Chapter 30

The real business cycle theory

Since the middle of the 1970s two quite different approaches to the explanation of business cycle fluctuations have been pursued. We may broadly classify them as either of a new-classical or a Keynesian orientation. The new-classical school attempts to explain output and employment fluctuations as movements in productivity and labor supply. The Keynesian approach attempts to explain them as movements in aggregate demand and the degree of capacity utilization.

Within the new-classical school the monetary mis-perception theory of Lucas (1972, 1975) was the dominating approach in the 1970s. We described this approach in Chapter 27. The theory came under serious empirical attack in the late 1970s.[1] From the early 1980s an alternative approach within new-classical thinking, the Real Business Cycle theory, gradually took over. This theory (RBC theory for short) was initiated by Nobel laureates Finn E. Kydland and Edward C. Prescott (1982) and is the topic of this chapter.[2]

The shared conception of new-classical approaches to business cycle analysis is that economic fluctuations can be explained by adding stochastic disturbances to the neoclassical framework with optimizing agents, rational expectations, and market clearing under perfect competition. Output and employment are seen as supply determined, the only difference compared with the standard neoclassical growth model being that there are fluctuations around the growth trend. These fluctuations are not viewed as deviations from a Walrasian equilibrium, but as a constituent part of a moving stochastic Walrasian equilibrium. In Lucas' monetary mis-perception theory from the 1970s shocks to the money supply were the driving force. When the RBC theory took over, the emphasis shifted to recurrent technology shocks and other supply shocks as a driving force behind economic

[1] For a survey, see Blanchard (1990).

[2] In 2004 Kydland and Prescott were awarded the Nobel prize, primarily for their contributions in two areas: policy implications of time inconsistency and quantitative business cycle research.

fluctuations. In fact, money is typically absent from the RBC models. The empirical positive correlation between money supply and output is attributed to reverse causation. The fluctuations in employment reflect fluctuations in labor supply triggered by real wage movements reflecting shocks to marginal productivity of labor. Government intervention with the purpose of stabilization is seen as likely to be counterproductive. Given the uncertainty due to shocks, the market forces establish a Pareto-optimal moving equilibrium. "Economic fluctuations are optimal responses to uncertainty in the rate of technological change", as Edward Prescott puts it (Prescott 1986).

Below we present a prototype RBC model.

30.1 A simple RBC model

The RBC theory is an extension of the non-monetary Ramsey growth model, usually in discrete time. The key point is that endogenous labor supply and exogenous stochastic recurrent productivity shocks are added. The presentation here is close to King and Rebelo (1999), available in *Handbook of Macroeconomics*, vol. 1B, 1999. As a rule, our notation is the same as that of King and Rebelo, but there will be a few exceptions in order not to diverge too much from our general notational principles. The notation appears in Table 29.1. The most precarious differences vis-a-vis King and Rebelo are that we use ρ in our customary meaning as a utility discount rate and θ for elasticity of marginal utility of consumption.

The firm

There are two categories of economic agents in the model: firms and households; the government sector is ignored. First we describe the firm.

Technology

The representative firm has the production function

$$Y_t = A_t F(K_t, X_t N_t), \tag{30.1}$$

where K_t and N_t are input of capital and labor in period t, while X_t is an exogenous deterministic labor-augmenting technology level, and A_t represents an exogenous random productivity factor. The production function F has constant returns to scale and is neoclassical (i.e., marginal productivity of each factor is positive, but decreasing in the same factor).

Table 29.1. Notation

Variable	*King & Rebelo*	*Here*
Aggregate consumption	C_t	same
Deterministic technology level	X_t	same
Growth corrected consumption	$c_t \equiv C_t/X_t$	same
Growth corrected investment	$i_t \equiv I_t/X_t$	same
Growth corrected output	$y_t \equiv Y_t/X_t$	same
Growth corrected capital	$k_t \equiv K_t/X_t$	same
Aggregate employment (hours)	N_t	same
Aggregate leisure (hours)	$L_t \equiv 1 - N_t$	same
Effective capital intensity	$\frac{k_t}{N_t}$	$\tilde{k}_t \equiv \frac{K_t}{X_t N_t}$
Real wage	$w_t X_t$	w_t
Technology-corrected real wage	w_t	$\tilde{w}_t \equiv w_t/X_t$
Real interest rate from end period t to end period $t+1$	r_t	r_{t+1}
Auto-correlation coefficient in technology process	ρ	ξ
Discount factor w.r.t. utility	b	$\frac{1}{1+\rho}$
Rate of time preference w.r.t. utility	$\frac{1}{b} - 1$	ρ
Elasticity of marginal utility of cons.	σ	θ
Elasticity of marginal utility of leisure	η	same
Elasticity of output w.r.t. labor	α	same
Steady state value of c_t	c	c^*
The natural logarithm	log	same
Log deviation of c_t from steady state value	$\hat{c}_t \equiv \log \frac{c_t}{c}$	$\hat{c}_t \equiv \log \frac{c_t}{c^*}$
Log deviation of N_t from steady state value	$\hat{L}_t \equiv \log \frac{L_t}{L}$	$\hat{N}_t \equiv \log \frac{N_t}{N^*}$

It is assumed that X_t grows deterministically at a constant rate, $\gamma - 1$, i.e.,

$$X_{t+1} = \gamma X_t, \qquad \gamma > 1; \tag{30.2}$$

so γ is a deterministic technology growth factor. The productivity variable A_t is stochastic and assumed to follow the process

$$A_t = A^{*1-\xi}(A_{t-1})^{\xi} e^{\varepsilon_t}.$$

This means that $\log A_t$ is an AR(1) process:

$$\log A_t = (1-\xi)\log A^* + \xi \log A_{t-1} + \varepsilon_t, \qquad 0 \leq \xi < 1. \tag{30.3}$$

The last term, ε_t, represents a productivity shock which is assumed to be *white noise* with variance σ_ε^2. The auto-correlation coefficient ξ measures the degree of *persistence* over time of the effect on $\log A$ of a shock. If $\xi = 0$, the effect is only temporary; if $\xi > 0$, there is some persistence. The unconditional expectation of $\log A_t$ is equal to $\log A^*$ (which is thus the expected value "in the long run"). The shocks, ε_t, may represent accidental events affecting productivity, perhaps technological changes that are not sustainable, including technological mistakes (think of the introduction and later abandonment of asbestos in the construction industry). Negative realizations of the noise term ε_t may represent technological regress. But it need not, since moderate negative values of ε_t are consistent with overall technological progress, though temporarily below the trend represented by the deterministic growth of X_t.

The reason we said "not sustainable" is that sustainability would require $\xi = 1$, which conflicts with (30.3). Yet $\xi = 1$, which turns (30.3) into a random walk with drift, would correspond better to our general conception of technological change as a *cumulative* process. Technical knowledge is cumulative in the sense that a technical invention continues to be known. But in the present version of the RBC model this cumulative part of technological change is represented by the deterministic trend γ in (30.2). Anyway, what the stochastic supply shock A_t really embodies remains somewhat vague. A broad interpretation includes abrupt structural changes, cartelization of markets, closures of industries, shifts in legal and political systems, harvest failures, wartime destruction, natural disasters, and strikes. For an open economy, shifts in terms of trade might be a possible interpretation for example due to oil price shocks.

Factor demand

The representative firm is assumed to maximize its value under perfect competition. Since there are no convex capital installation costs, the problem reduces to that of static maximization of profits each period. And since period t's technological conditions (F, X_t, and the realization of A_t) are assumed known to the firm in period t, the firm does not face any uncertainty. Profit maximization simply implies a standard factor demand (K_t, N_t), satisfying

$$A_t F_1(K_t, X_t N_t) = r_t + \delta, \qquad 0 \leq \delta \leq 1, \tag{30.4}$$

$$A_t F_2(K_t, X_t N_t) X_t = w_t, \tag{30.5}$$

where $r_t + \delta$ is the real cost per unit of the capital service and w_t is the real wage.

The household

There is a given number of households, or rather dynastic families, all alike and with infinite horizon. For simplicity we ignore population growth. Thus we consider a representative household of constant size. The household's saving in period t amounts to buying investment goods that in the next period are rented out to the firms at the rental rate $r_{t+1}+\delta$. Thus the household obtains a net rate of return on financial wealth equal to the interest rate r_{t+1}.

A decision problem under uncertainty

The preferences of the household are described by the expected discounted utility hypothesis. Both consumption, C_t, and leisure, L_t, enter the period utility function. The total time endowment of the household is 1 in all periods:

$$N_t + L_t = 1, \qquad t = 0, 1, 2, \ldots, \tag{30.6}$$

where N_t is labor supply in period t. The fact that N has now been used in two different meanings, in (30.1) as employment and in (30.6) as labor supply, should not cause problems since in the competitive equilibrium of the model the two are quantitatively the same.

The household has rational expectations and solves the problem:

$$\begin{aligned}
\max E_0 U_0 &= E_0[\sum_{t=0}^{\infty} u(C_t, 1-N_t)(1+\rho)^{-t}] \qquad \text{s.t.} && (30.7)\\
C_t &\geq 0, 0 \leq N_t \leq 1, \qquad \text{(control region)} && (30.8)\\
K_{t+1} &= (1+r_t)K_t + w_t N_t - C_t, \qquad K_0 \geq 0 \text{ given}, && (30.9)\\
K_{t+1} &\geq 0 \qquad \text{for } t = 0, 1, 2, , \ldots . && (30.10)
\end{aligned}$$

The period utility function u satisfies $u_1 > 0$, $u_2 > 0$, $u_{11} < 0$, $u_{22} < 0$ and is concave, which is equivalent to adding the assumption $u_{11}u_{22} - (u_{12})^2 \geq 0$. The decreasing marginal utility assumption reflects, first, a desire of smoothing over time both consumption and leisure; or we could say that there is aversion towards variation over time in these entities. Second, decreasing marginal utility reflects aversion towards variation in consumption and leisure over different "states of nature", i.e., risk aversion. The parameter ρ is the rate of time preference and is assumed positive (a further restriction on ρ will be introduced later).

The symbol E_0 signifies the expected value, conditional on information available in period 0. More generally, E_t is a shorthand for $E(\cdot\,|I_t)$, where I_t denotes information revealed up to and including period t. The only source of uncertainty derives from the stochastic productivity variable A_t. We assume the ex ante uncertainty about A_t is resolved at time t, by which we mean the beginning of period t, the latter being identified with the time interval $[t, t+1)$. Knowledge of the market clearing values of r_t and w_t is included in the conditioning information I_t. There is uncertainty about *future* values of r and w, however. Nonetheless, the household is assumed to know the stochastic processes which these variables follow. Indeed, the household is assumed to know the "true" model of the economy as well as the stochastic process followed by the productivity variable A_t).

First-order conditions and transversality condition

For each t there are three endogenous variables in the household's problem, the control variables C_t and N_t and the state variable K_{t+1}. The decision, as seen from period 0, is to choose a concrete *action* (C_0, N_0) and a series of *contingency plans* $(C(t, K_t), N(t, K_t))$ saying what to do in each of the future periods $t = 1, 2, \ldots$, as a function of the as yet unknown circumstances, including the financial wealth, K_t, at that time. The decision is made so that expected discounted utility is maximized. The pair of functions $(C(t, K_t), N(t, K_t))$ is named a *contingency plan* because it refers to what level of consumption and labor supply, respectively, will be chosen optimally in the future period t, contingent on the financial wealth at the beginning of period t. In turn this wealth, K_t, depends on the realized path, up to period $t-1$, of the ex ante unknown productivity factor A and the optimally chosen values of C and N. In order to choose the action (C_0, N_0) in a rational way, the household must take into account the whole future, including what the optimal contingent actions in the future will be.

Letting period t be an arbitrary period, i.e., $t \in \{0, 1, 2, ...\}$, we rewrite U_0 in the following way

$$\begin{aligned} U_0 &= \sum_{s=0}^{t-1} u(C_s, 1-N_s)(1+\rho)^{-s} + \sum_{s=t}^{\infty} u(C_s, 1-N_s)(1+\rho)^{-s} \\ &= \sum_{s=0}^{t-1} u(C_s, 1-N_s)(1+\rho)^{-s} + (1+\rho)^{-t} U_t, \end{aligned}$$

where $U_t = \sum_{s=t}^{\infty} u(C_s, 1-N_s)(1+\rho)^{-(s-t)}$. When deciding the "action" (C_0, N_0), the household knows that in every new period, it has to solve the remainder of the problem in a similar way, given the information revealed up to and including that period.

As seen from period t, the objective function can be written

$$\begin{aligned} E_t U_t &= u(C_t, 1-N_t) + (1+\rho)^{-1} E_t \left[u(C_{t+1}, 1-N_{t+1}) \right. \qquad (30.11) \\ &\left. + u(C_{t+2}, 1-N_{t+2})(1+\rho)^{-1} + ... \right]. \end{aligned}$$

since there is no uncertainty concerning the current period. To find first-order conditions we will use the *substitution method.* First, from (30.9) we have

$$\begin{aligned} C_t &= (1+r_t)K_t + w_t N_t - K_{t+1}, \qquad \text{and} \qquad (30.12) \\ C_{t+1} &= (1+r_{t+1})K_{t+1} + w_{t+1} N_{t+1} - K_{t+2}. \qquad (30.13) \end{aligned}$$

Substituting this into (30.11), the decision problem is reduced to an essentially unconstrained maximization problem, namely one of maximizing the function E_tU_t w.r.t. $(N_t, K_{t+1}), (N_{t+1}, K_{t+2}), \dots$. We first take the partial derivative w.r.t. N_t in (30.11), given (30.12), and set it equal to 0 (thus focusing on interior solutions):

$$\frac{\partial E_tU_t}{\partial N_t} = u_1(C_t, 1-N_t)w_t + u_2(C_t, 1-N_t)(-1) = 0,$$

which can be written

$$u_2(C_t, 1-N_t) = u_1(C_t, 1-N_t)w_t. \tag{30.14}$$

This first-order condition describes the trade-off between leisure in period t and consumption in the same period. The condition says that in the optimal plan, the opportunity cost (in terms of foregone current utility) associated with decreasing leisure by one unit equals the utility benefit of obtaining an increased labor income and using this increase for extra consumption. In brief, marginal cost = marginal benefit, both measured in current utility.

Secondly, in (30.11) we take the partial derivative w.r.t. K_{t+1}, given (30.12) and (30.13).[3] This gives the first-order condition

$$\frac{\partial E_tU_t}{\partial K_{t+1}} = u_1(C_t, 1-N_t)(-1) + (1+\rho)^{-1}E_t[u_1(C_{t+1}, 1-N_{t+1})(1+r_{t+1})] = 0,$$

which can be written

$$u_1(C_t, 1-N_t) = (1+\rho)^{-1}E_t[u_1(C_{t+1}, 1-N_{t+1})(1+r_{t+1})], \tag{30.15}$$

where r_{t+1} is unknown in period t. This first-order condition describes the trade-off between consumption in period t and the uncertain consumption in period $t+1$, as seen from period t. The optimal plan must satisfy that the current utility loss associated with decreasing consumption by one unit equals the discounted expected utility gain next period by having $1 + r_{t+1}$ extra units available for consumption, namely the gross return on saving one more unit. In brief, again marginal cost = marginal benefit in utility terms.

The condition (30.15) is an example of a *stochastic Euler equation*. If there is no uncertainty, the expectation operator E_t can be deleted. Then, apart from

[3] Generally speaking, for a given differentiable function $f(X, \alpha_1, \dots, \alpha_n)$, where X is a stochastic variable and $\alpha_1, \dots, \alpha_n$ are parameters, we have

$$\frac{\partial E(f(X, \alpha_1, \dots, \alpha_n))}{\partial \alpha_i} = E\frac{\partial f(X, \alpha_1, \dots, \alpha_n)}{\partial \alpha_i}, \quad i = 1, \dots, n.$$

leisure entering as a second argument, (30.15) is the standard discrete-time analogue to the Keynes-Ramsey rule in continuous time.

For completeness, let us also derive the first-order conditions w.r.t. the future pairs (N_{t+i}, K_{t+i+1}), $i = 1, 2, \dots$. We get

$$\frac{\partial E_t U_t}{\partial N_{t+i}} = (1+\rho)^{-1} E_t \left[u_1(C_{t+i}, 1 - N_{t+i}) w_{t+i} + u_2(C_{t+i}, 1 - N_{t+i})(-1) \right] = 0,$$

so that

$$E_t \left[u_2(C_{t+i}, 1 - N_{t+i}) \right] = E_t \left[u_1(C_{t+i}, 1 - N_{t+i}) w_{t+i} \right].$$

Similarly,

$$\begin{aligned} \frac{\partial E_t U_t}{\partial K_{t+i+1}} &= E_t \Big[u_1(C_{t+i}, 1 - N_{t+i})(-1) + (1+\rho)^{-1} u_1(C_{t+i+1}, 1 - N_{t+i+1}) \\ \cdot (1 + r_{t+i+1}) &= 0, \end{aligned}$$

so that

$$E_t \left[u_1(C_{t+i}, 1 - N_{t+i}) \right] = (1+\rho)^{-1} E_t \left[u_1(C_{t+i+1}, 1 - N_{t+i+1})(1 + r_{t+i+1}) \right]$$

So, it suffices to say that for the current period, t, the first-order conditions are (30.14) and (30.15), and for the future periods similar first-order conditions hold in *expected* values.

As usual in dynamic optimization problems the first-order conditions say something about optimal *relative* levels of consumption and leisure over time, not about the absolute initial levels of consumption and leisure. The absolute initial levels are determined as the highest possible levels consistent with the requirement that first-order conditions of form (30.14) and (30.15), together with the non-negativity in (30.10), hold for period t and, in terms of expected values as seen from period t, for all future periods. This requirement can be shown to be equivalent to requiring the transversality condition,

$$\lim_{t \to \infty} E_0 \left[K_t u_1(C_{t-1}, 1 - N_{t-1})(1+\rho)^{-(t-1)} \right] = 0,$$

satisfied in addition to the first-order conditions.[4] Finding the resulting consumption function requires specification of the period utility function. But to characterize the equilibrium path, the consumption function is in fact not needed.

[4] In fact, in the budget constraint of the household's optimization problem, we could replace K_t by financial wealth and allow borrowing, so that financial wealth could be negative. Then, instead of the non-negativity constraint (30.10), a No-Ponzi-Game condition in expected value would be relevant. In a representative agent model with infinite horizon, however, this does not change anything, since the non-negativity constraint (30.10) will never be binding.

The remaining elements in the model

It only remains to check market clearing conditions and determine equilibrium factor prices. Implicitly we have already assumed clearing in the factor markets, since we have used the same symbols for capital and employment, respectively, in the firm's problem (the demand side) as in the household's problem (the supply side). The equilibrium factor prices are given by (30.4) and (30.5). We will rewrite these two equations in a more convenient way. In view of constant returns to scale, we have

$$Y_t = A_t F(K_t, X_t N_t) = A_t X_t N_t F(\tilde{k}_t, 1) \equiv A_t X_t N_t f(\tilde{k}_t), \tag{30.16}$$

where $\tilde{k}_t \equiv K_t/(X_t N_t)$ is the effective capital-labor ratio. In terms of the intensive production function f, (30.4) and (30.5) yield

$$\begin{aligned} r_t + \delta &= A_t F_1(K_t, X_t N_t) = A_t f'(\tilde{k}_t), & (30.17)\\ w_t &= A_t F_2(K_t, X_t N_t) X_t = A_t \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \right] X_t. & (30.18) \end{aligned}$$

Finally, equilibrium in the output market requires that aggregate output equals aggregate demand, i.e., the sum of aggregate consumption and investment:

$$Y_t = C_t + I_t. \tag{30.19}$$

We now show that this equilibrium condition is automatically implied by previous equations. Indeed, adding δK_t on both sides of the budget constraint (30.9) of the representative household and rearranging, we get

$$\begin{aligned} K_{t+1} - K_t + \delta K_t &= (r_t + \delta) K_t + w_t N_t - C_t & (30.20)\\ &= A_t f'(\tilde{k}_t) K_t + A_t \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \right] X_t N_t - C_t \\ &= A_t X_t N_t f(\tilde{k}_t) - C_t = Y_t - C_t \equiv S_t, \end{aligned}$$

where the second equality comes from (30.17) and (30.18) and the fourth from (30.16). Now, in this model aggregate gross saving, S_t, is directly an act of investment so that $I_t = S_t$. From this follows (30.19).

Specification of technology and preferences

To quantify the model we have to specify the production function and the utility function. We abide by the standard assumption in the RBC literature and specify the production function to be Cobb-Douglas:

$$Y_t = A_t K_t^{1-\alpha} (X_t N_t)^{\alpha}, \qquad 0 < \alpha < 1. \tag{30.21}$$

We then get

$$\begin{aligned} f(\tilde{k}_t) &= A_t \tilde{k}_t^{1-\alpha}, && (30.22) \\ r_t + \delta &= (1-\alpha) A_t \tilde{k}_t^{-\alpha}, && (30.23) \\ w_t &= \alpha A_t \tilde{k}_t^{1-\alpha} X_t. && (30.24) \end{aligned}$$

As to the utility function we follow King and Rebelo (1999) and base the analysis on the additively separable CRRA case,

$$u(C_t, 1-N_t) = \frac{C_t^{1-\theta}}{1-\theta} + \omega \frac{(1-N_t)^{1-\eta}}{1-\eta}, \qquad \theta > 0, \eta > 0, \omega > 0. \qquad (30.25)$$

Here, θ is the (absolute) elasticity of marginal utility of consumption, equivalently the desire for consumption smoothing, η is the (absolute) elasticity of marginal utility of leisure, equivalently, the desire for leisure smoothing, and ω is the relative weight given to leisure. In case θ or η take on the value 1, the corresponding term in (30.25) should be replaced by $\log C_t$ or $\omega \log(1-N_t)$, respectively. In fact, most of the time King and Rebelo (1999) take both θ and η to be 1.

With (30.25) applied to (30.14) and (30.15), we get

$$\begin{aligned} \theta(1-N_t)^{-\eta} &= C_t^{-\theta} w_t, \quad \text{and} && (30.26) \\ C_t^{-\theta} &= \frac{1}{1+\rho} E_t\left[C_{t+1}^{-\theta}(1+r_{t+1})\right], && (30.27) \end{aligned}$$

respectively.

30.2 A deterministic steady state*

For a while, let us ignore shocks. That is, assume $A_t = A^*$ for all t.

The steady state solution

By a steady state we mean a path along which the growth-corrected variables like $\tilde{k}$ and $\tilde{w} \equiv w/X_t$ stay constant. With $A_t = A^*$ for all t, (30.23) and (30.24) return the steady-state relations between $\tilde{k}$, r, and $\tilde{w}$:

$$\begin{aligned} \tilde{k}^* &= \left[\frac{(1-\alpha)A^*}{r^* + \delta}\right]^{1/\alpha}, && (30.28) \\ \tilde{w}^* &= \alpha A^* \tilde{k}^{*1-\alpha}. && (30.29) \end{aligned}$$

We may write (30.27) as

$$1 + \rho = E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\theta}(1+r_{t+1})\right]. \qquad (30.30)$$

In the non-stochastic steady state the expectation operator E_t can be deleted, and r and C/X are independent of t. Hence, $C_{t+1}/C_t = \gamma$, by (30.2), and (30.30) takes the form

$$1 + r^* = (1 + \rho)\gamma^\theta. \tag{30.31}$$

In this expression we recognize the *modified golden rule* discussed in chapters 7 and 10.[5] Existence of general equilibrium in our Ramsey framework requires that the long-run real interest rate is larger than the long-run output growth rate, i.e., we need $r^* > \gamma - 1$. This condition is satisfied if and only if

$$1 + \rho > \gamma^{1-\theta}, \tag{30.32}$$

which we assume.[6] If we guess that $\theta = 1$ and $\rho = 0.01$, then with $\gamma = 1.004$ (taken from US national income accounting data 1947-96, using a quarter of a year as our time unit), we find the steady-state rate of return to be $r^* = 0.014$ or 0.056 per annum. Or, the other way round, observing the average return on the Standard & Poor 500 Index over the same period to be 6.5 per annum, given $\theta = 1$ and $\gamma = 1.004$, we estimate ρ to be 0.012.

Using that in steady state N_t is a constant, N^*, we can write (30.20) as

$$\gamma\tilde{k}_{t+1} - (1 - \delta)\tilde{k}_t = A^*\tilde{k}_t^{1-\alpha} - \tilde{c}_t, \tag{30.33}$$

where $\tilde{c}_t \equiv C_t/(X_t N^*)$. Given r^*, (30.28) yields the steady-state capital intensity $\tilde{k}^*$. Then, (30.33) returns

$$\tilde{c}^* \equiv \frac{c^*}{X_t} = A^*\tilde{k}^{*1-\alpha} - (\gamma + \delta - 1)\tilde{k}^*.$$

Consumption dynamics around the steady state in case of no uncertainty

The adjustment process for consumption, absent uncertainty, is given by (30.30) as

$$(\frac{C_{t+1}}{C_t})^{-\theta}(1 + r_{t+1}) = 1 + \rho,$$

or, taking logs,

$$\log \frac{C_{t+1}}{C_t} = \frac{1}{\theta}\left[\log(1 + r_{t+1}) - \log(1 + \rho)\right]. \tag{30.34}$$

[5] King and Rebelo, 1999, p. 947, express this in terms of the growth-adjusted discount factor $\beta \equiv (1 + \rho)^{-1}\gamma^{1-\theta}$, so that $1 + r^* = (1 + \rho)\gamma^\theta = \gamma/\beta$.

[6] Since $\gamma > 1$, only if $\theta < 1$ (which does not seem realistic, cf. Chapter 3), is $\rho > 0$ not sufficient for (30.32) to hold.

This is the deterministic Keynes-Ramsey rule in discrete time under separable CRRA utility. For any "small" x we have $\log(1+x) \approx x$ (from a first-order Taylor approximation of $\log(1+x)$ around 0). Hence, with $x = C_{t+1}/C_t - 1$, we have $\log(C_{t+1}/C_t) \approx C_{t+1}/C_t - 1$, so that (30.34) implies the approximate relation

$$\frac{C_{t+1} - C_t}{C_t} \approx \frac{1}{\theta}(r_{t+1} - \rho). \tag{30.35}$$

There is a supplementary way of writing the Keynes-Ramsey rule. Note that (30.31) implies $\log(1 + r^*) = \log(1 + \rho) + \theta \log \gamma$. Using first-order Taylor approximations, this gives $r^* \approx \rho + \theta \log \gamma \approx \rho + \theta g$, where $g \equiv \gamma - 1$ is the trend rate of technological progress. Thus $\rho \approx r^* - \theta g$, and inserting this into (30.35) we get

$$\frac{C_{t+1} - C_t}{C_t} \approx \frac{1}{\theta}(r_{t+1} - r^*) + g.$$

Then the technology-corrected consumption level, $c_t \equiv C_t/X_t$, moves according to

$$\frac{c_{t+1} - c_t}{c_t} \approx \frac{1}{\theta}(r_{t+1} - r^*),$$

since g is the growth rate of X_t.

30.3 On the approximate solution and numerical simulation*

In the special case $\theta = 1$ (the log utility case), still maintaining the Cobb-Douglas specification of the production function, the model can be solved analytically provided capital is non-durable (i.e., $\delta = 1$). It turns out that in this case the solution has consumption as a constant fraction of output. Further, in this special case labor supply equals a constant and is thus independent of the productivity shocks. Since in actual business cycles, employment fluctuates a lot, this might not seem to be good news for a business cycle model.

But assuming $\delta = 1$ for a period length of one quarter or one year is unreasonable anyway. Given a period length of one year, δ is generally estimated to be less than 0.1. With $\delta < 1$, labor supply *is* affected by the technology shocks, and an exact analytical solution can no longer be found.

One can find an *approximate* solution based on a log-linearization of the model around the steady state. Without dwelling on the more technical details we will make a few observations.

30.3.1 Log-linearization

If x^* is the steady-state value of the variable x_t in the non-stochastic case, then one defines the new variable, the log-deviation of x from x^* :

$$\hat{x}_t \equiv \log(\frac{x_t}{x^*}) = \log x_t - \log x^*. \tag{30.36}$$

That is, $\hat{x}_t$ is the logarithmic deviation of x_t from its steady-state value. But this is approximately the same as x's *proportionate* deviation from its steady-state value. This is because, when x_t is in a neighborhood of its steady-state value, a first-order Taylor approximation of $\log x_t$ around x^* yields

$$\log x_t \approx \log x^* + \frac{1}{x^*}(x_t - x^*),$$

so that

$$\hat{x}_t \approx \frac{x_t - x^*}{x^*}. \tag{30.37}$$

Working with the transformation $\hat{x}_t$ instead of x_t implies the convenience that

$$\begin{aligned}
\hat{x}_{t+1} - \hat{x}_t &= \log(\frac{x_{t+1}}{x^*}) - \log(\frac{x_t}{x^*}) = \log x_{t+1} - \log x_t \\
&\approx \frac{x_{t+1} - x_t}{x_t}.
\end{aligned}$$

That is, relative changes in x have been replaced by absolute changes in $\hat{x}$.

Some of the equations of interest are exactly log-linear from start. This is true for the equations (30.22), (30.23), and (30.24), as well as for the first-order condition (30.26) for the household. For other equations log-linearization requires approximation. Consider for instance the time constraint $N_t + L_t = 1$. This constraint implies

$$N^* \frac{N_t - N^*}{N^*} + L^* \frac{L_t - L^*}{L^*} = 0$$

or

$$N^* \hat{N}_t + L^* \hat{L}_t \approx 0, \tag{30.38}$$

by the principle in (30.37). From (30.26), taking into account that $1 - N_t = L_t$, we have

$$\begin{aligned}
\theta L_t{}^{-\eta} &= C_t^{-\theta} w_t \equiv (c_t X_t)^{-\theta} \tilde{w}_t X_t \\
&= c_t{}^{-\theta} \tilde{w}_t X_t^{1-\theta}.
\end{aligned} \tag{30.39}$$

In steady state this takes the form

$$\theta L^{*-\eta} = c^{*-\theta} \tilde{w}^* X_t^{1-\theta}. \tag{30.40}$$

We see that when there is sustained technological progress, $\gamma > 1$, we *need* $\theta = 1$ for a steady state to exist (which explains why in their calibration King and Rebelo assume $\theta = 1$). This quite "narrow" theoretical requirement is an unwelcome feature and is due to the additively separable utility function.

Combining (30.40) with (30.39) gives

$$\left(\frac{L_t}{L^*}\right)^{-\eta} = \left(\frac{c_t}{c^*}\right)^{-\theta} \frac{\tilde{w}_t}{\tilde{w}^*}.$$

Taking logs on both sides we get

$$-\eta \log \frac{L_t}{L^*} = \log \frac{\tilde{w}_t}{\tilde{w}^*} - \theta \log \frac{c_t}{c^*}$$

or

$$-\eta \hat{L}_t = \hat{w}_t - \theta \hat{c}_t.$$

In view of (30.38), this implies

$$\hat{N}_t = -\frac{L^*}{N^*}\hat{L}_t = \frac{1-N^*}{N^*\eta}\hat{w}_t - \frac{1-N^*}{N^*\eta}\theta\hat{c}_t. \tag{30.41}$$

This result tells us that the elasticity of labor supply w.r.t. a temporary change in the real wage depends negatively on η; this is not surprising, since η reflects the desire for leisure smoothing across time. Indeed, calling this elasticity ε, we have

$$\varepsilon = \frac{1-N^*}{N^*\eta}. \tag{30.42}$$

Departing from the steady state, a one per cent increase in the wage ($\hat{w}_t = 0.01$) leads to an ε per cent increase in the labor supply, by (30.41) and (30.42). The number ε measures a kind of compensated wage elasticity of labor supply (in an intertemporal setting), relevant for evaluating the pure substitution effect of a temporary rise in the wage. King and Rebelo (1999) reckon N^* in the US to be 0.2, that is, out of available time one fifth is working time. With $\eta = 1$, we then get $\varepsilon = 4$. This elasticity is much higher than what the micro-econometric evidence suggests, at least for men, namely typically an elasticity below 1 (Pencavel 1986). But with labor supply elasticity as low as 1, the RBC model is far from capable of generating a volatility in employment comparable to what the data show.

For some purposes it is convenient to have the endogenous time-dependent variables appearing separately in the stationary dynamic system. Then, to describe the supply of output in log-linear form, let $y_t \equiv Y_t/X_t \equiv A_t f(\tilde{k}_t)N_t$ and $k_t \equiv K_t/X_t \equiv \tilde{k}_t N_t$. From (30.21),

$$y_t = A_t k_t^{1-\alpha} N_t^{\alpha},$$

and dividing through by the corresponding expression in steady state, we get

$$\frac{y_t}{y^*} = \frac{A_t}{A^*}(\frac{k_t}{k^*})^{1-\alpha}(\frac{N_t}{N^*})^{\alpha}.$$

Taking logs on both sides, we end up with

$$\hat{y}_t = \hat{A}_t + (1-\alpha)\hat{k}_t + \alpha\hat{L}_t. \tag{30.43}$$

For the demand side we can obtain at least an approximate log-linear relation. Indeed, dividing trough by X_t in (30.19) we get

$$c_t + i_t = y_t,$$

where $i_t \equiv I_t/X_t$. Dividing through by y^* and reordering, this can also be written

$$\frac{c^*}{y^*}\frac{c_t - c^*}{c^*} + \frac{i^*}{y^*}\frac{i_t - i^*}{i^*} = \frac{y_t - y^*}{y^*},$$

which, using the hat notation from (30.37), can be written

$$\frac{c^*}{y^*}\hat{c}_t + \frac{i^*}{y^*}\hat{\imath}_t \approx \hat{y}_t. \tag{30.44}$$

to be equated with the right hand side of (30.43).

30.3.2 Numerical simulation

After log-linearization, the model can be reduced to two coupled *linear* stochastic first-order difference equations in k_t and c_t, where k_t is predetermined, and c_t is a jump variable. There are different methods available for solving such an approximate dynamic system analytically.[7] Alternatively, based on a specified set of parameter values one can solve the system by numerical simulation on a computer.

In any case, when it comes to checking the quantitative performance of the model, RBC theorists generally stick to *calibration,* that is, the method based on a choice of parameter values such that the model matches a list of data characteristics. In the present context this means that:

(a) the structural parameters $(\alpha, \delta, \rho, \theta, \eta, \omega, \gamma, N^*)$ are given values that are taken or constructed partly from national income accounting and similar data, partly from micro-econometric studies of households' and firms' behavior;

[7] For details one may consult Campbell (1994, p. 468 ff.), Obstfeld and Rogoff (1996, p. 503 ff.), or Uhlig (1999).

(b) the values of the parameters, ξ and σ_ε, in the stochastic process for the productivity variable A are chosen either on the basis of data for the Solow residual[8] over a long time period, or one or both values are chosen to yield, as closely as possible, a correspondence between the statistical moments (standard deviation, auto-correlation etc.) predicted by the model and those in the data.

The first approach to ξ and σ_ε is followed by, e.g., Prescott (1986). It has been severely criticized by, among others, Mankiw (1989). In the short and medium term, the Solow residual is very sensitive to labor hoarding and variations in the degree of utilization of capital equipment. It can therefore be argued that it is the business cycle fluctuations that explain the fluctuations in the Solow residual, rather than the other way round.[9] The second approach, used by, e.g., Hansen (1985) and Plosser (1989), has the disadvantage that it provides no independent information on the stochastic process for productivity shocks. Yet such information is necessary to assess whether the shocks can be the driving force behind business cycles.

As hitherto we abide to the approach of King and Rebelo (1999) which like Prescott's is based on the Solow residual. The parameters chosen are shown in Table 19.2. Remember that the time unit is a quarter of a year.

Table 29.2. Parameter values

α	δ	ρ	θ	η	ω	γ	N^*	ξ	σ_ε
0.667	0.025	0.0163	1	1	3.48	1.004	0.2	0.979	0.0072

Given these parameter values and initial values of k and A in conformity with the steady state, the simulation is ready to be started. The shock process is activated and the resulting evolution of the endogenous variables generated

[8] Given the Cobb-Douglas production function (30.21), take logs on both sides and rearrange to get

$$\log Y_t - (1-\alpha)\log K_t - \alpha \log L_t = \log A_t + \alpha \log X_t.$$

Based on time series for Y, K , and L, and estimating α by data on the labor income share, the left-hand side can be computed and used to uncover the productivity process $\log A_t + \alpha \log X_t$. In growth accounting the left-hand side makes up the "raw material" for calculating the Solow residual,

$$SR_t \equiv \Delta \log Y_t - ((1-\alpha)\Delta \log K_t + \alpha \Delta \log L_t).$$

Data on the degree of utilization is fragmentary. Hence, correction for variation over time in utilization is difficult.

[9] King and Rebelo (1999, p. 982-993) believe that the problem can be overcome by refinement of the RBC model.

through the propagation mechanism of the model calculated by the computer. From this evolution the analyst next calculates the different relevant statistics: standard deviation (as a measure of volatility), auto-correlation (as a measure of persistence), and cross correlations with different leads and lags (reflecting the co-movements and dynamic interaction of the different variables). These model-generated statistics can then be compared to those calculated on the basis of the empirical observations.

In order to visualize the economic mechanisms involved, *impulse-response functions* are calculated. Shocks before period 0 are ignored and the economy is assumed to be in steady state until this period. Then, a positive once-for-all shock to A occurs so that productivity is increased by, say, 1 % (i.e., given $A_{-1} = A^* = 1$, we put $\varepsilon_0 = 0.01$ in (30.3) with $t = 0$). The resulting path for the endogenous variables is calculated under the assumption that no further shocks occur (i.e., $\varepsilon_t = 0$ for $t = 1, 2, \ldots$). An inpulse-response diagram shows the implied time profiles for the different variables.

Remark. The text should here show some graphs of impulse-response functions. These graphs are not yet available. Instead the reader is referred to the graphs in King and Rebelo (1999), p. 966-970. As expected, the time profiles for output, consumption, employment, real wages, and other variables differ, depending on the size of ξ in (30.3). If we consider a purely temporary productivity shock, the case $\xi = 0$, we get the graphs in King and Rebelo (1999), p. 966. A highly persistent productivity shock, the case $\xi = 0.979$, gives rise to the responses on p. 968. We see that these responses are more drawn out over time. This persistence in the endogenous variables is, however, just inherited from the assumed persistence in the shock. And amplification is limited. In case of a permanent productivity shock, $\xi = 1$, wealth effects on labor supply are strong and tend to offset the substitution effect. □

30.4 The two basic propagation mechanisms

We have added technology shocks to a standard neoclassical growth model. The conclusion is that correlated fluctuations in output, consumption, investment, work hours, output per man-hour, real wages, and the real interest rate are generated. So far so good. Two basic propagation mechanisms drive the fluctuations:

1. *The capital accumulation mechanism.* To understand this mechanism in its pure form, let us abstract from the endogenous labor supply and assume an inelastic labor supply. A positive productivity shock increases marginal productivity of capital and labor. If the shock is not purely temporary, the household feels more wealthy. Both output, consumption and saving go

up, the latter due to the desire for consumption smoothing. The increased capital stock implies higher output also in the next periods. Hence output shows positive persistence. And output, consumption, and investment move together, i.e., there is co-movement.

2. *Intertemporal substitution in labor supply.* An immediate implication of increased marginal productivity of labor is a higher real wage. To the extent that this increased real wage is only temporary, the household is motivated to supply more labor in the current period and less later. This is the phenomenon of intertemporal substitution in leisure. By the adherents of the RBC theory the observed fluctuations in work hours are seen as reflecting this.

30.5 Limitations

During the last couple of decades there has been an increasing scepticism towards the RBC theory. The central limitation of the theory comes from its insistence upon interpreting fluctuations in employment as reflecting fluctuations in labor supply. The critics maintain that, starting from market clearing based on flexible prices, it is not surprising that it becomes difficult to match the business cycle facts arise.

We may summarize the objections to the theory in the following four points:

a. *Where are the productivity shocks?* As some critics ask: "If productivity shocks are so important, why don't we read about them in the Wall Street Journal or in The Economist?" Indeed, technology shocks occur within particular lines of a multitude of businesses and sum up, at the aggregate level, to an upward *trend* in productivity, relevant for growth theory. It is not easy to see they should be able to drive the business cycle component of the data. Moreover, it seems hard to interpret the absolute economic contractions (decreases in GDP) that sometimes occur in the real world as due to productivity shocks. If the elasticity of output w.r.t. productivity shocks does not exceed one (as it does not seem to, empirically, according to Campbell (1994)), then a *backward* step in technology at the aggregate level is needed. Although genuine technological knowledge as such is inherently increasing, mistakes could be made in choosing technologies. At the disaggregate level, one can sometimes identify technological mistakes, cf. the use of DDT and its subsequent ban in the 1960's due to its damaging effects on health. But it is hard to think of technological drawbacks at the *aggregate* level, capable of explaining the observed economic recessions.

Think of the large and long-lasting contraction of GDP in the US during the Great Depression (27 % reduction between 1929 and 1933 according to Romer (2001), p. 171). Sometimes the adherents of the RBC theory have referred also to other kinds of supply shocks: changes in taxation, changes in environmental legislation etc. (Hansen and Prescott, 1993). But significant changes in taxation and regulation occur rather infrequently.

b. *Lack of internal propagation.* Given the available micro-econometric evidence, the two mechanisms above seem far from capable at generating the *large* fluctuations in output and employment that we observe. Both mechanisms imply little amplification of the shocks. This means that to replicate the stylized business cycle facts, standard RBC models must rely heavily on exogenous shocks dynamics. Indeed, the intertemporal substitution in labor supply as described above is not able to generate much amplification. This is related to the fact that changes in real wages tend to be permanent rather than purely transitory. Permanent wage increases tend to have little or no effect on labor supply (the wealth effect tends to offset the substitution and income effects). Given the very minor temporary movements in the real wage that occur at the empirical level, a high intertemporal elasticity of substitution in labor supply is required to generate large fluctuations in employment as observed in the data. But the empirical evidence suggests that this requirement is not met. Micro-econometric studies of labor supply indicate that this elasticity, at least for men, is quite small (in the range 0 to 1.5, typically below 1).[10] Yet, Prescott (1986) and Plosser (1989) assume it is around 4.

c. *Correlation puzzles.* Sometimes the sign, sometimes the size of correlation coefficients seem persevering wrong (see King and Rebelo, p. 957, 961). As Akerlof (2003, p. 414) points out, there is a conflict between the empirically observed pro-cyclical behavior of workers' quits[11] and the theory's prediction that quits should increase in cyclical *downturns* (since variation in employment is voluntary according to the theory). Considering a dozen of OECD countries, Danthine and Donaldson (1993) find that the required positive correlation between labor productivity and output is visible only in data for the U.S. (and not strong), whereas the correlation is markedly negative for the majority of the other countries.

d. *Disregard of non-neutrality of money.* According to many critics, the RBC

[10] *Handbook of Labor Economics*, vol. 1, 1986, Table 1.22, last column. See also Hall (1999, p. 1148 ff.).

[11] See Chapter 29.

theory conflicts with the empirical evidence of the real effects of monetary policy.

Numerous, and more and more imaginative, attempts at overcoming the criticisms have been made. King and Rebelo (1999, p. 974-993) present some of these. In particular, adherents of the RBC approach have looked for mechanisms that may raise the size of labor supply elasticities at the aggregate level over and above that at the individual level found in micro-econometric studies.

30.6 Technological change as a random walk with drift

Above we have considered technical change as a mean-reverting process with a deterministic trend. This is the approach followed by Prescott (1986) and King and Rebelo (1999). In contrast, Plosser (1989) assumes that technological change is a random walk with drift. The representative firm has the production function

$$Y_t = Z_t F(K_t, N_t),$$

where Z_t is a measure of the level of technology, and the production function F has constant returns to scale. In the numerical simulation Plosser used a Cobb-Douglas specification.

The total factor productivity, Z_t, is an exogenous stochastic variable. In contrast to the process for the logarithm of A_t in the Prescott version above, where we had $\xi < 1$, we now assume that $\xi = 1$ so that the process assumed for $z_t \equiv \log Z_t$ is

$$z_t = \beta + z_{t-1} + \varepsilon_t, \tag{30.45}$$

which is a *random walk.* This corresponds to our general conception of technical knowledge as *cumulative.* If the deterministic term $\beta \neq 0$, the process is called a random walk *with drift.* In the present setting we can interpret β as some underlying deterministic component in the productivity trend, suggesting $\beta > 0$.[12] A stochastic trend component, which can go both ways, is generated by the noise term ε_t. Negative occurrences of this term need not represent *technological regress*, but just a technology development below trend (which will occur when $-\beta \leq \varepsilon_t < 0$). In an open economy, adverse shocks to terms of trade is a candidate interpretation.

[12] The growth rate in total factor productivity is $(Z_t - Z_{t-1})/Z_{t-1}$. From (30.45) we have $E_{t-1}(z_t - z_{t-1}) = \beta$, and $z_t - z_{t-1} = \log Z_t - \log Z_{t-1} \approx (Z_t - Z_{t-1})/Z_{t-1}$ by a 1. order Taylor approximation of $\log Z_t$ about Z_{t-1}. Hence, $E_{t-1}(Z_t - Z_{t-1})/Z_{t-1} \approx \beta$. In Plosser's model all technological change is represented by changes in Z_t, i.e., in (30.2) Plosser has $\gamma \equiv 1$.

Embedded in a Walrasian equilibrium framework the specification (30.45) tends to generate too little fluctuation in employment and output. This is because, when shocks are permanent, large wealth effects offset the intertemporal substitution in labor supply. On top of this comes limitations similar to points a, c, and d in the previous section.

30.7 Concluding remarks

It is advisory to make a distinction between on the one hand *RBC theory* (based on fully flexible prices and market clearing in an environment where productivity shocks are the driving force behind the fluctuations) and on the other hand the broader quantitative modeling framework known as *DSGE models.* A significant amount of research on business cycle fluctuations has left the RBC *theory* and the predominant emphasis on productivity shocks but applies similar *quantitative methods.* This approach is nowadays known as of an attempt at building *Dynamic Stochastic General Equilibrium (DSGE)* modeling. The economic contents of such a model *can* be new-classical (as in the tradition of Kydland and Prescott), emphasizing technology shocks and similar supply side effects. Alternatively it can be new-Keynesian of some variety, based on a combination of imperfect competition with nominal and real price rigidities and with emphasis on monetary policy and demand shocks (see, e.g., Jeanne, 1998, Smets and Wouters, 2003 and 2007, and Danthine and Kurmann, 2004, Gali, 2008). There are many varieties of these new-Keynesian models, some small and analytically oriented, some large and simulation- and forecasting-oriented. We consider an example of the "small" type in Chapter 32.

The aim of medium-run theory is to throw light on business cycle fluctuations and to clarify what kinds of countra-cyclical economic policy, if any, may be functional. This seems to be the area within macroeconomics where there is most disagreement − and has been so for a long time. Some illustrating quotations (TO BE UPDATED):

> Indeed, if the economy did not display the business cycle phenomena, there would be a puzzle. ... costly efforts at stabilization are likely to be counterproductive. Economic fluctuations are optimal responses to uncertainty in the rate of technological change (Prescott 1986).
>
> My view is that real business cycle models of the type urged on us by Prescott have nothing to do with the business cycle phenomena observed in the United States or other capitalist economies. ... The image of a big loose tent flapping in the wind comes to mind (Summers 1986).

30.8 Literature notes

The RBC theory was initiated by Finn E. Kydland and Edward C. Prescott (1982), where a complicated time-to-build aspect was part of the model. A simpler version of the RBC theory was given in Prescott (1986) where also the "economic philosophy" behind the theory was proclaimed. The King and Rebelo (1999) exposition followed here builds on Prescott's 1986 version which has become the prototype RBC model. Plosser's version (Plosser 1989), briefly sketched in Section 30.6, makes up an alternative regarding the modeling of the technology shocks.

In dealing with the intertemporal decision problem of the household we applied the substitution method. More advanced approaches include the discrete time Maximum Principle (see Chapter 8), the Lagrange method (see, e.g., King and Rebelo, 1999), or Dynamic Programming (see, e.g., Ljungqvist and Sargent, 2004).

The empirical approach, *calibration,* is different from econometric estimation and testing in the formal sense. Criteria for what constitutes a good fit are not clear. The calibration method can be seen as a first check whether the model is logically capable of matching main features of the data (say the first and second moments of key variables). Calibration delivers a quantitative example of the working of the model. It does not deliver an econometric test of the validity of the model or of hypotheses based on the model. Whether it provides a useful guide as to what aspects of the model should be revised is debated, see Hoover, 1995, pp. 24-44, Quah, 1995.

30.9 Exercises