in (12.86)). Even *at* the $\tilde{k}$-axis, this holds true. Such paths violate the individual transversality conditions. Indeed, along these paths consumption will in finite time be zero at the same time as both financial wealth and human wealth are far from zero. This indicates that people consume less than what their intertemporal budget constraint allows, which is equivalent to the transversality condition not being satisfied: people over-accumulate. Any individual expecting an evolution of w and r implied by such a path will *deviate* from the consumption level along the path by choosing higher consumption. Hence, the path can not be a perfect foresight equilibrium path.

What about paths starting from above the stable arm in Fig. 12.4? These paths will violate the NPG condition of the individuals (the argument is similar to that used for the Ramsey model in Appendix C of Chapter 10). An individual expecting an evolution of w and r implied by such a path will *deviate* from the consumption level along the path by choosing a *lower* consumption level in order to remain solvent, i.e., comply with the NPG condition.

E. Saddle-point stability in the perpetual youth model

First, we construct the Jacobian matrix of the right-hand sides of the differential equations (12.26) and (12.27), that is, the matrix

$$J(\tilde{k}, \tilde{c}) = \begin{bmatrix} \partial\dot{\tilde{k}}/\partial\tilde{k} & \partial\dot{\tilde{k}}/\partial\tilde{c} \\ \partial\dot{\tilde{c}}/\partial\tilde{k} & \partial\dot{\tilde{c}}/\partial\tilde{c} \end{bmatrix} = \begin{bmatrix} f'(\tilde{k}) - (\delta + g + b - m) & -1 \\ f''(\tilde{k})\tilde{c} - b(\rho + m) & f'(\tilde{k}) - \delta - \rho - g \end{bmatrix}$$

The determinant, evaluated at the steady state point $(\tilde{k}^*, \tilde{c}^*)$, is $\det(J(\tilde{k}^*, \tilde{c}^*))$

$$\begin{aligned} &= \left[f'(\tilde{k}^*) - (\delta + g + b - m)\right]\left[f'(\tilde{k}^*) - \delta - \rho - g\right] + \left[f''(\tilde{k}^*)\tilde{c}^* - b(\rho + m)\right] \\ &= \left(\frac{f'(\tilde{k}^*) - (\delta + g + b - m)}{\frac{f(\tilde{k}^*)}{\tilde{k}^*} - (\delta + g + b - m)} - 1\right) b(\rho + m) + f''(\tilde{k}^*)\tilde{c}^* < f''(\tilde{k}^*)\tilde{c}^* < 0, \end{aligned}$$

where the second equality follows from (12.32) whereas the last inequality follows from $f'' < 0$ and the second last from $f'(\tilde{k}^*) < f(\tilde{k}^*)/\tilde{k}^*$.[30] Since the determinant is negative, $J(\tilde{k}^*, \tilde{c}^*)$ has one positive and one negative eigenvalue. Hence the steady state is a saddle point.[31] The remaining needed conditions for (local) saddle-point stability include that the saddle path should not be parallel to the jump-variable axis. This condition holds here, since if the saddle path were parallel to the jump-variable axis, the element in the first row and last column of $J(\tilde{k}^*, \tilde{c}^*)$ would vanish, which it never does here. The remaining conditions for saddle-point stability were confirmed in the text.

An argument analogue to that for the Ramsey model, see Appendix A to Chapter 10, shows that the steady state is in fact *globally* saddle-point stable.

F. The upper bound for r^* in the Blanchard model with retirement

An algebraic proof of the right-hand inequality in (12.54) is given here. Because f satisfies the Inada conditions and $f'' < 0$, the equation

$$f'(\tilde{k}) - \delta = \rho + g + b$$

has a unique solution in $\tilde{k}$. Let this solution be denoted $\underline{\tilde{k}}$.

Since the inequality $r^* < \rho + g + b$ is equivalent with $\underline{\tilde{k}} < \tilde{k}^*$, it is enough to prove the latter inequality. Suppose that on the contrary we have $\underline{\tilde{k}} \geq \tilde{k}^*$. Then, $f'(\tilde{k}^*) - \delta \geq \rho + g + b$ and there exists an $\varepsilon \geq 0$ such that

$$f'(\tilde{k}^*) = \delta + \rho + g + b(1 + \varepsilon). \tag{12.87}$$

This equation is equivalent to

$$\rho + m = f'(\tilde{k}^*) - \delta - g - b(1 + \varepsilon) + m = f'(\tilde{k}^*) - \delta - g - n - b\varepsilon, \tag{12.88}$$

since $n \equiv b - m$. In steady state

$$\tilde{c}^* = \frac{b}{\lambda + b}\left[f(\tilde{k}^*) - (\delta + g + n)\tilde{k}^*\right] = \frac{b\,(\rho + m)\,\tilde{k}^*}{f'(\tilde{k}^*) - \delta - \rho - g + \lambda},$$

which implies

$$\frac{f(\tilde{k}^*)}{\tilde{k}^*} - (\delta + g + n) = \frac{(\lambda + b)\,(\rho + m)}{f'(\tilde{k}^*) - \delta - \rho - g + \lambda}.$$

[30] To convince yourself of this last inequality, draw a graph of the function $f(\tilde{k})$, reflecting the properties $f' > 0$, $f'' < 0$, and $f(0) \geq 0$.

[31] This also holds in the case with retirement, $\lambda > 0$.

Inserting (12.87) on the right-hand side gives

$$\begin{aligned}
\frac{f(\tilde{k}^*)}{\tilde{k}^*} - (\delta + g + n) &= \frac{(\lambda + b)\,(\rho + m)}{b(1+\varepsilon) + \lambda} \\
&= \frac{\lambda + b}{b(1+\varepsilon) + \lambda}(f'(\tilde{k}^*) - \delta - g - n - b\varepsilon) \quad \text{(from 12.88))} \\
&\leq f'(\tilde{k}^*) - \delta - g - n - b\varepsilon \qquad \text{(since } \varepsilon \geq 0\text{)}.
\end{aligned}$$

This inequality implies

$$\frac{f(\tilde{k}^*)}{\tilde{k}^*} \leq f'(\tilde{k}^*) - b\varepsilon.$$

But this last inequality is impossible because of strict concavity of f. Indeed, $f'' < 0$ together with $f(0) = 0$ implies $f(\tilde{k})/\tilde{k} > f'(\tilde{k})$ for all $\tilde{k} > 0$. Thus, from the assumption that $\underline{\tilde{k}} \geq \tilde{k}^*$ we arrive at a contradiction; hence, the assumption must be rejected. It follows that $\underline{\tilde{k}} < \tilde{k}^*$. □

12.9 Exercises