Chapter 9

Solving the intertemporal consumption/saving problem in discrete and continuous time

In the next two chapters we shall discuss the continuous-time version of the basic representative agent model, the Ramsey model, and some of its applications. As a preparation for this, the present chapter gives an account of the transition from discrete time to continuous time analysis and of the application of optimal control theory for solving the household's consumption/saving problem in continuous time.

There are many fields of study where a setup in continuous time is preferable to one in discrete time. One reason is that continuous time opens up for application of the mathematical apparatus of differential equations; this apparatus is more powerful than the corresponding apparatus of difference equations. Another reason is that optimal control theory is more developed and potent in its continuous time version than in its discrete time version, considered in Chapter 8. In addition, many formulas in continuous time are simpler than the corresponding ones in discrete time (cf. the growth formulas in Appendix A).

As a vehicle for comparing continuous time analysis with discrete time analysis we consider a standard household consumption/saving problem. How does the household assess the choice between consumption today and consumption in the future? In contrast to the preceding chapters we allow for an arbitrary number of periods within the time horizon of the household. The period length may thus be much shorter than in the previous models. This opens up for capturing additional aspects of economic behavior and for carrying out the transition to continuous time in a smooth way.

First, we shall specify the market environment in which the optimizing household operates.

9.1 Market conditions

In the Diamond OLG model no loan market was active and wealth effects on consumption through changes in the interest rate were absent. It is different in a setup where agents live for many periods and realistically have a hump-shaped income profile through life. This motivates a look at the financial market and more refined notions related to intertemporal choice.

We maintain the assumption of perfect competition in all markets, i.e., households take all prices as given from the markets. Ignoring uncertainty, the various assets (real capital, stocks, loans etc.) in which households invest give the same rate of return in equilibrium. To begin with we consider time as discrete.

A perfect loan market Consider a given household. Suppose it can at any date take a loan or provide loans to others at the going interest rate, i_t, measured in money terms. That is, monitoring, administration, and other transaction costs are absent so that (a) the household faces the same interest rate whether borrowing or lending; (b) the household can not influence this rate; and (c) there are no borrowing restrictions other than the requirement on the part of the borrower to comply with her financial commitments. The borrower can somehow be forced to repay the debt with interest and so the lender faces no default risk. A loan market satisfying these idealized conditions is called a *perfect loan market*. The implications of such a market are:

1. various payment streams can be subject to comparison; if they have the same present value (PV for short), they are equivalent;

2. any payment stream can be converted into another one with the same present value;

3. payment streams can be compared with the value of stocks.

Consider a payment stream $\{x_t\}_{t=0}^{T-1}$ over T periods, where x_t is the payment in currency at the *end* of period t. As in the previous chapters, period t runs from time t to time $t+1$ for $t = 0, 1, ..., T-1$; and i_t is defined as the interest rate on a loan from time t to time $t+1$. Then the present value, PV_0, as seen from the beginning of period 0, of the payment stream is defined as[1]

$$PV_0 = \frac{x_0}{1+i_0} + \frac{x_1}{(1+i_0)(1+i_1)} + \cdots + \frac{x_{T-1}}{(1+i_0)(1+i_1)\cdots(1+i_{T-1})}. \quad (9.1)$$

If Ms. Jones is entitled to the income stream $\{x_t\}_{t=0}^{T-1}$ and at time 0 wishes to buy a durable consumption good of value PV_0, she can borrow this amount and use the income stream $\{x_t\}_{t=0}^{T-1}$ to repay the debt over the periods $t = 0, 1, 2, ..., T-1$. In general, when Jones wishes to have a time profile on the payment stream different from the income stream, she can attain this through appropriate transactions in the loan market, leaving her with any stream of payments of the same present value as the income stream.

The good which is traded in the loan market will here be referred to as a *bond*. The borrower issues bonds and the lender buys them. In this chapter all bonds are assumed to be short-term, i.e., one-period bonds. For every unit of account borrowed in the form of a one-period loan at the end of period $t-1$, the borrower pays back with certainty (1 + short-term interest rate) units of account at the end of period t. If a borrower wishes to maintain debt through several periods, new bonds are issued and the obtained loans are spent rolling over the older loans at the going market interest rate. For the lender, who lends in several periods, this is equivalent to a variable-rate demand deposit in a bank.[2]

Real versus nominal rate of return As in the preceding chapters our analysis will be in real terms, also called inflation-corrected terms. In principle the unit of account is a fixed bundle of consumption goods. In the simple macroeconomic models to be studied in this and subsequent chapters, such a bundle is reduced to *one* consumption good because the models assume there *is* only one consumption good in the economy. Moreover, there will only be *one produced good*, "the" output good, which can be used for both consumption and capital investment. Then, whether we say our unit of account is the consumption good or the output good does not matter. To fix our language, we will say the latter.

The *real* (net) rate of return on an investment is the rate of return in units of the output good. More precisely, the *real rate of return* in period t, r_t, is the (proportionate) rate at which the *real* value of an investment, made at the end of period $t-1$, has grown after one period.

[1] We use "present value" as synonymous with "present discounted value". As usual our timing convention is such that PV_0 denotes the time-0 value of the payment stream, including the discounted value of the payment (or dividend) indexed by 0.

[2] Unless otherwise specified, we use terms like "loan market", "credit market", and "bond market" interchangeably.

The link between this rate of return and the more commonplace concept of a nominal rate of return is the following. Imagine that at the end of period $t-1$ you make a deposit of value V_t euro on an account in a bank. The *real value* of the deposit when you invest is then V_t/P_{t-1}, where P_{t-1} is the price in euro of the output good at the end of period $t-1$. If the nominal short-term interest rate is i_t, the deposit is worth $V_{t+1} = V_t(1+i_t)$ euro at the end of period t. By definition of r_t, the factor by which the deposit in real terms has expanded is

$$1 + r_t = \frac{V_{t+1}/P_t}{V_t/P_{t-1}} = \frac{V_{t+1}/V_t}{P_t/P_{t-1}} = \frac{1+i_t}{1+\pi_t}, \tag{9.2}$$

where $\pi_t \equiv (P_t - P_{t-1})/P_{t-1}$ is the inflation rate in period t. So the real (net) *rate* of return on the investment is $r_t = (i_t - \pi_t)/(1+\pi_t) \approx i_t - \pi_t$ for i_t and π_t "small". The number $1 + r_t$ is called the real interest *factor* and measures the rate at which current units of output can be traded for units of output one period later.

In the remainder of this chapter we will think in terms of *real* values and completely ignore monetary aspects of the economy.

9.2 Maximizing discounted utility in discrete time

We assume that the consumption/saving problem faced by the household involves only one consumption good. So the composition of consumption in each period is not part of the problem. What remains is the question how to distribute consumption over time.

The intertemporal utility function

A plan for consumption in the periods $0, 1, ..., T-1$ is denoted $\{c_t\}_{t=0}^{T-1}$, where c_t is the consumption in period t. We say the plan has *time horizon* T. We assume the preferences of the household can be represented by a time-separable intertemporal utility function with a constant utility discount rate and no utility from leisure. The latter assumption implies that the labor supply of the household in each period is inelastic. The time-separability itself just means that the intertemporal utility function is additive, i.e., $U(c_0, c_1, \ldots, c_{T-1}) = u^{(0)}(c_0) + u^{(1)}(c_1) + \ldots + u^{(T-1)}(c_{T-1})$, where $u^{(t)}(c_t)$ is the utility contribution from period-t consumption, $t = 0, 1, \ldots, T-1$. But in addition we assume there is a constant utility discount rate $\rho > -1$, implying that $u^{(t)}(c_t) = u(c_t)(1+\rho)^{-t}$, where $u(c)$ is a time-independent period utility function. Together these two assumptions amount to

$$U(c_0, c_1, \cdots, c_{T-1}) = u(c_0) + \frac{u(c_1)}{1+\rho} + \ \ldots + \frac{u(c_{T-1})}{(1+\rho)^{T-1}} = \sum_{t=0}^{T-1} \frac{u(c_t)}{(1+\rho)^t}. \quad (9.3)$$

The period utility function is assumed to satisfy $u'(c) > 0$ and $u''(c) < 0$.

Box 9.1. Admissible transformations of the period utility function

When preferences, as assumed here, can be represented by *discounted utility*, the concept of utility appears at two levels. The function $U(\cdot)$ in (9.3) is defined on the set of alternative feasible consumption paths and corresponds to an ordinary utility function in general microeconomic theory. That is, $U(\cdot)$ will express the same ranking between alternative consumption paths as any increasing transformation of $U(\cdot)$. The period utility function, $u(\cdot)$, defined on the consumption in a single period, is a less general concept, requiring that reference to "utility units" is legitimate. That is, the *size* of the difference in terms of period utility between two outcomes has significance for choices. Indeed, the essence of the discounted utility hypothesis is that we have, for example,

$$u(c_0) - u(c_0') > 0.95\left[u(c_1') - u(c_1)\right] \Leftrightarrow (c_0, c_1) \succ (c_0', c_1'),$$

meaning that the household, having a utility discount factor $1/(1+\rho) = 0.95$, strictly prefers consuming (c_0, c_1) to (c_0', c_1') in the first two periods, if and only if the utility differences satisfy the indicated inequality. (The notation $x \succ y$ means that x is strictly preferred to y.)

Only a *linear* positive transformation of the utility function $u(\cdot)$, that is, $v(c) = au(c) + b$, where $a > 0$, leaves the ranking of all possible alternative consumption paths, $\{c_t\}_{t=0}^{T-1}$, unchanged. This is because a linear positive transformation does not affect the *ratios* of marginal period utilities (the marginal rates of substitution across time).

To avoid corner solutions we impose the No Fast Assumption $\lim_{c\to 0} u'(c) = \infty$. As (9.3) indicates, the number $1+\rho$ tells how many extra units of utility in the next period the household insists on to compensate for a decrease of one unit of utility in the current period. So, a $\rho > 0$ will reflect that if the chosen level of consumption is the same in two periods, then the individual always appreciates a marginal unit of consumption higher if it arrives in the earlier period. This explains why ρ is named the *rate of time preference* or even more to the point the *rate of impatience.* The *utility discount factor,*

$1/(1+\rho)^t$, indicates how many units of utility the household is at most willing to give up in period 0 to get one additional unit of utility in period t.[3]

It is generally believed that human beings are impatient and that ρ should therefore be assumed positive; indeed, it seems intuitively reasonable that the distant future does not matter much for current private decisions.[4] There is, however, a growing body of evidence suggesting that the utility discount rate is generally not constant, but declining with the time distance from the current period to the future periods within the horizon. Since this last point complicates the models considerably, macroeconomics often, as a first approach, ignores it and assumes a constant ρ to keep things simple. Here we follow this practice. Except where needed, we shall not, however, impose any other constraint on ρ than the definitional requirement in discrete time that $\rho > -1$.

As explained in Box 9.1, only *linear* positive transformations of the period utility function are admissible.

The saving problem in discrete time

Suppose the household considered has income from two sources: work and financial wealth. Let a_t denote the real value of financial wealth held by the household at the beginning of period t (a for "assets"). We treat a_t as predetermined at time t and in this respect similar to a variable-interest deposit in a bank. The initial financial wealth, a_0, is thus *given*, independently of whatever might happen to expected future interest rates. And a_0 can be positive as well as negative (in the latter case the household is initially in debt).

The labor income of the household in period t is denoted $w_t \geq 0$ and may follow a typical life-cycle pattern with labor income first rising, then more or less stationary, and finally vanishing due to retirement. Thus, in contrast to previous chapters where w_t denoted the real wage per unit of labor, here a broader interpretation of w_t is allowed. Whatever the time profile of the amount of labor delivered by the household through life, in this chapter, where the focus is on individual saving, we regard this time profile, as well as the hourly wage as exogenous. The present interpretation of w_t will coincide

[3]Multiplying through in (9.3) by $(1+\rho)^{-1}$ would make the objective function appear in a way similar to (9.1) in the sense that also the first term in the sum becomes discounted. At the same time the ranking of all possible alternative consumption paths would remain unaffected. For ease of notation, however, we use the form (9.3) which is more standard.

[4]If uncertainty were included in the model, $(1+\rho)^{-1}$ might be seen as reflecting the probability of surviving to the next period and in this perspective $\rho > 0$ seems a plausible assumption.

with the one in the other chapters if we imagine that the household in each period delivers one unit of labor.

Since uncertainty is by assumption ruled out, the problem is to choose a plan $(c_0, c_1, \ldots, c_{T-1})$ so as to maximize

$$\begin{aligned} U &= \sum_{t=0}^{T-1} u(c_t)(1+\rho)^{-t} \qquad \text{s.t.} & (9.4)\\ c_t &\geq 0, & (9.5)\\ a_{t+1} &= (1+r_t)a_t + w_t - c_t, \qquad a_0 \text{ given}, & (9.6)\\ a_T &\geq 0, & (9.7) \end{aligned}$$

where r_t is the interest rate. The control region (9.5) reflects the definitional non-negativity of consumption. The dynamic equation (9.6) is an accounting relation telling how financial wealth moves over time. Indeed, income in period t is $r_t a_t + w_t$ and saving is then $r_t a_t + w_t - c_t$. Since saving is by definition the same as the increase in financial wealth, $a_{t+1} - a_t$, we obtain (9.6). Finally, the terminal condition (9.7) is a solvency requirement that no financial debt be left over at the terminal date, T. We shall refer to this decision problem as the *standard discounted utility maximization problem without uncertainty.*

Solving the problem

To solve the problem, let us use the *substitution method.*[5] From (9.6) we have $c_t = (1 + r_t)a_t + w_t - a_{t+1}$, for $t = 0, 1, \ldots, T-1$. Substituting this into (9.4), we obtain a function of $a_1, a_2, \ldots, a_T$. Since $u' > 0$, saturation is impossible and so an optimal solution cannot have $a_T > 0$. Hence we can put $a_T = 0$ and the problem is reduced to an essentially unconstrained problem of maximizing a function $\tilde{U}$ w.r.t. $a_1, a_2, \ldots, a_{T-1}$. Thereby we indirectly choose $c_0, c_1, \ldots, c_{T-2}$. Given a_{T-1}, consumption in the last period is trivially given as

$$c_{T-1} = (1 + r_{T-1})a_{T-1} + w_{T-1},$$

ensuring $a_T = 0$.

To obtain first-order conditions we put the partial derivatives of $\tilde{U}$ w.r.t. a_{t+1}, $t = 0, 1, \ldots, T-2$, equal to 0:

$$\frac{\partial \tilde{U}}{\partial a_{t+1}} = (1+\rho)^{-t}\left[u'(c_t)\cdot(-1) + (1+\rho)^{-1}u'(c_{t+1})(1+r_{t+1})\right] = 0.$$

[5] Alternative methods include the *Maximum Principle* as described in the previous chapter or *Dynamic Programming* as described in the appendix to Chapter 29.

Reordering gives the Euler equations describing the trade-off between consumption in two succeeding periods,

$$u'(c_t) = (1+\rho)^{-1}u'(c_{t+1})(1+r_{t+1}), \qquad t = 0, 1, 2, ..., T-2. \tag{9.8}$$

Interpretation The interpretation of (30.14) is as follows. Let the consumption path $(c_0, c_1, \ldots, c_{T-1})$ be our "reference path". Imagine an alternative path which coincides with the reference path except for the periods t and $t+1$. If it is possible to obtain a higher total discounted utility than in the reference path by varying c_t and c_{t+1} within the constraints (9.5), (9.6), and (9.7), at the same time as consumption in the other periods is kept unchanged, then the reference path cannot be optimal. That is, "local optimality" is a necessary condition for "global optimality". So the optimal plan must be such that the current utility loss by decreasing consumption c_t by one unit equals the discounted expected utility gain next period by having $1+r_{t+1}$ extra units available for consumption, namely the gross return on saving one more unit in the current period.

A more concrete interpretation, avoiding the notion of "utility units", is obtained by rewriting (30.14) as

$$\frac{u'(c_t)}{(1+\rho)^{-1}u'(c_{t+1})} = 1 + r_{t+1}. \tag{9.9}$$

The left-hand side indicates the marginal rate of substitution, MRS, of period-$(t+1)$ consumption for period-t consumption, namely the increase in period-$(t+1)$ consumption needed to compensate for a one-unit marginal decrease in period-t consumption:

$$MRS_{t+1,t} = -\frac{dc_{t+1}}{dc_t}|_{U=\bar{U}} = \frac{u'(c_t)}{(1+\rho)^{-1}u'(c_{t+1})}. \tag{9.10}$$

And the right-hand side of (9.9) indicates the marginal rate of transformation, MRT, which is the rate at which the loan market allows the household to shift consumption from period t to period $t+1$. In an optimal plan MRS must equal MRT.

The formula (9.10) for MRS indicates why the assumption of a constant utility discount rate is convenient (but also restrictive). The marginal rate of substitution between consumption this period and consumption next period is independent of the level of consumption as long as this level is the same in the two periods.

Moreover, the formula for MRS between consumption this period and consumption *two* periods ahead is

$$MRS_{t+2,t} = -\frac{dc_{t+2}}{dc_t}|_{U=\bar{U}} = \frac{u'(c_t)}{(1+\rho)^{-2}u'(c_{t+2})}.$$

This displays one of the reasons that the time-separability of the intertemporal utility function is a strong assumption. It implies that the trade-off between consumption this period and consumption two periods ahead is independent of consumption in the interim.

Deriving a consumption function The first-order conditions (30.14) tell us about the relative consumption levels over time, not the absolute level. The latter is determined by the condition that initial consumption, c_0, must be highest possible, given that the first-order conditions *and* the constraints (9.6) and (9.7) must be satisfied.

To find an explicit solution we have to specify the period utility function. As an example we choose the CRRA function $u(c) = c^{1-\theta}/(1-\theta)$, where $\theta > 0$.[6] Moreover we simplify by assuming $r_t = r$, a constant > -1. Then the Euler equations take the form $(c_{t+1}/c_t)^{\theta} = (1+r)(1+\rho)^{-1}$ so that

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+r}{1+\rho}\right)^{1/\theta} \equiv \gamma, \tag{9.11}$$

and thereby $c_t = \gamma^t c_0$, $t = 0, 1, \ldots, T-1$. Substituting into the accounting equation (9.6), we thus have $a_{t+1} = (1+r)a_t + w_t - \gamma^t c_0$. By backward substitution we find the solution of this difference equation to be

$$a_t = (1+r)^t \left[a_0 + \sum_{i=0}^{t-1}(1+r)^{-(i+1)}(w_i - \gamma^i c_0)\right].$$

Optimality requires that the left-hand side of this equation vanishes for $t = T$. So we can solve for c_0 :

$$c_0 = \frac{1+r}{\sum_{i=0}^{T-1}\left(\frac{\gamma}{1+r}\right)^i}\left[a_0 + \sum_{i=0}^{T-1}(1+r)^{-(i+1)}w_i\right] = \frac{1+r}{\sum_{i=0}^{T-1}\left(\frac{\gamma}{1+r}\right)^i}(a_0 + h_0), \tag{9.12}$$

where we have inserted the human wealth of the household (present value of expected lifetime labor income) as seen from time zero:

$$h_0 = \sum_{i=0}^{T-1}(1+r)^{-(i+1)}w_i. \tag{9.13}$$

[6] In later sections of this chapter we will let the time horizon of the decision maker go to infinity. To ease convergence of an infinite sum of discounted utilities, it is an advantage not to have to bother with additive constants in the period utilities and therefore we write the CRRA function as $c^{1-\theta}/(1-\theta)$ instead of the form, $(c^{1-\theta}-1)/(1-\theta)$, introduced in Chapter 3. As implied by Box 9.1, the two forms represent the same preferences.

Thus (9.12) says that initial consumption is proportional to initial total wealth, the sum of financial wealth and human wealth at time 0. To allow positive consumption we need $a_0 + h_0 > 0$. This can be seen as a solvency condition which we assume satisfied.

As (9.12) indicates, the propensity to consume out of total wealth depends on:

$$\sum_{i=0}^{T-1}\left(\frac{\gamma}{1+r}\right)^i = \begin{cases} \frac{1-\left(\frac{\gamma}{1+r}\right)^T}{1-\frac{\gamma}{1+r}} & \text{when } \gamma \neq 1+r, \\ T & \text{when } \gamma = 1+r. \end{cases} \tag{9.14}$$

where the result for $\gamma \neq 1+r$ follows from the formula for the sum of a finite geometric series. Inserting this together with (9.11) into (9.12), we end up with a *candidate* consumption function,

$$c_0 = \begin{cases} \frac{(1+r)\left[1-(1+\rho)^{-1/\theta}(1+r)^{(1-\theta)/\theta}\right]}{1-(1+\rho)^{-T/\theta}(1+r)^{(1-\theta)T/\theta}}(a_0+h_0) & \text{when } \left(\frac{1+r}{1+\rho}\right)^{1/\theta} \neq 1+r, \\ \frac{1+r}{T}(a_0+h_0) & \text{when } \left(\frac{1+r}{1+\rho}\right)^{1/\theta} = 1+r. \end{cases} \tag{9.15}$$

For the subsequent periods we have from (9.11) that $c_t = ((1+r)/(1+\rho))^{t/\theta} c_0$, $t = 1,\dots,T-1$.

EXAMPLE 1 Consider the special case $\theta = 1$ (i.e., $u(c) = \ln c$) together with $\rho > 0$. The upper case in (9.15) is here the relevant one and period-0 consumption will be

$$c_0 = \frac{(1+r)(1-(1+\rho)^{-1})}{1-(1+\rho)^{-T}}(a_0+h_0) \qquad \text{for } \theta = 1.$$

We see that $c_0 \to (1+r)\rho(1+\rho)^{-1}(a_0+h_0)$ for $T \to \infty$.

We have assumed that payment for consumption occurs at the end of the period at the price 1 per consumption unit. To compare with the corresponding result in continuous time with continuous compounding (see Section 9.4), we might want to have initial consumption in the same present value terms as a_0 and h_0. That is, we consider $\tilde{c}_0 \equiv c_0(1+r)^{-1} = \rho(1+\rho)^{-1}(a_0+h_0)$. □

That our candidate consumption function is indeed an optimal solution when $a_0 + h_0 > 0$ follows by concavity of the objective function (or by concavity of the Hamiltonian if one applies the Maximum Principle of the previous chapter). The conclusion is that under the idealized conditions assumed, including a perfect loan market and perfect foresight, it is only initial wealth and the interest rate that affect the time profile of consumption. The time profile of income does not matter because consumption can be smoothed over time by drawing on the bond market. *Consumers look beyond current income.*

EXAMPLE 2 Consider the special case $\rho = r > 0$. Again the upper case in (9.15) is the relevant one and period-0 consumption will be

$$c_0 = \frac{r}{1-(1+r)^{-T}}(a_0 + h_0).$$

We see that $c_0 \to r(a_0 + h_0)$ for $T \to \infty$. So, with an infinite time horizon current consumption equals the interest on total current wealth. By consuming this the individual or household maintains total wealth intact. This consumption function provides an interpretation of Milton Friedman's *permanent income hypothesis*. Friedman defined "permanent income" as "the amount a consumer unit could consume (or believes it could) while maintaining its wealth intact" (Friedman, 1957). The key point of Friedman's theory was the idea that a random change in current income only affects current consumption to the extent that it affects "permanent income". □

Alternative approach based on the intertemporal budget constraint

There is another approach to the household's saving problem. With its choice of consumption plan the household must act in conformity with its intertemporal budget constraint (IBC for short). The present value of the consumption plan $(c_1, ..., c_{T-1})$, as seen from time zero, is

$$PV(c_0, c_1, ..., c_{T-1}) \equiv \sum_{t=0}^{T-1} \frac{c_t}{\Pi_{\tau=0}^{t}(1+r_\tau)}. \tag{9.16}$$

This value cannot exceed the household's total initial wealth, $a_0 + h_0$. So the household's *intertemporal budget constraint* is

$$\sum_{t=0}^{T-1} \frac{c_t}{\Pi_{\tau=0}^{t}(1+r_\tau)} \leq a_0 + h_0. \tag{9.17}$$

In this setting the household's problem is to choose its consumption plan so as to maximize U in (9.4) subject to this budget constraint.

This way of stating the problem is equivalent to the approach above based on the dynamic budget condition (9.6) and the solvency condition (9.7). Indeed, given the accounting equation (9.6), the consumption plan of the household will satisfy the intertemporal budget constraint (9.17) if and only if it satisfies the solvency condition (9.7). And there will be strict equality in the intertemporal budget constraint if and only if there is strict equality in the solvency condition (the proof is similar to that of a similar claim relating to the government sector in Chapter 6.2).

Moreover, since in our specific saving problem saturation is impossible, an optimal solution must imply strict equality in (9.17). So it is straightforward to apply the substitution method also within the IBC approach. Alternatively one can introduce the *Lagrange function* associated with the problem of maximizing $U = \sum_{t=0}^{T-1}(1+\rho)^{-t}u(c_t)$ s.t. (9.17) with strict equality.

9.3 Transition to continuous time analysis

In the discrete time framework the run of time is divided into successive periods of equal length, taken as the time-unit. Let us here index the periods by $i = 0, 1, 2, ...$ Thus financial wealth accumulates according to

$$a_{i+1} - a_i = s_i, \qquad a_0 \text{ given},$$

where s_i is (net) saving in period i.

Multiple compounding per year

With time flowing continuously, we let $a(t)$ refer to financial wealth at time t. Similarly, $a(t+\Delta t)$ refers to financial wealth at time $t+\Delta t$. To begin with, let Δt equal one time unit. Then $a(i\Delta t)$ equals $a(i)$ and is of the same value as a_i. Consider the *forward* first difference in a, $\Delta a(t) \equiv a(t+\Delta t) - a(t)$. It makes sense to consider this change in a in relation to the length of the time interval involved, that is, to consider the *ratio* $\Delta a(t)/\Delta t$. As long as $\Delta t = 1$, with $t = i\Delta t$ we have $\Delta a(t)/\Delta t = (a_{i+1} - a_i)/1 = a_{i+1} - a_i$. Now, keep the time unit unchanged, but let the length of the time interval $[t, t+\Delta t)$ approach zero, i.e., let $\Delta t \to 0$. When $a(\cdot)$ is a differentiable function, we have

$$\lim_{\Delta t \to 0} \frac{\Delta a(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{a(t+\Delta t) - a(t)}{\Delta t} = \frac{da(t)}{dt},$$

where $da(t)/dt$, often written $\dot{a}(t)$, is known as the *derivative of* $a(\cdot)$ at the point t. Wealth accumulation in continuous time can then be written

$$\dot{a}(t) = s(t), \qquad a(0) = a_0 \text{ given}, \tag{9.18}$$

where $s(t)$ is the saving flow at time t. For Δt "small" we have the approximation $\Delta a(t) \approx \dot{a}(t)\Delta t = s(t)\Delta t$. In particular, for $\Delta t = 1$ we have $\Delta a(t) = a(t+1) - a(t) \approx s(t)$.

As time unit choose one year. Going back to discrete time we have that if wealth grows at a constant rate $g > 0$ per year, then after i periods of length one year, with annual compounding, we have

$$a_i = a_0(1+g)^i, \quad i = 0, 1, 2, \ldots . \tag{9.19}$$

If instead compounding (adding saving to the principal) occurs n times a year, then after i periods of length $1/n$ year and a growth rate of g/n per such period,

$$a_i = a_0(1 + \frac{g}{n})^i. \tag{9.20}$$

With t still denoting time measured in years passed since date 0, we have $i = nt$ periods. Substituting into (9.20) gives

$$a(t) = a_{nt} = a_0(1 + \frac{g}{n})^{nt} = a_0 \left[(1 + \frac{1}{m})^m\right]^{gt}, \qquad \text{where } m \equiv \frac{n}{g}.$$

We keep g and t fixed, but let $n \to \infty$ and thus $m \to \infty$. Then, in the limit there is continuous compounding and it can be shown that

$$a(t) = a_0 e^{gt}, \tag{9.21}$$

where e is a mathematical constant called the base of the natural logarithm and defined as $e \equiv \lim_{m\to\infty}(1 + 1/m)^m \simeq 2.7182818285....$

The formula (9.21) is the continuous-time analogue to the discrete time formula (9.19) with annual compounding. A geometric growth factor is replaced by an exponential growth factor.

We can also view the formulas (9.19) and (9.21) as the solutions to a difference equation and a differential equation, respectively. Thus, (9.19) is the solution to the linear difference equation $a_{i+1} = (1+g)a_i$, given the initial value a_0. And (9.21) is the solution to the linear differential equation $\dot{a}(t) = ga(t)$, given the initial condition $a(0) = a_0$. Now consider a time-dependent growth rate, $g(t)$. The corresponding differential equation is $\dot{a}(t) = g(t)a(t)$ and it has the solution

$$a(t) = a(0)e^{\int_0^t g(\tau)d\tau}, \tag{9.22}$$

where the exponent, $\int_0^t g(\tau)d\tau$, is the definite integral of the function $g(\tau)$ from 0 to t. The result (9.22) is called the *basic accumulation formula* in continuous time and the factor $e^{\int_0^t g(\tau)d\tau}$ is called the *growth factor* or the *accumulation factor*.[7]

Compound interest and discounting in continuous time

Let $r(t)$ denote the *short-term real interest rate in continuous time* at time t. To clarify what is meant by this, consider a deposit of $V(t)$ euro on a drawing account in a bank at time t. If the general price level in the economy at time

[7] Sometimes the accumulation factor with time-dependent growth rate is written in a different way, see Appendix B.

t is $P(t)$ euro, the *real* value of the deposit is $a(t) = V(t)/P(t)$ at time t. By definition the *real rate of return* on the deposit in continuous time (with continuous compounding) at time t is the (proportionate) instantaneous rate at which the real value of the deposit expands per time unit when there is no withdrawal from the account. Thus, if the instantaneous nominal interest rate is $i(t)$, we have $\dot{V}(t)/V(t) = i(t)$ and so, by the fraction rule in continuous time (cf. Appendix A),

$$r(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\dot{V}(t)}{V(t)} - \frac{\dot{P}(t)}{P(t)} = i(t) - \pi(t), \tag{9.23}$$

where $\pi(t) \equiv \dot{P}(t)/P(t)$ is the instantaneous inflation rate. In contrast to the corresponding formula in discrete time, this formula is exact. Sometimes $i(t)$ and $r(t)$ are referred to as the nominal and real *interest intensity*, respectively, or the nominal and real *force of interest.*

Calculating the terminal value of the deposit at time $t_1 > t_0$, given its value at time t_0 and assuming no withdrawal in the time interval $[t_0, t_1]$, the accumulation formula (9.22) immediately yields

$$a(t_1) = a(t_0)e^{\int_{t_0}^{t_1} r(t)dt}.$$

When calculating *present values* in continuous time analysis, we use compound discounting. We simply reverse the accumulation formula and go from the compounded or terminal value to the present value $a(t_0)$. Similarly, given a consumption plan, $(c(t))_{t=t_0}^{t_1}$, the present value of this plan as seen from time t_0 is

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-rt} dt, \tag{9.24}$$

presupposing a constant interest rate. Instead of the geometric discount factor, $1/(1 + r)^t$, from discrete time analysis, we have here an exponential discount factor, $1/(e^{rt}) = e^{-rt}$, and instead of a sum, an integral. When the interest rate varies over time, (9.24) is replaced by

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-\int_{t_0}^{t} r(\tau)d\tau} dt.$$

In (9.24) $c(t)$ is discounted by $e^{-rt} \approx (1 + r)^{-t}$ for r "small". This might not seem analogue to the discrete-time discounting in (9.16) where it is c_{t-1} that is discounted by $(1 + r)^{-t}$, assuming a constant interest rate. When taking into account the timing convention that payment for c_{t-1} in period $t - 1$ occurs at the end of the period (= time t), there is no discrepancy, however, since the continuous-time analogue to this payment is $c(t)$.

The allowed range for parameter values

The allowed range for parameters may change when we go from discrete time to continuous time with continuous compounding. For example, the usual equation for aggregate capital accumulation in continuous time is

$$\dot{K}(t) = I(t) - \delta K(t), \qquad K(0) = K_0 \text{ given}, \tag{9.25}$$

where $K(t)$ is the capital stock, $I(t)$ is the gross investment at time t and $\delta \geq 0$ is the (physical) capital depreciation rate. Unlike in discrete time, here $\delta > 1$ is conceptually allowed. Indeed, suppose for simplicity that $I(t) = 0$ for all $t \geq 0$; then (9.25) gives $K(t) = K_0 e^{-\delta t}$. This formula is meaningful for any $\delta \geq 0$. Usually, the time unit used in continuous time macro models is one year (or, in business cycle theory, rather a quarter of a year) and then a realistic value of δ is of course < 1 (say, between 0.05 and 0.10). However, if the time unit applied to the model is large (think of a Diamond-style OLG model), say 30 years, then $\delta > 1$ may fit better, empirically, if the model is converted into continuous time with the same time unit. Suppose, for example, that physical capital has a half-life of 10 years. With 30 years as our time unit, inserting into the formula $1/2 = e^{-\delta/3}$ gives $\delta = (\ln 2) \cdot 3 \simeq 2$.

In many simple macromodels, where the level of aggregation is high, the relative price of a unit of physical capital in terms of the consumption good is 1 and thus constant. More generally, if we let the relative price of the capital good in terms of the consumption good at time t be $p(t)$ and allow $\dot{p}(t) \neq 0$, then we have to distinguish between the physical depreciation of capital, δ, and the *economic depreciation*, that is, the loss in economic value of a machine per time unit. The economic depreciation will be $d(t) = p(t)\delta - \dot{p}(t)$, namely the economic value of the physical wear and tear (and technological obsolescence, say) minus the capital gain (positive or negative) on the machine.

Other variables and parameters that by definition are bounded from below in discrete time analysis, but not so in continuous time analysis, include rates of return and discount rates in general.

Stocks and flows

An advantage of continuous time analysis is that it forces the analyst to make a clear distinction between *stocks* (say wealth) and *flows* (say consumption or saving). Recall, a *stock* variable is a variable measured as a quantity at a given point in time. The variables $a(t)$ and $K(t)$ considered above are stock variables. A *flow* variable is a variable measured as quantity *per time unit* at a given point in time. The variables $s(t)$, $\dot{K}(t)$ and $I(t)$ are flow variables.

One can not add a stock and a flow, because they have *different denominations*. What exactly is meant by this? The elementary measurement units in economics are *quantity units* (so many machines of a certain kind or so many liters of oil or so many units of payment, for instance) and *time units* (months, quarters, years). On the basis of these we can form *composite measurement units*. Thus, the capital stock, K, has the denomination "quantity of machines", whereas investment, I, has the denomination "quantity of machines per time unit" or, shorter, "quantity/time". A growth rate or interest rate has the denomination "(quantity/time)/quantity" = "time^{-1}". If we change our time unit, say from quarters to years, the value of a flow variable as well as a growth rate is changed, in this case quadrupled (presupposing annual compounding).

In continuous time analysis expressions like $K(t)+I(t)$ or $K(t)+\dot{K}(t)$ are thus illegitimate. But one can write $K(t+\Delta t) \approx K(t)+(I(t)-\delta K(t))\Delta t$, or $\dot{K}(t)\Delta t \approx (I(t)-\delta K(t))\Delta t$. In the same way, suppose a bath tub at time t contains 50 liters of water and that the tap pours $\frac{1}{2}$ liter per second into the tub for some time. Then a sum like $50\ \ell + \frac{1}{2}\ (\ell/\text{sec})$ does not make sense. But the *amount* of water in the tub after one minute is meaningful. This amount would be $50\ \ell + \frac{1}{2} \cdot 60\ ((\ell/\text{sec})\times\text{sec}) = 80\ \ell$. In analogy, economic flow variables in continuous time should be seen as *intensities* defined for every t in the time interval considered, say the time interval $[0,\ T)$ or perhaps $[0,\ \infty)$. For example, when we say that $I(t)$ is "investment" at time t, this is really a short-hand for "investment intensity" at time t. The actual investment in a time interval $[t_0, t_0+\Delta t)$, i.e., the invested amount *during* this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} I(t)dt \approx I(t_0)\Delta t$. Similarly, the flow of individual saving, $s(t)$, should be interpreted as the saving *intensity* at time t. The actual saving in a time interval $[t_0, t_0+\Delta t)$, i.e., the saved (or accumulated) amount during this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} s(t)dt$. If Δt is "small", this integral is approximately equal to the product $s(t_0)\cdot\Delta t$, cf. the hatched area in Fig. 9.1.

The notation commonly used in discrete time analysis blurs the distinction between stocks and flows. Expressions like $a_{i+1} = a_i + s_i$, without further comment, are usual. Seemingly, here a stock, wealth, and a flow, saving, are added. In fact, however, it is wealth at the beginning of period i and the saved *amount during* period i that are added: $a_{i+1} = a_i + s_i \cdot \Delta t$. The tacit condition is that the period length, Δt, is the time unit, so that $\Delta t = 1$. But suppose that, for example in a business cycle model, the period length is one quarter, but the time unit is one year. Then saving in quarter i is $s_i = (a_{i+1} - a_i)\cdot 4$ per year.

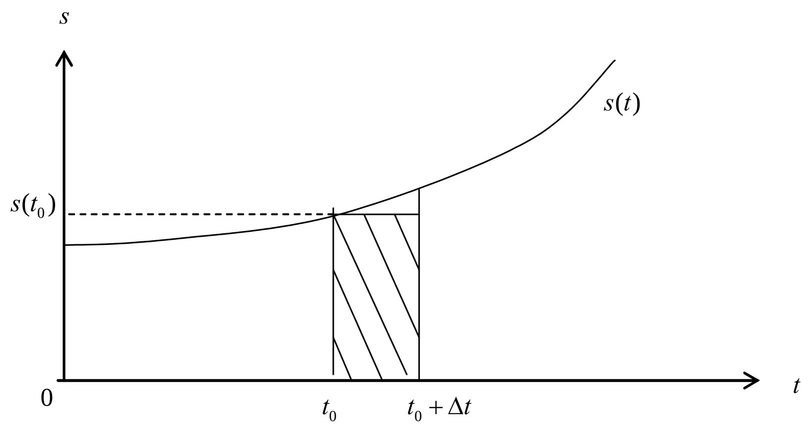

Figure 9.1: With Δt "small" the integral of $s(t)$ from t_0 to $t_0 + \Delta t$ is $\approx$ the hatched area.

The choice between discrete and continuous time analysis

In empirical economics, data typically come in discrete time form and data for flow variables typically refer to periods of constant length. One could argue that this discrete form of the data speaks for discrete time rather than continuous time modelling. And the fact that economic actors often think and plan in period terms, may seem a good reason for putting at least microeconomic analysis in period terms. Nonetheless real time is continuous. And it can hardly be said that the *mass* of economic actors think and plan with one and the same period. In macroeconomics we consider the *sum* of the actions. In this perspective the continuous time approach has the advantage of allowing variation *within* the usually artificial periods in which the data are chopped up. And for example centralized asset markets equilibrate almost instantaneously and respond immediately to new information. For such markets a formulation in continuous time seems preferable.

There is also a risk that a discrete time model may generate *artificial* oscillations over time. Suppose the "true" model of some mechanism is given by the differential equation

$$\dot{x} = \alpha x, \qquad \alpha < -1. \tag{9.26}$$

The solution is $x(t) = x(0)e^{\alpha t}$ which converges in a monotonic way toward 0 for $t \to \infty$. However, the analyst takes a discrete time approach and sets up the seemingly "corresponding" discrete time model

$$x_{t+1} - x_t = \alpha x_t.$$

This yields the difference equation $x_{t+1} = (1+\alpha)x_t$, where $1+\alpha < 0$. The solutuion is $x_t = (1+\alpha)^t x_0$, $t = 0, 1, 2, \ldots$. As $(1+\alpha)^t$ is positive when t is even and negative when t is odd, oscillations arise in spite of the "true" model generating monotonous convergence towards the steady state $x^* = 0$.

It should be added, however, that this potential problem *can* always be avoided within discrete time models by choosing a sufficiently *short* period length. Indeed, the solution to a differential equation can always be obtained as the limit of the solution to a corresponding difference equation for the period length approaching zero. In the case of (9.26) the approximating difference equation is $x_{i+1} = (1+\alpha\Delta t)x_i$, where Δt is the period length, $i = t/\Delta t$, and $x_i = x(i\Delta t)$. By choosing Δt small enough, the solution comes arbitrarily close to the solution of (9.26). It is generally more difficult to go in the opposite direction and find a differential equation that approximates a given difference equation. But the problem is solved as soon as a differential equation has been found that has the initial difference equation as an approximating difference equation.

From the point of view of the economic contents, the choice between discrete time and continuous time may be a matter of taste. From the point of view of mathematical convenience, the continuous time formulation, which has worked so well in the natural sciences, seems preferable. At least this is so in the absence of uncertainty. For problems where uncertainty is important, discrete time formulations are easier to work with unless one is familiar with stochastic calculus.

9.4 Maximizing discounted utility in continuous time

9.4.1 The saving problem in continuous time

In continuous time the analogue to the intertemporal utility function, (9.3), is

$$U_0 = \int_0^T u(c(t))e^{-\rho t}dt. \tag{9.27}$$

In this context it is common to name the utility flow, $u(\cdot)$, the *instantaneous utility function*. We still assume that $u' > 0$ and $u'' < 0$. The analogue to the intertemporal budget constraint in Section 9.2 is

$$\int_0^T c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq a_0 + h_0, \tag{9.28}$$

where, as above, a_0 is the historically given initial financial wealth (the value of the stock of short-term bonds held at time 0), while h_0 is the given human wealth,

$$h_0 = \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt. \tag{9.29}$$

The household's problem is then to choose a consumption plan $(c(t))_{t=0}^T$ so as to maximize discounted utility, U_0, subject to the budget constraint (9.28).

Infinite time horizon In the Ramsey model of the next chapter the idea is used that households may have an *infinite* time horizon. One interpretation of this is that parents care about their children's future welfare and leave bequests accordingly. This gives rise to a series of intergenerational links. The household is then seen as a family dynasty with a time horizon far beyond the lifetime of the current members of the family; Barro's bequest model in Chapter 7 is a discrete time application of this idea. Introducing a positive constant utility discount rate, less weight is attached to circumstances further away in the future and it may be ensured that the improper integral of achievable discounted utilities over an infinite horizon is bounded from above.

One could say, of course, that infinity is a long time. The sun will eventually, in some billion years, burn out and life on earth become extinct. Nonetheless an infinite time horizon may provide a useful substitute for finite but remote horizons. This is because the solution to an optimization problem for T "large" will in many cases in a large part of $[0, T]$ be close to the solution for $T \rightarrow \infty$.[8] And an infinite time horizon may make aggregation easier because at any future point in time, remaining time is still infinite. An infinite time horizon can also sometimes be a convenient notion when in any given period there is a positive probability that there will also be a next period to be concerned about. If this probability is low, it can simply be reflected in a high effective utility discount rate. This idea is applied in chapters 12 and 13.

We perform the transition to infinite horizon by letting $T \rightarrow \infty$ in (9.27), (9.28), and (9.29). In the limit the household's, or dynasty's, problem becomes one of choosing a plan, $(c(t))_{t=0}^{\infty}$, which maximizes

$$U_0 = \int_0^\infty u(c(t))e^{-\rho t}dt \qquad \text{s.t.} \tag{9.30}$$

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq a_0 + h_0, \tag{IBC}$$

where h_0 emerges by letting T in (9.29) approach ∞. Working with infinite horizons, there may exist technically feasible paths along which the improper

[8] The turnpike proposition in Chapter 8 exemplifies this.

integrals in (9.27), (9.28), and (9.29) go to ∞ for $T \to \infty$. In that case maximization is not well-defined. However, the assumptions that we are going to make when working with the Ramsey model will guarantee that the integrals converge as $T \to \infty$ (or at least that *some* feasible paths have $-\infty < U_0 < \infty$, while the remainder have $U_0 = -\infty$ and are thus clearly inferior). The essence of the matter is that the rate of time preference, ρ, must be assumed sufficiently high relative to the potential growth in instantaneous utility so as to ensure that the interest rate becomes higher than the long-run growth rate of income.

Generally we define a person as *solvent* if she is able to meet her financial obligations as they fall due. Each person is considered "small" relative to the economy as a whole. As long as all agents in the economy remain "small", they will in general equilibrium remain solvent if and only if their *gross* debt does not exceed their gross assets. Because of our assumption of a perfect loan market, "assets" should here be understood in the broadest possible sense, that is, including the present value of the expected future labor income. Considering *net* debt, d_0, the solvency requirement becomes

$$d_0 \leq \int_0^\infty (w(t) - c(t))e^{-\int_0^t r(\tau)d\tau} dt,$$

where the right-hand side of the inequality is the present value of the expected future primary saving.[9] By use of the definition in (9.29), it can be seen that this requirement is identical to the intertemporal budget constraint (IBC) which consequently expresses solvency.

The budget constraint in flow terms

The method which is particularly apt for solving intertemporal decision problems in continuous time is based on the mathematical discipline *optimal control theory*. To apply the method, we have to convert the household's budget constraint from the present value formulation considered above into flow terms.

By mere accounting, in every short time interval $(t, t+\Delta t)$ the household's consumption plus saving equals the household's total income, that is,

$$(c(t) + \dot{a}(t))\Delta t = (r(t)a(t) + w(t))\Delta t.$$

Here, $\dot{a}(t) \equiv da(t)/dt$ is saving and thus the same as the increase per time unit in financial wealth. If we divide through by Δt and isolate saving on

[9] By *primary* saving is meant the difference between current *earned* income and current consumption, where earned income means income before interest transfers.

the left-hand side of the equation, we get for all $t \geq 0$

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t), \qquad a(0) = a_0 \text{ given.} \tag{9.31}$$

This equation in itself is just a dynamic budget identity. It tells us by how much and in which direction the financial wealth is changing due to the difference between current income and current consumption. The equation *per se* does not impose any restriction on consumption over time. If this equation were the only "restriction", one could increase consumption indefinitely by incurring an increasing debt without limits. It is not until we add the requirement of solvency that we get a *constraint.* When $T < \infty$, the relevant solvency requirement is $a(T) \geq 0$ (that is, no debt left over at the terminal date). This is equivalent to satisfying the intertemporal budget constraint (9.28). When $T = \infty$, the relevant solvency requirement is a No-Ponzi-Game condition:

$$\lim_{t \to \infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0, \tag{NPG}$$

i.e., the present value of *debts,* measured as $-a(t)$, infinitely far out in the future, is not permitted to be positive. Indeed, we have the following equivalency:

PROPOSITION 1 *(equivalence of NPG condition and intertemporal budget constraint)* Let the time horizon be infinite and assume that the integral (9.29) remains finite for $T \to \infty$. Then, given the accounting relation (9.31), we have:
(i) the requirement (NPG) is satisfied if and only if the intertemporal budget constraint, (IBC), is satisfied; and
(ii) there is strict equality in (NPG) if and only if there is strict equality in (IBC).

Proof. See Appendix C.

The condition (NPG) does not preclude that the household (or rather the family dynasty) can remain in debt. This would also be an unnatural requirement as the dynasty is infinitely-lived. The condition does imply, however, that there is an upper bound for the speed whereby debts can increase in the long term. The NPG condition says that in the long term, debts are not allowed to grow at a rate greater than or equal to the interest rate.

To understand the implication, let us look at the case where the interest rate is a constant, $r > 0$. Assume that the household at time t has net debt $d(t) > 0$, i.e., $a(t) \equiv -d(t) < 0$. If $d(t)$ were persistently growing at a rate equal to or greater than the interest rate, (NPG) would be violated.[10] Equivalently, one can interpret (NPG) as an assertion that lenders will only issue loans if the borrowers in the long run are able to cover at least part of their interest payments by other means than by taking up new loans. In this way, it is avoided that $\dot{d}(t) \geq rd(t)$ in the long run, that is, the debt does not explode.

As mentioned in Chapter 6 the name "No-Ponzi-Game condition" refers to a guy, Charles Ponzi, who in Boston in the 1920s temporarily became very rich by a loan arrangement based on the chain letter principle. The fact that debts grow without bounds is irrelevant for the lender *if* the borrower can always find new lenders and use their outlay to pay off old lenders. In the real world, endeavours to establish this sort of financial eternity machine tend sooner or later to break down because the flow of new lenders dries up. Such financial arrangements, in everyday speech known as pyramid companies, are universally illegal.[11] It is exactly such arrangements the constraint (NPG) precludes.

9.4.2 Solving the saving problem

The household's consumption/saving problem is one of choosing a path for the *control variable* $c(t)$ so as to maximize a *criterion function,* in the form of an integral, subject to constraints that include a first-order differential equation. This equation determines the evolution of the *state variable*, $a(t)$. Optimal control theory, which in Chapter 8 was applied to a related discrete time problem, offers a well-suited apparatus for solving this kind of optimization problem. We will make use of a special case of Pontryagin's *Maximum Principle* (the basic tool of optimal control theory) in its continuous time version. We shall consider the case with a finite as well as infinite time horizon.

[10] Starting from a given initial positive debt, d_0, when $\dot{d}(t)/d(t) \geq r > 0$, we have $d(t) \geq d_0 e^{rt}$ so that $d(t)e^{-rt} \geq d_0 > 0$ for all $t \geq 0$. Consequently, $a(t)e^{-rt} = -d(t)e^{-rt} \leq -d_0 < 0$ for all $t \geq 0$, that is, $\lim_{t\to\infty} a(t)e^{-rt} < 0$, which violates (NPG).

[11] A related Danish instance, though on a modest scale, could be read in the Danish newpaper *Politiken* on the 21st of August 1992. "A twenty-year-old female student from Tylstrup in Northern Jutland is charged with fraud. In an ad, she offered 200 DKK to tell you how to make easy money. Some hundred people responded and received the reply: Do like me."

A more serious present day example is the American stockbroker, Bernard Madoff, who admitted a Ponzi scheme that is considered to be the largest financial fraud in U.S. history. In 2009 Madoff was sentenced to 150 years in prison.

For $T < \infty$ the problem is: choose a plan $(c(t))_{t=0}^{T}$ that maximizes

$$\begin{aligned}
U_0 &= \int_0^T u(c(t))e^{-\rho t}dt \qquad \text{s.t.} && (9.32)\\
c(t) &\geq 0, \qquad \text{(control region)} && (9.33)\\
\dot{a}(t) &= r(t)a(t) + w(t) - c(t), \quad a(0) = a_0 \text{ given}, && (9.34)\\
a(T) &\geq 0. && (9.35)
\end{aligned}$$

With an infinite time horizon, T in (9.32) is interpreted as ∞ and the solvency condition (9.35) is replaced by

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0. \qquad \text{(NPG)}$$

Let I denote the time interval $[0,T]$ if $T < \infty$ and the time interval $[0,\infty)$ if $T = \infty$. If $c(t)$ and the corresponding evolution of $a(t)$ fulfil (9.33) and (9.34) for all $t \in I$ as well as the relevant solvency condition, we call $(a(t), c(t))_{t=0}^{T}$ an *admissible path*. If a given admissible path $(a(t), c(t))_{t=0}^{T}$ solves the problem, it is referred to as an *optimal path*.[12] We assume that $w(t)$ and $r(t)$ are piecewise continuous functions of t and that $w(t)$ is positive for all t. No condition on the impatience parameter, ρ, is imposed (in this chapter).

First-order conditions

The solution procedure for this problem is as follows:[13]

1. We set up the *current-value Hamiltonian function*:

$$H(a, c, \lambda, t) \equiv u(c) + \lambda(ra + w - c),$$

where λ is the *adjoint variable* (also called the *co-state variable*) associated with the dynamic constraint (9.34).[14] That is, λ is an auxiliary variable which is a function of t and is analogous to the Lagrange multiplier in static optimization.

[12] The term "path", sometimes "trajectory", is common in the natural sciences for a solution to a differential equation because one may think of this solution as the path of a particle moving in two- or three-dimensional space.

[13] The four-step solution procedure below is applicable to a large class of dynamic optimization problems in continuous time, see Math tools.

[14] The explicit dating of the time-dependent variables a, c, and λ is omitted where not needed for clarity.

2. At every point in time, we maximize the Hamiltonian w.r.t. the *control variable*, in the present case c. So, focusing on an *interior* optimal path,[15] we calculate
$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0.$$
For every $t \in I$ we thus have the condition
$$u'(c(t)) = \lambda(t). \tag{9.36}$$

3. We calculate the partial derivative of H with respect to the *state variable*, in the present case a, and put it equal to the difference between the discount rate (as it appears in the integrand of the criterion function) multiplied by λ and the time derivative of λ :
$$\frac{\partial H}{\partial a} = \lambda r = \rho\lambda - \dot{\lambda}.$$
That is, for all $t \in I$, the adjoint variable λ should fulfil the differential equation
$$\dot{\lambda}(t) = (\rho - r(t))\lambda(t). \tag{9.37}$$

4. We now apply the *Maximum Principle* which applied to this problem says: an interior optimal path $(a(t), c(t))_{t=0}^{T}$ will satisfy that there exits a continuous function $\lambda = \lambda(t)$ such that for all $t \in I$, (9.36) and (9.37) hold along the path, and the *transversality condition,*
$$\begin{aligned} a(T)\lambda(T)e^{-\rho T} &= 0, \text{ if } T < \infty, \text{ or} \\ \lim_{t\to\infty} a(t)\lambda(t)e^{-\rho t} &= 0, \text{ if } T = \infty, \end{aligned} \tag{TVC}$$
is satisfied.

An optimal path is thus characterized as a path that for every t maximizes the Hamiltonian associated with the problem. The intuition is that the Hamiltonian weighs the direct contribution of the marginal unit of the control variable to the criterion function in the "right" way relative to the indirect contribution, which comes from the generated change in the state variable (here financial wealth); "right" means in accordance with the opportunities offered by the rate of return vis-a-vis the time preference rate, ρ. The optimality condition (9.36) can be seen as a $MC = MB$ condition: on the margin one unit of account (here the consumption good) must be equally valuable in its two uses: consumption and wealth accumulation. Together

[15] A path, $(a_t, c_t)_{t=0}^{T}$, is an *interior* path if $c_t > 0$ for all $t \geq 0$.

with the optimality condition (9.37) this signifies that the adjoint variable λ can be interpreted as the *shadow price* (measured in units of current utility) of financial wealth along the optimal path.[16]

Remark. The current-value Hamiltonian function is often just called the *current-value Hamiltonian.* More importantly the prefix "current-value" is used to distinguish it from what is known as the *present-value Hamiltonian.* The latter is defined as $\hat{H} \equiv He^{-\rho t}$ with $\lambda e^{-\rho t}$ substituted by μ, which is the associated (discounted) adjoint variable. The solution procedure is similar except that step 3 is replaced by $\partial \hat{H}/\partial a = -\dot{\mu}$ and $\lambda(t)e^{-\rho t}$ in the transversality condition is replaced by $\mu(t)$. The two methods are equivalent. But for many economic problems the *current-value* Hamiltonian has the advantage that it makes both the calculations and the interpretation slightly simpler. The adjoint variable, $\lambda(t)$, which as mentioned acts as a shadow price of the state variable, is a *current* price along with the other prices in the problem, $w(t)$ and $r(t)$, in contrast to $\mu(t)$ which is a *discounted* price. $\square$

Reordering (9.37) gives

$$\frac{r\lambda + \dot{\lambda}}{\lambda} = \rho. \tag{9.38}$$

This can be interpreted as a no-arbitrage condition. The left-hand side gives the *actual* rate of return, measured in utility units, on the marginal unit of saving. Indeed, $r\lambda$ can be seen as a dividend and $\dot{\lambda}$ as a capital gain. The-right hand side is the *required* rate of return measured in utility terms, ρ. Along an optimal path the two must coincide. The household is willing to save the marginal unit only up to the point where the actual rate of return on saving equals the required rate.

We may alternatively write the no-arbitrage condition as

$$r = \rho - \frac{\dot{\lambda}}{\lambda}. \tag{9.39}$$

On the left-hand-side appears the actual *real* rate of return on saving and on the right-hand-side the *required real* rate of return. The intuition behind this condition can be seen in the following way. Suppose Ms. Jones makes a deposit of V utility units in a "bank" that offers a proportionate rate of expansion of the utility value of the deposit equal to i (assuming no withdrawal occurs), i.e.,

$$\frac{\dot{V}}{V} = i.$$

[16] Recall, a *shadow price* (measured in some unit of account) of a good is the number of units of account that the optimizing agent is just willing to offer for one extra unit of the good.

This is the actual *utility* rate of return, a kind of "nominal interest rate". To calculate the corresponding "real interest rate" let the "nominal price" of a consumption good be λ utility units. Dividing the number of invested utility units, V, by λ, we get the *real* value, $m = V/\lambda$, of the deposit at time t. The actual *real* rate of return on the deposit is therefore

$$r = \frac{\dot{m}}{m} = \frac{\dot{V}}{V} - \frac{\dot{\lambda}}{\lambda} = i - \frac{\dot{\lambda}}{\lambda}. \tag{9.40}$$

Ms. Jones is then just willing to save the marginal unit of income if this real rate of return on saving equals the required real rate, that is, the right-hand side of (9.39); in turn this necessitates that the "nominal interest rate" i equals the required nominal rate, ρ. The formula (9.40) is analogue to the discrete-time formula (9.2) except that the unit of account in (9.40) is current utility while in (9.2) it is currency.

Substituting (9.36) into the transversality condition for the case $T < \infty$, gives

$$a(T)e^{-\rho T}u'(c(T)) = 0. \tag{9.41}$$

Our solvency condition, $a(T) \geq 0$, can be seen as an example of a general inequality constraint, $a(T) \geq a_T$, where here a_T happens to equal 0. So (9.41) can be read as a standard complementary slackness condition. The (discounted) "price", $e^{-\rho T}u'(c(T))$, is always positive, hence an optimal plan must satisfy $a(T) = a_T (= 0)$. The alternative, $a(T) > 0$, would imply that consumption, and thereby U_0, could be increased by a decrease in $a(T)$ without violating the solvency requirement.

Now let $T \to \infty$. Then in the limit the solvency requirement is (NPG), and (9.41) is replaced by

$$\lim_{T\to\infty} a(T)e^{-\rho T}u'(c(T)) = 0. \tag{9.42}$$

This is the same as (TVC) (replace T by t). Intuitively, a plan that violates this condition by having ">" instead "=" indicates scope for improvement and thus cannot be optimal. There would be "purchasing power left for eternity" which could be transferred to consumption on earth at an earlier date.

Generally, care must be taken when extending a necessary transversality condition from a finite to an infinite horizon. But for the present problem, the extension *is* valid. Indeed, (TVC) is just a requirement that the NPG condition is not "over-satisfied":

PROPOSITION 2 *(the household's transversality condition with infinite time horizon)* Let $T \to \infty$ and assume the integral (9.29) remains finite for $T \to$

∞. Provided the adjoint variable, $\lambda(t)$, satisfies the optimality conditions (9.36) and (9.37), (TVC) holds if and only if (NPG) holds with strict equality.

Proof. See Appendix D.

In view of this proposition, we can write the transversality condition for $T \to \infty$ as the NPG condition with strict equality:

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} = 0. \tag{TVC'}$$

9.4.3 The Keynes-Ramsey rule

The first-order conditions have interesting implications. Differentiate both sides of (9.36) w.r.t. t to get $u''(c)\dot{c} = \dot{\lambda}$ which can be written as $u''(c)\dot{c}/u'(c) = \dot{\lambda}/\lambda$ by drawing on (9.36) again. Applying (9.37) now gives

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta(c(t))}(r(t) - \rho), \tag{9.43}$$

where $\theta(c)$ is the (absolute) *elasticity of marginal utility* w.r.t. consumption,

$$\theta(c) \equiv -\frac{c}{u'(c)}u''(c) > 0. \tag{9.44}$$

As in discrete time, $\theta(c)$ indicates the strength of the consumer's desire to smooth consumption. The inverse of $\theta(c)$ measures the *instantaneous intertemporal elasticity of substitution* in consumption, which in turn indicates the willingness to accept variation in consumption over time when the interest rate changes, see Appendix F.

The result (9.43) says that an optimal consumption plan is characterized in the following way. The household will completely smooth consumption over time if the rate of time preference equals the real interest rate. The household will choose an upward-sloping time path for consumption if and only if the rate of time preference is less than the real interest rate. Indeed, in this case the household would accept a relatively low level of current consumption with the purpose of enjoying more consumption in the future. The lower the rate of time preference relative to the real interest rate, the more favorable it becomes to defer consumption. Moreover, by (9.43) we see that the greater the elasticity of marginal utility (that is, the greater the curvature of the utility function), the greater the incentive to smooth consumption for a given value of $r(t) - \rho$. The reason for this is that a large curvature means that the marginal utility will drop sharply if consumption increases, and will rise sharply if consumption falls. Fig. 9.2 illustrates this

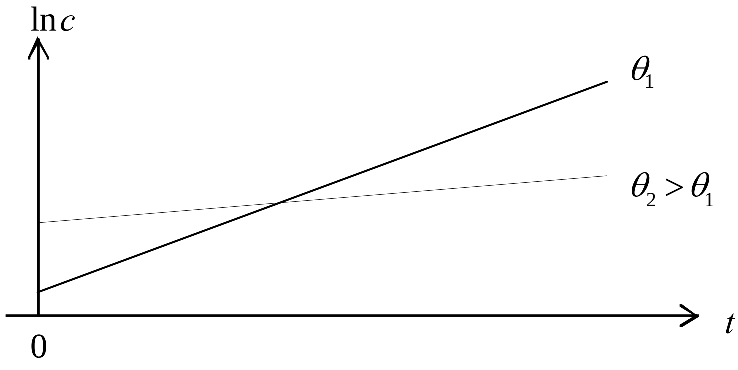

Figure 9.2: Optimal consumption paths for a low and a high constant θ, given a constant $r > \rho$.

in the CRRA case where $\theta(c) = \theta$, a positive constant. For a given constant $r > \rho$, the consumption path chosen when θ is high has lower slope, but starts from a higher level, than when θ is low.

The condition (9.43), which holds for a finite as well as an infinite time horizon, is referred to as the *Keynes-Ramsey rule*. The name springs from the English mathematician Frank Ramsey who derived the rule in as early as 1928, while John Maynard Keynes suggested a simple and intuitive way of presenting it. The rule is the continuous-time counterpart to the consumption Euler equation in discrete time.

The Keynes-Ramsey rule reflects the general microeconomic principle that the consumer equates the marginal rate of substitution between any two goods with the corresponding price ratio. In the present context the principle is applied to a situation where the "two goods" refer to the same consumption good delivered at two different dates. In Section 9.2 we used the principle to interpret the optimal saving behavior in discrete time. How can the principle be translated into a continuous time setting?

Local optimality in continuous time Let $(t, t+\Delta t)$ and $(t+\Delta t, t+2\Delta t)$ be two short successive time intervals. The marginal rate of substitution, $MRS_{t+\Delta t,t}$, of consumption in the second time interval for consumption in the first is[17]

$$MRS_{t+\Delta t,t} \equiv -\frac{dc(t+\Delta t)}{dc(t)}\Big|_{U=\bar{U}} = \frac{u'(c(t))}{e^{-\rho\Delta t}u'(c(t+\Delta t))}, \tag{9.45}$$

[17] The underlying analytical steps can be found in Appendix E.

approximately. On the other hand, by saving $-\Delta c(t)$ more per time unit (where $\Delta c(t) < 0$) in the short time interval $(t,\, t + \Delta t)$, one can via the market transform $-\Delta c(t) \cdot \Delta t$ units of consumption in this time interval into

$$\Delta c(t + \Delta t) \cdot \Delta t \approx -\Delta c(t)\Delta t \; e^{\int_t^{t+\Delta t} r(\tau)d\tau} \tag{9.46}$$

units of consumption in the time interval $(t + \Delta t,\, t + 2\Delta t)$. The marginal rate of transformation is therefore

$$\begin{aligned} MRT_{t+\Delta t,t} &\equiv -\frac{dc(t + \Delta t)}{dc(t)}|_{U=\bar{U}} \approx \\ &= e^{\int_t^{t+\Delta t} r(\tau)d\tau}. \end{aligned}$$

In the optimal plan we must have $MRS_{t+\Delta t,t} = MRT_{t+\Delta t,t}$ which gives

$$\frac{u'(c(t))}{e^{-\rho\Delta t}u'(c(t + \Delta t))} = e^{\int_t^{t+\Delta t} r(\tau)d\tau}, \tag{9.47}$$

approximately. When $\Delta t = 1$ and ρ and $r(t)$ are small, this relation can be approximated by (9.9) from discrete time (generally, by a first-order Taylor approximation, $e^x \approx 1 + x$, when x is close to 0).

Taking logs on both sides of (9.47), dividing through by Δt, inserting (9.46), and letting $\Delta t \to 0$, we get (see Appendix E)

$$\rho - \frac{u''(c(t))}{u'(c(t))}\dot{c}(t) = r(t). \tag{9.48}$$

With the definition of $\theta(c)$ in (9.44), this is exactly the same as the Keynes-Ramsey rule (9.43) which, therefore, is merely an expression of the general optimality condition $MRS = MRT$. When $\dot{c}(t) > 0$, the household is willing to sacrifice some consumption today for more consumption tomorrow only if it is compensated by an interest rate sufficiently above ρ. Naturally, the required compensation is higher, the faster marginal utility declines with rising consumption, i.e., the larger is $(-u''/u')\dot{c}$ already. Indeed, a higher c_t in the future than today implies a lower marginal utility of consumption in the future than of consumption today. So saving of the marginal unit of income today is only warranted if the rate of return is sufficiently above ρ, and this is what (9.48) indicates.

9.4.4 Mangasarian's sufficient conditions

The Maximum Principle delivers a set of first-order and transversality conditions that as such are only *necessary* conditions for an interior path to be optimal. Hence, up to this point we have only claimed that if the consumption-saving problem has an interior solution, then it satisfies the Keynes-Ramsey rule and a transversality condition, (TVC'). But are these conditions also *sufficient*? The answer is yes in the present case. This follows from *Mangasarian's sufficiency theorem* (see Math tools) which applied to the present problem tells us that if the Hamiltonian is *concave* in (a, c) for every t, then the listed necessary conditions, including the transversality condition, are also sufficient. Because the instantaneous utility function, the first term in the Hamiltonian, is strictly concave and the second term is linear in (a, c), the Hamiltonian *is* concave in (a, c). Thus if we have found a path satisfying the Keynes-Ramsey rule and the (TVC'), we have a *candidate solution.* And by the Mangasarian theorem this candidate *is* an optimal solution. In fact the strict concavity of the Hamiltonian with respect to the control variable ensures that the optimal solution is unique (Exercise 9.?).

9.5 The consumption function

We have not yet fully solved the saving problem. The Keynes-Ramsey rule gives only the optimal rate of *change* of consumption over time. It says nothing about the *level* of consumption at any given time. In order to determine, for instance, the level $c(0)$, we implicate the solvency condition which limits the amount the household can borrow in the long term. Among the infinitely many consumption paths satisfying the Keynes-Ramsey rule, the household will choose the "highest" one that also fulfils the solvency requirement (NPG). Thus, the household acts so that strict equality in (NPG) obtains. As we saw, this is equivalent to the transversality condition being satisfied.

To avoid any misunderstanding, the examples below should not be interpreted such that for *any* evolution of wages and interest rates there exists a solution to the household's maximization problem with infinite horizon. There is generally no guarantee that integrals have an upper bound for $T \to \infty$. The evolution of wages and interest rates which prevails in *general equilibrium* is not arbitrary, however. It is determined by the requirement of equilibrium. In turn, of course *existence* of an equilibrium imposes restrictions on the utility discount rate relative to the potential growth in instantaneous utility. We shall return to these issues in the next chapter.

EXAMPLE 1 *(constant elasticity of marginal utility; infinite time horizon).* In the problem in Section 9.4.2 with $T = \infty$, we consider the case where the elasticity of marginal utility $\theta(c)$, as defined in (9.44), is a constant $\theta > 0$. From Appendix A of Chapter 3 we know that this requirement implies that up to a positive linear transformation the utility function must be of the form:

$$u(c) = \begin{cases} \frac{c^{1-\theta}}{1-\theta}, & \text{when } \theta > 0, \theta \neq 1, \\ \ln c, & \text{when } \theta = 1. \end{cases} \tag{9.49}$$

The Keynes-Ramsey rule then implies $\dot{c}(t) = \theta^{-1}(r(t) - \rho)c(t)$. Solving this linear differential equation yields

$$c(t) = c(0)e^{\frac{1}{\theta}\int_0^t (r(\tau)-\rho)d\tau}, \tag{9.50}$$

cf. the general accumulation formula, (9.22).

We know from Proposition 2 that the transversality condition is equivalent to the intertemporal budget constraint being satisfied with strict equality, i.e.,

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt = a_0 + h_0, \tag{IBC'}$$

where h_0 is the human wealth,

$$h_0 = \int_0^\infty w(t)e^{-\int_0^t r(\tau)d\tau}dt. \tag{9.51}$$

This result can be used to determine $c(0)$.[18] Substituting (9.50) into (IBC') gives

$$c(0)\int_0^\infty e^{\int_0^t [\frac{1}{\theta}(r(\tau)-\rho)-r(\tau)]d\tau}dt = a_0 + h_0.$$

The consumption function is thus

$$\begin{aligned} c(0) &= \beta_0(a_0 + h_0), \qquad \text{where} \\ \beta_0 &\equiv \frac{1}{\int_0^\infty e^{\int_0^t [\frac{1}{\theta}(r(\tau)-\rho)-r(\tau)]d\tau}dt} = \frac{1}{\int_0^\infty e^{\frac{1}{\theta}\int_0^t [(1-\theta)r(\tau)-\rho]d\tau}dt} \end{aligned} \tag{9.52}$$

is the marginal propensity to consume out of wealth. We have here assumed that these improper integrals over an infinite horizon are bounded from above for all admissible paths.

Generally, an increase in the interest rate level, for given total wealth, $a_0 + h_0$, can effect $c(0)$ both positively and negatively.[19] On the one hand, such an increase makes future consumption cheaper in present value terms. This change in the trade-off between current and future consumption entails

[18] The method also applies if instead of $T = \infty$, we have $T < \infty$.

[19] By an increase in the interest rate *level* we mean an upward shift in the time-profile of the interest rate. That is, there is at least one time interval within $[0, \infty)$ where the interest rate is higher than in the original situation and no time interval within $[0, \infty)$ where the interest rate is lower.

a negative *substitution effect* on $c(0)$. On the other hand, the increase in the interest rates decreases the present value of a given consumption plan, allowing for higher consumption both today and in the future, for given total wealth, cf. (IBC'). This entails a positive *pure income effect* on consumption today as consumption is a normal good. If $\theta < 1$ (small curvature of the utility function), the substitution effect will dominate the pure income effect, and if $\theta > 1$ (large curvature), the reverse will hold. This is because the larger is θ, the stronger is the propensity to smooth consumption over time.

In the intermediate case $\theta = 1$ (the logarithmic case) we get from (9.52) that $\beta_0 = \rho$, hence

$$c(0) = \rho(a_0 + h_0). \tag{9.53}$$

In this special case the marginal propensity to consume is time independent and equal to the rate of time preference. For a given *total* wealth, $a_0 + h_0$, current consumption is thus independent of the expected path of the interest rate. That is, in the logarithmic case the *substitution* and *pure income effects* on current consumption exactly offset each other. Yet, on top of this comes the negative *wealth effect* on current consumption of an increase in the interest rate level. The present value of future wage incomes becomes lower (similarly with expected future dividends on shares and future rents in the housing market in a more general setup). Because of this, h_0 (and so $a_0 + h_0$) becomes lower, which adds to the negative substitution effect.[20] Thus, even in the logarithmic case, and *a fortiori* when $\theta < 1$, the *total effect* of an increase in the interest rate level is unambiguously negative on $c(0)$.

If, for example, $r(t) = r$ and $w(t) = w$ (positive constants), we get $\beta_0 = [(\theta - 1)r + \rho]/\theta$ and $a_0 + h_0 = a_0 + w/r$. When $\theta = 1$, the negative effect of a higher r on h_0 is decisive. When $\theta < 1$, a higher r reduces both β_0 and h_0, hence the total effect on $c(0)$ is even "more negative". When $\theta > 1$, a higher r implies a higher β_0 which more or less offsets the lower h_0, so that the total effect on $c(0)$ becomes ambiguous. As referred to in Chapter 3, available empirical studies generally suggest a value of θ somewhat above 1. □

EXAMPLE *2* (*constant absolute semi-elasticity of marginal utility; infinite time horizon*). In the problem in Section 9.4.2 with $T = \infty$, we consider the case where the sensitivity of marginal utility, measured by the absolute value of the semi-elasticity of marginal utility, $-u''(c)/u'(c) \approx -(\Delta u'/u')/\Delta c$, is a positive constant, α. The utility function must then, up to a positive linear

[20] If $a_0 < 0$ and this net debt is not a variable-rate loan (as hitherto assumed), but like a fixed-rate mortgage loan, then a rise in the interest rate level implies a lowering of the present value of the debt and thereby raises total wealth (ceteris paribus) and *counteracts* the negative substitution effect on current consumption.

transformation, be of the form,

$$u(c) = -\alpha^{-1}e^{-\alpha c}, \alpha > 0. \tag{9.54}$$

The Keynes-Ramsey rule becomes $\dot{c}(t) = \alpha^{-1}(r(t) - \rho)$. When the interest rate is a constant $r > 0$, we find, through (IBC') and partial integration, $c(0) = r(a_0 + h_0) - (r - \rho)/(\alpha r)$, presupposing $r \geq \rho$ and $a_0 + h_0 > (r - \rho)/(ar^2)$.

This hypothesis of a "constant absolute variability aversion" implies that the degree of *relative* variability aversion is $\theta(c) = \alpha c$ and thus greater, the larger is c. In the theory of behavior under uncertainty, (9.54) is referred to as the CARA function ("Constant Absolute Risk Aversion"). One of the theorems of expected utility theory is that the degree of absolute risk aversion, $-u''(c)/u'(c)$, is proportional to the risk premium which the economic agent will require to be willing to exchange a specified amount of consumption received with certainty for an uncertain amount having the same mean value. Empirically this risk premium seems to be a decreasing function of the level of consumption. Therefore the CARA function is generally considered less realistic than the CRRA function of the previous example. □

EXAMPLE 3 *(logarithmic utility; finite time horizon; retirement).* We consider a life-cycle saving problem. A worker enters the labor market at time 0 with a financial wealth of 0, has finite lifetime T (assumed known) and does not wish to pass on bequests. For simplicity, we assume that $r_t = r > 0$ for all $t \in [0, T]$ and $w(t) = w > 0$ for $t \leq t_1 \leq T$, while $w(t) = 0$ for $t > t_1$ (no wage income after retirement, which takes place at time t_1). The decision problem is

$$\begin{aligned} \max_{(c(t))_{t=0}^T} U_0 &= \int_0^T (\ln c(t))e^{-\rho t}dt \quad \text{s.t.} \\ c(t) &\geq 0, \\ \dot{a}(t) &= ra(t) + w(t) - c(t), \qquad\qquad a(0) = 0, \\ a(T) &\geq 0. \end{aligned}$$

The Keynes-Ramsey rule becomes $\dot{c}_t/c_t = r - \rho$. A solution to the problem will thus fulfil

$$c(t) = c(0)e^{(r-\rho)t}. \tag{9.55}$$

Inserting this into the differential equation for a, we get a first-order linear differential equation the solution of which (for $a(0) = 0$) can be reduced to

$$a(t) = e^{rt}\left[\frac{w}{r}(1 - e^{-rz}) - \frac{c_0}{\rho}(1 - e^{-\rho t})\right], \tag{9.56}$$

where $z = t$ if $t \leq t_1$, and $z = t_1$ if $t > t_1$. We need to determine $c(0)$. The transversality condition implies $a(T) = 0$. Having $t = T$, $z = t_1$ and $a_T = 0$ in (9.56), we get

$$c(0) = (\rho w/r)(1 - e^{-rt_1})/(1 - e^{-\rho T}). \tag{9.57}$$

Substituting this into (9.55) gives the optimal consumption plan.[21]

If $r = \rho$, consumption is constant over time at the level given by (9.57). If, in addition, $t_1 < T$, this consumption level is less than the wage income per year up to t_1 (in order to save for retirement); in the last years the level of consumption is maintained although there is no wage income; the retired person uses up both the return on financial wealth and this wealth itself. □

The examples illustrate the importance of *forward-looking expectations*, here the expected evolution of interest rates and wages. The expectations affect $c(0)$ both through their impact on the marginal propensity to consume (cf. β_0 in Example 1) and through their impact on the present value, h_0, of expected future labor income (or of expected future dividends on shares in a more general setup).[22] Yet the examples – and the consumption theory in this chapter in general – should only be seen as a first, crude approximation to consumption/saving behavior. Real world factors such as uncertainty and narrow credit constraints (absence of perfect loan markets) also affect the behavior. Including these factors in the analysis tend to make current income an additional determinant of the consumption by a large fraction of the population, as is recognized in many short- and medium-run macro models.

9.6 Literature notes

(incomplete)

Loewenstein and Thaler (1989) survey the evidence suggesting that the utility discount rate is generally not constant, but declining with the time distance from the current period to the future periods within the horizon. This is known as *hyperbolic discounting.*

The (strong) assumptions regarding the underlying intertemporal preferences which allow them to be represented by the present value of period

[21] For $t_1 = T$ and $T \to \infty$ we get in the limit $c(0) = \rho w/r \equiv \rho h_0$, which is also what (9.52) gives when $a(0) = 0$ and $\theta = 1$.

[22] How to treat cases where, due to new information, a shift in expectations occurs so that a discontinuity in a responding endogenous variable results is dealt with in Chapter 11.

utilities discounted at a constant rate are dealt with by Koopmans (1960), Fishburn and Rubinstein (1982), and – in summary form – by Heal (1998).

Rigorous and more general treatments of the Maximum Principle in continuous time applied in economic analysis are available in, e.g., Seierstad and Sydsaeter (1987) and Sydsaeter et al. (2008).

Allen (1967). Goldberg (1958).

9.7 Appendix

A. Growth arithmetic in continuous time

Let the variables z, x, and y be differentiable functions of time t. Suppose $z(t)$, $x(t)$, and $y(t)$ are positive for all t. Then:

PRODUCT RULE $z(t) = x(t)y(t) \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} + \frac{\dot{y}(t)}{y(t)}$.

Proof. Taking logs on both sides of the equation $z(t) = x(t)y(t)$ gives $\ln z(t) = \ln x(t) + \ln y(t)$. Differentiation w.r.t. t, using the chain rule, gives the conclusion. □

The procedure applied in this proof is called *logarithmic differentiation* w.r.t. t.

FRACTION RULE $z(t) = \frac{x(t)}{y(t)} \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} - \frac{\dot{y}(t)}{y(t)}$.

The proof is similar.

POWER FUNCTION RULE $z(t) = x(t)^\alpha \Rightarrow \frac{\dot{z}(t)}{z(t)} = \alpha\frac{\dot{x}(t)}{x(t)}$.

The proof is similar.

In continuous time these simple formulas are exactly true. In discrete time the analogue formulas are only approximately true and the approximation can be quite bad unless the growth rates of x and y are small, cf. Appendix A to Chapter 4.

B. The cumulative mean of growth and interest rates

Sometimes in the literature the basic accumulation formula, (9.22), is expressed in terms of the arithmetic average (also called the cumulative mean) of the growth rates in the time interval $[0, t]$. This average is $\bar{g}_{0,t} = (1/t)\int_0^t g(\tau)d\tau$. So we can write

$$a(t) = a(0)e^{\bar{g}_{0,t}t},$$

which has form similar to (9.21). Similarly, let $\bar{r}_{0,t}$ denote the arithmetic average of the (short-term) interest rates from time 0 to time t, i.e., $\bar{r}_{0,t}$

$= (1/t)\int_0^t r(\tau)d\tau$. Then we can write the present value of the consumption stream $(c(t))_{t=0}^T$ as $PV = \int_0^T c(t)e^{-\bar{r}_{0,t}t}dt$.

In discrete time the arithmetic average of growth rates can at best be used as an approximation. Let $\hat{g}_{0,t}$ be the the average compound growth rate from year 0 to year t, that is, $1+\hat{g}_{0,t} = ((1+g_0)(1+g_1)\cdots(1+g_{t-1}))^{1/t}$. If the period length is short, however, say a quarter of a year, then the growth rates g_1, g_2,..., hence also $\hat{g}_{0,t}$, will generally be not far from zero so that the approximation $\ln(1+g_t) \approx g_t$ is acceptable. Then $\hat{g}_{0,t} \approx \frac{1}{t}(g_0+g_1+...+g_{t-1})$, a simple *arithmetic* average. As compounding is left out, this approximation is not good if there are many periods unless the growth rates are very small numbers.

Similarly with interest rates in discrete time.

C. Notes on Proposition 1 (equivalence between the No-Ponzi-Game condition and the intertemporal budget constraint)

We consider the book-keeping relation

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t), \tag{9.58}$$

where $a(0) = a_0$ (given), and the solvency requirement

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0. \tag{NPG}$$

Technical remark. The expression in (NPG) is understood to include the possibility that $a(t)e^{-\int_0^t r(\tau)d\tau} \to \infty$ for $t \to \infty$. Moreover, if full generality were aimed at, we should allow for infinitely fluctuating paths in both the (NPG) and (TVC) and therefore replace "$\lim_{t\to\infty}$" by "$\liminf_{t\to\infty}$", i.e., the *limit inferior*. The limit inferior for $t \to \infty$ of a function $f(t)$ on $[0,\infty)$ is defined as $\lim_{t\to\infty} \inf\{f(s)|\ s \geq t\}$.[23] As noted in Appendix E of the previous chapter, however, undamped infinitely fluctuating paths never turn up in the optimization problems considered in this book, whether in discrete or continuous time. Hence, we apply the simpler concept "lim" rather than "lim inf". □

On the background of (9.58) Proposition 1 claimed that (NPG) is equivalent with the intertemporal budget constraint,

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq h_0 + a_0, \tag{IBC}$$

[23] By "inf" is meant *infinum* of the set, that is, the largest number less than or equal to all numbers in the set.

being satisfied, where h_0 is defined as in (9.51) and is assumed to be a finite number. In addition, Proposition 1 in Section 9.4 claimed that there is strict equality in (IBC) if and only there is strict equality in (NPG). We now prove these claims.

Proof. Isolate $c(t)$ in (9.58) and multiply through by $e^{-\int_0^t r(\tau)d\tau}$ to obtain

$$c(t)e^{-\int_0^t r(\tau)d\tau} = w(t)e^{-\int_0^t r(\tau)d\tau} - (\dot{a}(t) - r(t)a(t))e^{-\int_0^t r(\tau)d\tau}.$$

Integrate from 0 to $T > 0$ to get $\int_0^T c(t)e^{-\int_0^t r(\tau)d\tau}dt$

$$\begin{aligned}
&= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - \int_0^T \dot{a}(t)e^{-\int_0^t r(\tau)d\tau}dt + \int_0^T r(t)a(t)e^{-\int_0^t r(\tau)d\tau}dt \\
&= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - \left(\left[a(t)e^{-\int_0^t r(\tau)d\tau}\right]_0^T - \int_0^T a(t)e^{-\int_0^t r(\tau)d\tau}(-r(t))dt\right) \\
&+ \int_0^T r(t)a(t)e^{-\int_0^t r(\tau)d\tau}dt \\
&= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - (a(T)e^{-\int_0^T r(\tau)d\tau} - a(0)),
\end{aligned}$$

where the second last equality follows from integration by parts. If we let $T \to \infty$ and use the definition of h_0 and the initial condition $a(0) = a_0$, we get (IBC) if and only if (NPG) holds. It follows that when (NPG) is satisfied with strict equality, so is (IBC), and vice versa. □

An alternative proof is obtained by using the general solution to a linear inhomogeneous first-order differential equation and then let $T \to \infty$. Since this is a more generally applicable approach, we will show how it works and use it for Claim 1 below (an extended version of Proposition 1) and for the proof of Proposition 2. Claim 1 will for example prove useful in Exercise 9.1 and in the next chapter.

CLAIM 1 Let $f(t)$ and $g(t)$ be given continuous functions of time, t. Consider the differential equation

$$\dot{x}(t) = g(t)x(t) + f(t), \tag{9.59}$$

with $x(t_0) = x_{t_0}$, a given initial value. Then the inequality

$$\lim_{t\to\infty} x(t)e^{-\int_{t_0}^t g(s)ds} \geq 0 \tag{9.60}$$

is equivalent to

$$-\int_{t_0}^{\infty} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds} d\tau \leq x_{t_0}. \tag{9.61}$$

Moreover, if and only if (9.60) is satisfied with strict equality, then (9.61) is also satisfied with strict equality.

Proof. The linear differential equation, (9.59), has the solution

$$x(t) = x(t_0)e^{\int_{t_0}^{t} g(s)ds} + \int_{t_0}^{t} f(\tau)e^{\int_{\tau}^{t} g(s)ds} d\tau. \tag{9.62}$$

Multiplying through by $e^{-\int_{t_0}^{t} g(s)ds}$ yields

$$x(t)e^{-\int_{t_0}^{t} g(s)ds} = x(t_0) + \int_{t_0}^{t} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds} d\tau.$$

By letting $t \to \infty$, it can be seen that if and only if (9.60) is true, we have

$$x(t_0) + \int_{t_0}^{\infty} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds} d\tau \geq 0.$$

Since $x(t_0) = x_{t_0}$, this is the same as (9.61). We also see that if and only if (9.60) holds with strict equality, then (9.61) also holds with strict equality. □

COROLLARY Let n be a given constant and let

$$h_{t_0} \equiv \int_{t_0}^{\infty} w(\tau)e^{-\int_{t_0}^{\tau}(r(s)-n)ds} d\tau, \tag{9.63}$$

which we assume is a finite number. Then, given

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t), \text{ where } a(t_0) = a_{t_0}, \tag{9.64}$$

it holds that

$$\lim_{t\to\infty} a(t)e^{-\int_{t_0}^{t}(r(s)-n)ds} \geq 0 \Leftrightarrow \int_{t_0}^{\infty} c(\tau)e^{-\int_{t_0}^{\tau}(r(s)-n)ds} d\tau \leq a_{t_0} + h_{t_0}, \tag{9.65}$$

where a strict equality on the left-hand side of "$\Leftrightarrow$" implies a strict equality on the right-hand side, and vice versa.

Proof. Let $x(t) = a(t)$, $g(t) = r(t) - n$ and $f(t) = w(t) - c(t)$ in (9.59), (9.60) and (9.61). Then the conclusion follows from Claim 1. □

By setting $t_0 = 0$ in the corollary and replacing τ by t and n by 0, we have hereby provided an alternative proof of Proposition 1.

D. Proof of Proposition 2 (the transversality condition with an infinite time horizon)

In the differential equation (9.59) we let $x(t) = \lambda(t)$, $g(t) = -(r(t) - \rho)$, and $f(t) = 0$. This gives the linear differential equation $\dot{\lambda}(t) = (\rho - r(t))\lambda(t)$, which is identical to the first-order condition (9.37) in Section 9.3. The solution is

$$\lambda(t) = \lambda(t_0)e^{-\int_{t_0}^{t}(r(s)-\rho)ds}.$$

Substituting this into (TVC) in Section 9.3 yields

$$\lambda(t_0) \lim_{t\to\infty} a(t)e^{-\int_{t_0}^{t}(r(s)-n)ds} = 0. \tag{9.66}$$

From the first-order condition (9.36) in Section 9.3 we have $\lambda(t_0) = u'(c(t_0)) > 0$ so that $\lambda(t_0)$ in (9.66) can be ignored. Thus (TVC) in Section 9.3 is equivalent to the condition (NPG) in that section being satisfied with strict equality (let $t_0 = 0 = n$). $\square$

E. Intertemporal consumption smoothing

We claimed in Section 9.4 that equation (9.45) gives approximately the marginal rate of substitution of consumption in the time interval $(t+\Delta t,\, t+2\Delta t)$ for consumption in $(t,\, t+\Delta t)$. This can be seen in the following way. To save notation we shall write our time-dependent variables as c_t, r_t, etc., even though they are continuous functions of time. The contribution from the two time intervals to the criterion function is

$$\begin{aligned}
\int_t^{t+2\Delta t} u(c_\tau)e^{-\rho\tau}d\tau &\approx e^{-\rho t}\left(\int_t^{t+\Delta t} u(c_t)e^{-\rho(\tau-t)}d\tau + \int_{t+\Delta t}^{t+2\Delta t} u(c_{t+\Delta t})e^{-\rho(\tau-t)}d\tau\right)\\
&= e^{-\rho t}\left(u(c_t)\left[\frac{e^{-\rho(\tau-t)}}{-\rho}\right]_t^{t+\Delta t} + u(c_{t+\Delta t})\left[\frac{e^{-\rho(\tau-t)}}{-\rho}\right]_{t+\Delta t}^{t+2\Delta t}\right)\\
&= \frac{e^{-\rho t}(1-e^{-\rho\Delta t})}{\rho}\left[u(c_t) + u(c_{t+\Delta t})e^{-\rho\Delta t}\right].
\end{aligned}$$

Requiring unchanged utility integral $U_0 = \bar{U}_0$ is thus approximately the same as requiring $\Delta[u(c_t) + u(c_{t+\Delta t})e^{-\rho\Delta t}] = 0$, which by carrying through the differentiation and rearranging gives (9.45).

The instantaneous local optimality condition, equation (9.48), can be interpreted on the basis of (9.47). Take logs on both sides of (9.47) to get

$$\ln u'(c_t) + \rho\Delta t - \ln u'(c_{t+\Delta t}) = \int_t^{t+\Delta t} r_\tau d\tau.$$

Dividing by Δt, substituting (9.46), and letting $\Delta t \to 0$ we get

$$\rho - \lim_{\Delta t \to 0} \frac{\ln u'(c_{t+\Delta t}) - \ln u'(c_t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{R_{t+\Delta t} - R_t}{\Delta t}, \tag{9.67}$$

where R_t is the antiderivative of r_t. By the definition of a time derivative, (9.67) can be written

$$\rho - \frac{d \ln u'(c_t)}{dt} = \frac{dR_t}{dt}.$$

Carrying out the differentiation, we get

$$\rho - \frac{1}{u'(c_t)} u''(c_t)\dot{c}_t = r_t,$$

which was to be shown.

F. Elasticity of intertemporal substitution in continuous time

The relationship between the elasticity of marginal utility and the concept of *instantaneous elasticity of intertemporal substitution* in consumption can be exposed in the following way: consider an indifference curve for consumption in the non-overlapping time intervals $(t,\ t + \Delta t)$ and $(s,\ s + \Delta t)$. The indifference curve is depicted in Fig. 9.3. The consumption path outside the two time intervals is kept unchanged. At a given point $(c_t \Delta t, c_s \Delta t)$ on the indifference curve, the marginal rate of substitution of s-consumption for t-consumption, MRS_{st}, is given by the absolute slope of the tangent to the indifference curve at that point. In view of $u''(c) < 0$, MRS_{st} is rising along the curve when c_t decreases (and thereby c_s increases).

Conversely, we can consider the ratio c_s/c_t as a function of MRS_{st} along the given indifference curve. The elasticity of this consumption ratio w.r.t. MRS_{st} as we move along the given indifference curve then indicates the *elasticity of substitution* between consumption in the time interval $(t,\ t+\Delta t)$ and consumption in the time interval $(s,\ s + \Delta t)$. Denoting this elasticity by $\sigma(c_t, c_s)$, we thus have:

$$\sigma(c_t, c_s) = \frac{MRS_{st}}{c_s/c_t} \frac{d(c_s/c_t)}{dMRS_{st}} \approx \frac{\frac{\Delta(c_s/c_t)}{c_s/c_t}}{\frac{\Delta MRS_{st}}{MRS_{st}}}.$$

At an optimum point, MRS_{st} equals the ratio of the discounted prices of good t and good s. Thus, the elasticity of substitution can be interpreted as approximately equal to the percentage increase in the ratio of the chosen goods, c_s/c_t, generated by a one percentage increase in the inverse price

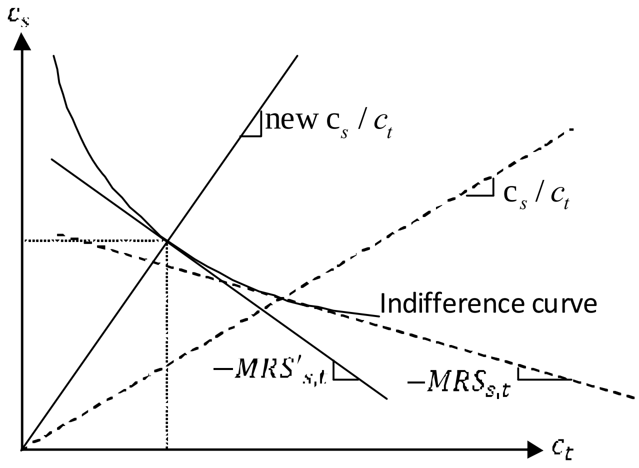

Figure 9.3: Substitution of s-consumption for t-consumption as MRS_{st} increases til MRS'_{st}.

ratio, holding the utility level and the amount of other goods unchanged. If $s = t + \Delta t$ and the interest rate from date t to date s is r, then (with continuous compounding) this price ratio is $e^{r\Delta t}$, cf. (9.47). Inserting MRS_{st} from (9.45) with $t + \Delta t$ replaced by s, we get

$$\begin{aligned} \sigma(c_t, c_s) &= \frac{u'(c_t)/[e^{-\rho(s-t)}u'(c_s)]}{c_s/c_t} \frac{d(c_s/c_t)}{d\{u'(c_t)/[e^{-\rho(s-t)}u'(c_s)]\}} \\ &= \frac{u'(c_t)/u'(c_s)}{c_s/c_t} \frac{d(c_s/c_t)}{d(u'(c_t)/u'(c_s))}, \end{aligned} \tag{9.68}$$

since the factor $e^{-\rho(t-s)}$ cancels out.

We now interpret the d's in (9.68) as differentials (recall, the differential of a differentiable function $y = f(x)$ is denoted dy and defined as $dy = f'(x)dx$ where dx is some arbitrary real number). Calculating the differentials we get

$$\sigma(c_t, c_s) \approx \frac{u'(c_t)/u'(c_s)}{c_s/c_t} \frac{(c_t dc_s - c_s dc_t)/c_t^2}{[u'(c_s)u''(c_t)dc_t - u'(c_t)u''(c_s)dc_s]/u'(c_s)^2}.$$

Hence, for $s \to t$ we get $c_s \to c_t$ and

$$\sigma(c_t, c_s) \to \frac{c_t(dc_s - dc_t)/c_t^2}{u'(c_t)u''(c_t)(dc_t - dc_s)/u'(c_t)^2} = -\frac{u'(c_t)}{c_t u''(c_t)} \equiv \tilde{\sigma}(c_t).$$

This limiting value is known as the *instantaneous elasticity of intertemporal substitution* of consumption. It reflects the opposite of the desire for consumption smoothing. Indeed, we see that $\tilde{\sigma}(c_t) = 1/\theta(c_t)$, where $\theta(c_t)$ is the elasticity of marginal utility at the consumption level $c(t)$.

9.8 Exercises

9.1 We look at a household (or dynasty) with infinite time horizon. The household's labor supply is inelastic and grows at the constant rate $n > 0$. The household has a constant rate of time preference $\rho > n$ and the individual instantaneous utility function is $u(c) = c^{1-\theta}/(1-\theta)$, where θ is a positive constant. There is no uncertainty. The household maximizes the integral of per capita utility discounted at the rate $\rho - n$. . Set up the household's optimization problem. Show that the optimal consumption plan satisfies

$$\begin{aligned} c(0) &= \beta_0(a_0 + h_0), \qquad \text{where} \\ \beta_0 &= \frac{1}{\int_0^\infty e^{\int_0^t (\frac{(1-\theta)r(\tau)-\rho}{\theta} + n)d\tau} dt}, \qquad \text{and} \\ h_0 &= \int_0^\infty w(t) e^{-\int_0^t (r(\tau)-n)d\tau} dt, \end{aligned}$$

where $w(t)$ is the real wage per unit of labor and otherwise the same notation as in this chapter is used. *Hint:* use the corollary to Claim 1 in Appendix C and the method of Example 1 in Section 9.5.

Chapter 10

The Ramsey model

As early as 1928 a sophisticated model of a society's optimal saving was published by the British mathematician and economist Frank Ramsey (1903-1930). Ramsey's contribution was mathematically demanding and did not experience much response at the time. Three decades had to pass until his contribution was taken up seriously (Samuelson and Solow, 1956). The model was merged with Solow's simpler growth model (Solow 1956) and became a cornerstone in neoclassical growth theory from the mid 1960s. The version of the model which we present below was completed by the work of Cass (1965) and Koopmans (1965). Hence the model is also known as the *Ramsey-Cass-Koopmans model*.

The model is one of the basic workhorse models in macroeconomics. It can be seen as placed at one end of a line segment. At the other end appears another basic workhorse model, namely Diamond's overlapping generations model. In the Diamond model there is an *infinite* number of agents (since in every new period a new generation enters the economy) and these have a *finite* time horizon. In the Ramsey model there is a *finite* number of agents with an *infinite* time horizon and these agents are completely alike. The Ramsey model is thus a *representative agent* model, whereas the Diamond model has heterogeneous agents, young and old, interacting in every period. There are important economic questions where these differences in the model setup lead to salient differences in the answers. Along the line segment where these two frameworks are polar cases, less abstract models are scattered, some being closer to the one pole and others closer to the other.

The present chapter presents the continuous-time version of the Ramsey framework. We first study the framework under the conditions of a perfectly competitive market economy. In this context we will see, for example, that the Solow growth model comes out as a special case of the Ramsey model. Next we consider the Ramsey framework in a setting with an "all-knowing and all-powerful" social planner.

10.1 Preliminaries

We consider a closed economy. Time is continuous. We assume that the households own the capital goods and hire them out to firms at a market rental rate, $\hat{r}$. This is just to have something concrete in mind. If instead the capital goods were owned by the firms using them in production and the capital investment by these firms were financed by issuing shares and bonds, the conclusions would remain unaltered as long as we ignore uncertainty.

The variables in the model are considered as (piecewise) continuous and differentiable functions of time, t. Yet, to save notation, we shall write them as w_t, $\hat{r}_t$, etc. instead of $w(t)$, $\hat{r}(t)$, etc. In every short time interval $(t, t+\Delta t)$, the individual firm employs labor at the market wage w_t and rents capital goods at the rental rate $\hat{r}_t$. The combination of labor and capital produces the homogeneous output good. This good can be used for consumption as well as investment. So in every short time interval there are at least three active markets, one for the "all-purpose" output good, one for labor, and one for capital services (the rental market for capital goods). For the sake of intuition it may be useful to imagine that there is also a market for loans. There is a time-dependent short-term interest rate, r_t, on these loans. As all households are alike, however, the loan market will not be active in general equilibrium. There is perfect competition in all markets, that is, prices are exogenous to the individual households and firms. Any need for means of payment − money − is abstracted away. Prices are measured in units of the homogeneous current output.

There are no stochastic elements in the model. We assume households understand exactly how the economy works and can predict the future path of wages and interest rates. That is, we assume "rational expectations" which in our non-stochastic setting amounts to *perfect foresight*. In this way the results that emerge will be the outcome of economic mechanisms in isolation from expectational errors.

As uncertainty is by assumption absent, rates of return on alternative assets must in equilibrium be the same. The latent real interest rate, r_t, on the not active loan market will satisfy the no-arbitrage condition

$$\hat{r}_t - \delta = r_t, \tag{10.1}$$

where the left-hand side is the rate of return on holding capital goods, δ (≥ 0) being a constant rate of capital depreciation.

Below we describe, first, the households' behavior and next the firms' behavior. After this, the interaction between households and firms in general equilibrium and the resulting dynamics will be analyzed.

10.2 The agents

10.2.1 Households

There is a fixed number of identical households with an infinite time horizon. This feature makes aggregation very simple: we just have to multiply the behavior of a single household with the number of households. Every household has L_t (adult) members and L_t changes over time at a constant rate, n :

$$L_t = L_0 e^{nt}, \qquad L_0 > 0. \tag{10.2}$$

Indivisibility problems are ignored.

Each household member supplies inelastically one unit of labor per time unit. Equation (10.2) therefore describes the growth of the population as well as the labor force. Since there is only one consumption good, the only decision problem is how to distribute current income between consumption and saving.

Intertemporal utility function

From now we consider a single household. Its preferences can be represented by an additive intertemporal utility function with a constant rate of time preference, ρ. Seen from time 0, the intertemporal utility function is

$$U_0 = \int_0^\infty u(c_t) L_t e^{-\rho t} dt,$$

where $c_t \equiv C_t/L_t$ is consumption per family member. The instantaneous utility function, $u(c)$, has $u'(c) > 0$ and $u''(c) < 0$, i.e., positive but diminishing marginal utility of consumption. The utility contribution from consumption per family member is weighted by the number of family members, L_t. So it is the sum of the family members' utility that counts. Such a utility function is called a *classical-utilitarian* utility function (with discounting).

The infinite horizon of the "household" may be seen as reflecting an altruistic bequest motive. That is, the household is seen as an infinitely-lived family, a family dynasty. The current members of the dynasty act in unity and are concerned about the utility from own consumption as well as the utility of the future generations within the dynasty.[1] Note that in this setup, births (into adult life) do not reflect the emergence of *new* economic agents with independent interests. Births and population growth are seen as just an expansion of the size of already existing infinitely-lived households. In contrast, in the Diamond OLG model births imply entrance of new economic decision makers whose preferences no-one has cared about in advance.

Because of (10.2), U_0 can be written as

$$U_0 = \int_0^\infty u(c_t)e^{-(\rho-n)t}dt, \tag{10.3}$$

where the unimportant positive factor L_0 has been eliminated. Here $\bar{\rho} \equiv \rho - n$ is known as the *effective* rate of time preference while ρ is the *pure* rate of time preference. We later introduce a restriction on $\rho - n$ to ensure upward boundedness of the utility integral in general equilibrium.

The household chooses a consumption-saving plan which maximizes U_0 subject to its budget constraint. Let $A_t \equiv a_t L_t$ be the household's (net) financial wealth in real terms at time t. It is of no consequence whether we imagine the components of this wealth are capital goods or loans to other agents in the economy. We have

$$\dot{A}_t = r_t A_t + w_t L_t - c_t L_t, \qquad A_0 \text{ given.} \tag{10.4}$$

This equation is a book-keeping relation telling how financial wealth or debt $(-A)$ is evolving depending on how consumption relates to current income. The equation merely says that the increase in financial wealth per time unit equals saving which equals income minus consumption. Income is the sum of the net return on financial wealth, $r_t A_t$, and labor income, $w_t L_t$, where w_t is the real wage.[2] Saving can be negative. In that case the household "dissaves" and does so simply by selling a part of its stock of capital goods or by taking loans in the loan market.[3]

When the dynamic budget identity (10.4) is combined with a requirement of solvency, we have a budget *constraint.* The relevant solvency requirement is the No-Ponzi-Game condition (NPG for short):

$$\lim_{t\to\infty} A_t e^{-\int_0^t r_s ds} \geq 0. \tag{10.5}$$

[1] The Barro model of Chapter 7 exemplifies such a structure in discrete time. In that chapter we also discussed some of the shortcomings of the dynasty setup.

[2] Since the technology exhibits constant returns to scale, in competitive equilibrium the firms make no (pure) profit to pay out to their owners (presumably the households).

[3] The market prices, w_t and r_t, faced by the household are assumed to be piecewise continuous functions of time.

This condition says that financial wealth far out in the future cannot have a negative present value. That is, in the long run, debt is at most allowed to rise at a rate *less* than the real interest rate r. The NPG condition thus precludes permanent financing of the interest payments by new loans.[4]

The decision problem is: choose a plan $(c_t)_{t=0}^{\infty}$ so as to achieve a maximum of U_0 subject to non-negativity of the control variable, c, and the constraints (10.4) and (10.5). The problem is a slight generalization of the problem studied in Section 9.4 of the previous chapter.

To solve the problem we shall apply the Maximum Principle. This method can be applied directly to the problem as stated above or to an equivalent problem with constraints expressed in per capita terms. Let us follow the latter approach. From the definition $a_t \equiv A_t/L_t$ we get by differentiation w.r.t. t

$$\dot{a}_t = \frac{L_t \dot{A}_t - A_t \dot{L}_t}{L_t^2} = \frac{\dot{A}_t}{L_t} - a_t n.$$

Substitution of (10.4) gives the dynamic budget identity in per capita terms:

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t, \qquad a_0 \text{ given.} \tag{10.6}$$

By inserting $A_t \equiv a_t L_t = a_t L_0 e^{nt}$, the No-Ponzi-Game condition (10.5) can be rewritten as

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0, \tag{10.7}$$

where the unimportant factor L_0 has been eliminated.

We see that in both (10.6) and (10.7) a kind of corrected interest rate appears, namely the interest rate, r, minus the family size growth rate, n. Although deferring consumption gives a real interest rate of r, this return is diluted on a per head basis because it will have to be shared with more members of the family when $n > 0$. In the form (10.7) the NPG condition requires that debt, if any, in the long run rises at most at a rate *less* than $r - n$.

Solving the consumption/saving problem

The decision problem is now: choose $(c_t)_{t=\infty}^{0}$ so as to a maximize U_0 subject to the constraints: $c_t \geq 0$, (10.6), and (10.7). To solve the problem we use Pontryagin's Maximum Principle. The solution procedure is similar to that in the slightly simpler problem in Section 9.4 of the previous chapter:

[4] In the previous chapter we saw that the NPG condition, in combination with (10.4), is equivalent to an ordinary *intertemporal* budget constraint which says that the present value of the planned consumption path cannot exceed initial total wealth, i.e., the sum of the initial financial wealth and the present value of expected future labor income.

1) Set up the current-value Hamiltonian

$$H(a, c, \lambda, t) = u(c) + \lambda\left[(r - n)\,a + w - c\right],$$

where λ is the *adjoint variable* associated with the dynamic constraint (10.6).

2) Differentiate H partially w.r.t. the control variable, c, and put the result equal to zero:

$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0. \tag{10.8}$$

3) Differentiate H partially w.r.t. the state variable, a, and put the result equal to the effective discount rate (appearing in the integrand of the criterion function) multiplied by λ minus the time derivative of λ:

$$\frac{\partial H}{\partial a} = \lambda(r - n) = (\rho - n)\lambda - \dot{\lambda}. \tag{10.9}$$

4) Apply the Maximum Principle: an interior optimal path $(a_t, c_t)_{t=0}^{\infty}$ will satisfy that there exists a continuous function $\lambda = \lambda(t)$ such that for all $t \geq 0$, (10.8) and (10.9) hold along the path and the transversality condition,

$$\lim_{t\to\infty} a_t\lambda_t e^{-(\rho-n)t} = 0, \tag{10.10}$$

is satisfied.

The interpretation of these optimality conditions is as follows. The condition (10.8) can be considered a $MC = MB$ condition (in utility terms). It illustrates together with (10.9) that the adjoint variable, λ, constitutes the shadow price, measured in current utility, of per head financial wealth along the optimal path. Rearranging (10.9) gives, $r_t = \rho - \dot{\lambda}_t/\lambda_t$; the left-hand-side of this equation is the market rate of return on saving while the right-hand-side is the *required* rate of return. The household is willing to save the marginal unit up to the point where the actual return equals the required return.

The transversality condition (10.10) says that for $t \to \infty$ the present shadow value of per head financial wealth should go to zero. Combined with (10.8), the condition is that

$$\lim_{t\to\infty} a_t u'(c_t) e^{-(\rho-n)t} = 0 \tag{10.11}$$

must hold along the optimal path. This requirement is not surprising if we compare with the case where instead $\lim_{t\to\infty} a_t u'(c_t) e^{-(\rho-n)t} > 0$. In this case there would be over-saving; U_0 could be increased by reducing the "ultimate" a_t and thereby, before eternity, consume more and save less. The opposite case, $\lim_{t\to\infty} a_t u'(c_t)e^{-(\rho-n)t} < 0$, will not even satisfy the NPG condition in view of Proposition 2 of the previous chapter. In fact, from that proposition we know that the transversality condition (10.11) is equivalent with the NPG condition (10.7) being satisfied with strict equality, i.e.,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s-n)ds} = 0. \tag{10.12}$$

Recall that the Maximum Principle gives only *necessary* conditions for an optimal plan. But since the Hamiltonian is jointly concave in (a, c) for every t, the necessary conditions are also *sufficient*, by Mangasarian's sufficiency theorem.

The first-order conditions (10.8) and (10.9) give the Keynes-Ramsey rule:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta(c_t)}(r_t - \rho), \tag{10.13}$$

where $\theta(c_t)$ is the (absolute) elasticity of marginal utility,

$$\theta(c_t) \equiv -\frac{c_t}{u'(c_t)}u''(c_t) > 0. \tag{10.14}$$

As we know from previous chapters, this elasticity indicates the consumer's wish to smooth consumption over time. The inverse of $\theta(c_t)$ is the elasticity of intertemporal substitution in consumption. It indicates the willingness to incur variation in consumption over time in response to a change in the interest rate.

Interestingly, the population growth rate, n, does not appear in the Keynes-Ramsey rule. Going from $n = 0$ to $n > 0$ implies that r_t is replaced by $r_t - n$ in the dynamic budget identity and ρ is replaced by $\rho - n$ in the criterion function. This implies that n cancels out in the Keynes-Ramsey rule. Yet n appears in the transversality condition and thereby also in the *level* of consumption for given wealth, cf. (10.18) below.

CRRA utility

In order that the model can accommodate Kaldor's stylized facts, it should be able to generate a balanced growth path. When the population grows at the same constant rate as the labor force, here n, by definition balanced growth requires that per capita output, per capita capital, and per capita consumption grow at constant rates. This will generally require that the real interest rate is constant in the process. But (11.29) shows that having a constant per capita consumption growth rate at the same time as r is constant, is only possible if the elasticity of marginal utility does *not* vary with c. Hence, it makes sense to assume that the right-hand-side of (10.14) is a positive constant, θ. So we will assume that the instantaneous utility function is of CRRA form:

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \qquad \theta > 0; \tag{10.15}$$

here, for $\theta = 1$, the right-hand side should be interpreted as $\ln c$. So our Keynes-Ramsey rule simplifies to

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho). \tag{10.16}$$

The Keynes-Ramsey rule characterizes the optimal *rate of change* of consumption. The optimal initial *level* of consumption, c_0, will be the highest feasible c_0 which is compatible with both the Keynes-Ramsey rule and the NPG condition. And for this reason the choice will exactly comply with the transversality condition (10.12). Although an explicit determination of c_0 is not actually necessary to pin down the equilibrium path of the economy, we note in passing that c_0 can be found by the method described at the end of Chapter 9. Indeed, given the book-keeping relation (10.6), we have by Proposition 1 of Chapter 9 that the transversality condition (10.12) is equivalent with satisfying the following intertemporal budget constraint (with strict equality):

$$\int_0^\infty c_t e^{-\int_0^t (r_s - n) ds} dt = a_0 + h_0. \tag{10.17}$$

Solving the differential equation (10.16) we get $c_t = c_0 e^{\frac{1}{\theta}\int_0^t (r_s - \rho) ds}$, which we substitute for c_t in (10.17). Isolating c_0 now gives[5]

$$\begin{aligned} c_0 &= \beta_0(a_0 + h_0), \qquad \text{where} \\ \beta_0 &= \frac{1}{\int_0^\infty e^{\int_0^t (\frac{(1-\theta)r_s - \rho}{\theta} + n) ds} dt}, \qquad \text{and} \\ h_0 &= \int_0^\infty w_t e^{-\int_0^t (r_s - n) ds} dt. \end{aligned} \tag{10.18}$$

Thus, the entire expected future evolution of wages and interest rates determines c_0. The marginal propensity to consume out of wealth, β_0, is less, the greater is the population growth rate, n.[6] The explanation is that the

[5] These formulas can also be derived directly from Example 1 of Section 9.5 of Chapter 9 by replacing $r(\tau)$ and ρ by $r(\tau) - n$ and $\rho - n$, respectively.

[6] This holds also if $\theta = 1$, since in that case $\beta_0 = \rho - n$.

effective utility discount rate, $\rho - n$, is less, the greater is n. The propensity to save is greater the more mouths to feed in the future. The initial saving level will be $r_0a_0 + w_0 - c_0 = r_0a_0 + w_0 - \beta_0(a_0 + h_0)$.

In the Solow growth model the saving-income ratio is a parameter, a given constant. The Ramsey model endogenizes the saving-income ratio. Solow's parametric saving rate is replaced by two "deeper" parameters, the rate of impatience, ρ, and the desire for consumption smoothing, θ. As we shall see the resulting saving-income ratio will not generally be constant outside the steady state of the dynamic system implied by the Ramsey model.

10.2.2 Firms

There is a large number of firms which maximize profits under perfect competition. All firms have the same neoclassical production function with CRS,

$$Y_t = F(K_t^d, T_t L_t^d) \tag{10.19}$$

where Y_t is supply of output, K_t^d is capital input, and L_t^d is labor input, all measured per time unit, at time t. The superscript d on the two inputs indicates that these inputs are seen from the demand side. The factor T_t represents the economy-wide level of technology as of time t and is exogenous. We assume there is technological progress at a constant rate g (≥ 0) :

$$T_t = T_0 e^{gt}, \qquad T_0 > 0. \tag{10.20}$$

Thus the economy features Harrod-neutral technological progress, as is needed for compliance with Kaldor's stylized facts.

Necessary and sufficient conditions for the factor combination (K_t^d, L_t^d), where $K_t^d > 0$ and $L_t^d > 0$, to maximize profits are that

$$F_1(K_t^d, T_t L_t^d) = \hat{r}_t, \tag{10.21}$$

$$F_2(K_t^d, T_t L_t^d) T_t = w_t. \tag{10.22}$$

10.3 General equilibrium

We now consider the economy as a whole and thereby the interaction between households and firms in the various markets. For simplicity, we assume that the number of households is the same as the number of firms. We normalize this number to one so that $F(\cdot, \cdot)$ can from now on be interpreted as the aggregate production function and C_t as aggregate consumption.

Factor markets

In the short term, that is, for fixed t, the available quantities of labor, $L_t = L_0 e^{nt}$, and capital, K_t, are predetermined. The factor markets clear at all points in time, that is,

$$K_t^d = K_t, \quad \text{and} \quad L_t^d = L_t, \tag{10.23}$$

for all $t \geq 0$. It is the rental rate, $\hat{r}_t$, and the wage rate, w_t, which adjust (immediately) so that this is achieved. Aggregate output can be written

$$Y_t = F(K_t, T_t L_t) = T_t L_t F(\tilde{k}_t, 1) \equiv T_t L_t f(\tilde{k}_t), \qquad f' > 0, f'' < 0, \tag{10.24}$$

where $\tilde{k}_t \equiv K_t/(T_t L_t)$. Substituting (10.23) into (10.21) and (10.22), we find the equilibrium interest rate and wage rate:

$$r_t = \hat{r}_t - \delta = \frac{\partial(T_t L_t f(\tilde{k}_t))}{\partial K_t} - \delta = f'(\tilde{k}_t) - \delta, \tag{10.25}$$

$$w_t = \frac{\partial(T_t L_t f(\tilde{k}_t))}{\partial L_t} T_t = \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \right] T_t \equiv \tilde{w}(\tilde{k}_t) T_t, \tag{10.26}$$

where $\tilde{k}_t$ is at any point in time predetermined and where in (10.25) we have used the no-arbitrage condition (10.1).

Capital accumulation

From now we leave out the explicit dating of the variables when not needed for clarity. By national product accounting we have

$$\dot{K} = Y - C - \delta K. \tag{10.27}$$

Let us check whether we get the same result from the wealth accumulation equation of the household. Because physical capital is the only asset in the economy, aggregate financial wealth, A, at time t equals the total quantity of capital, K, at time t.[7] From (10.4) we thus have

$$\begin{aligned}
\dot{K} &= rK + wL - cL \\
&= (f'(\tilde{k}) - \delta)K + (f(\tilde{k}) - \tilde{k} f'(\tilde{k}))TL - cL \qquad \text{(from (10.25) and (10.26))} \\
&= f(\tilde{k})TL - \delta K - cL \qquad \text{(by rearranging and use of } K \equiv \tilde{k}TL) \\
&= F(K, TL) - \delta K - C = Y - C - \delta K \qquad \text{(by } C \equiv cL).
\end{aligned}$$

[7] Whatever financial claims on each other the households might have, they net out for the household sector as a whole.

Hence, the book-keeping is in order (the national income account is consistent with the national product account).

We now face a fundamental difference as compared with models where households have a finite horizon, such as the Diamond OLG model. Current consumption cannot be determined independently of the expected *long-term* evolution of the economy. This is because consumption and saving, as we saw in Section 10.2, depend on the expectations of the entire future evolution of wages and interest rates. And given the presumption of perfect foresight, the households' expectations are identical to the prediction that can be calculated from the model. In this way there is interdependence between expectations and the level and evolution of consumption. We can determine the level of consumption only in the context of the overall dynamic analysis. In fact, the economic agents are in some sense in the same situation as the outside analyst. They, too, have to think through the entire dynamics of the economy in order to form their rational expectations.

The dynamic system

We get a concise picture of the dynamics by reducing the model to the minimum number of coupled differential equations. This minimum number is two. The key endogenous variables are $\tilde{k} \equiv K/(TL)$ and $\tilde{c} \equiv C/(TL) \equiv c/T$. Using the rule for the growth rate of a fraction, we get

$$\begin{aligned}
\frac{\dot{\tilde{k}}}{\tilde{k}} &= \frac{\dot{K}}{K} - \frac{\dot{T}}{T} - \frac{\dot{L}}{L} = \frac{\dot{K}}{K} - (g+n) \qquad \text{(from (10.2) and (10.20))} \\
&= \frac{F(K,TL) - C - \delta K}{K} - (g+n) \qquad \text{(from (10.27))} \\
&= \frac{f(\tilde{k}) - \tilde{c}}{\tilde{k}} - (\delta + g + n) \qquad \text{(from (10.24))}.
\end{aligned}$$

The associated differential equation for $\tilde{c}$ is obtained in a similar way:

$$\begin{aligned}
\frac{\dot{\tilde{c}}}{\tilde{c}} &= \frac{\dot{c}}{c} - \frac{\dot{T}}{T} = \frac{1}{\theta}(r_t - \rho) - g \quad \text{(from the Keynes-Ramsey rule)} \\
&= \frac{1}{\theta}\left[f'(\tilde{k}) - \delta - \rho - \theta g\right] \qquad \text{(from (10.25))}.
\end{aligned}$$

We thus end up with the dynamic system

$$\dot{\tilde{k}} = f(\tilde{k}) - \tilde{c} - (\delta + g + n)\tilde{k}, \qquad \tilde{k}_0 > 0 \quad \text{given}, \tag{10.28}$$

$$\dot{\tilde{c}} = \frac{1}{\theta}\left[f'(\tilde{k}) - \delta - \rho - \theta g\right]\tilde{c}. \tag{10.29}$$

We have no given initial value of c. Instead we have the transversality condition (10.12).

The lower panel of Fig. 10.1 shows the *phase diagram* of the system. The curve OEB represents the points where $\dot{\tilde{k}} = 0$ and is called the *nullcline* for the differential equation (10.28). We see from (10.28) that

$$\dot{\tilde{k}} = 0 \text{ for } \tilde{c} = f(\tilde{k}) - (\delta + g + n)\tilde{k} \equiv \tilde{c}(\tilde{k}). \tag{10.30}$$

The upper panel of Fig. 10.1 displays the value of $\tilde{c}(\tilde{k})$ as the vertical distance between the curve $\tilde{y} = f(\tilde{k})$ and the line $\tilde{y} = (\delta + g + n)\tilde{k}$ (to save space the proportions are distorted).[8] The maximum value of $\tilde{c}(\tilde{k})$, if it exists, is reached at the point where the tangent to the OEB curve in the lower panel is horizontal, i.e., where $\tilde{c}'(\tilde{k}) = f'(\tilde{k}) - (\delta + g + n) = 0$ or $f'(\tilde{k}) - \delta = g + n$. The value of $\tilde{k}$ which satisfies this is the golden rule capital intensity, $\tilde{k}_{GR}$:

$$f'(\tilde{k}_{GR}) - \delta = g + n. \tag{10.31}$$

From (10.28) we see that for points above the $\dot{\tilde{k}} = 0$ locus we have $\dot{\tilde{k}} < 0$, whereas for points below the $\dot{\tilde{k}} = 0$ locus, $\dot{\tilde{k}} > 0$. The horizontal arrows in the figure indicate these directions of movement.

We also need the nullcline for the differential equation (10.29). We see from (10.29) that

$$\dot{\tilde{c}} = 0 \text{ for } f'(\tilde{k}) = \delta + \rho + \theta g \qquad \text{or} \qquad \tilde{c} = 0. \tag{10.32}$$

Let $\tilde{k}^* > 0$ satisfy the equation $f'(\tilde{k}^*) - \delta = \rho + \theta g$. Then the vertical line $\tilde{k} = \tilde{k}^*$ represents points where $\dot{\tilde{c}} = 0$ (and so does of course the horizontal half-line $\tilde{c} = 0, \tilde{k} \geq 0$). For points to the left of the $\tilde{k} = \tilde{k}^*$ line we have, according to (10.29), $\dot{\tilde{c}} > 0$ and for points to the right of the $\tilde{k} = \tilde{k}^*$ line we have $\dot{\tilde{c}} < 0$. The vertical arrows in Fig. 10.1 indicate these directions of movement. Four illustrative examples of solution curves (*I*, *II*, *III*, and *IV*) are drawn in the figure.

[8] As the graph is drawn, $f(0) = 0$, i.e., capital is assumed essential. But none of the conclusions we are going to consider depends on this.

Steady state

The point E has coordinates $(\tilde{k}^*, \tilde{c}^*)$ and represents the unique steady state.[9] From (10.32) and (10.30) follows that

$$f'(\tilde{k}^*) = \delta + \rho + \theta g, \qquad \text{and} \tag{10.33}$$

$$\tilde{c}^* = f(\tilde{k}^*) - (\delta + g + n)\tilde{k}^*. \tag{10.34}$$

From (10.33) it can be seen that the real interest rate in steady state is

$$r^* = f'(\tilde{k}^*) - \delta = \rho + \theta g. \tag{10.35}$$

The capital intensity satisfying this equation is known as the *modified golden rule* capital intensity, $\tilde{k}_{MGR}$. The modified golden rule is the rule saying that for a representative agent economy to be in steady state, the capital intensity must be such that the net marginal product of capital equals the required rate of return, taking into account the pure rate of time preference, ρ, and the desire for consumption smoothing, measured by θ.[10]

We show below that the steady state is, in a specific sense, asymptotically stable. First we have to make sure, however, that the steady state exists and is consistent with general equilibrium. This consistency requires that the household's transversality condition (10.12) holds in the point E. Using $a_t = K_t/L_t \equiv \tilde{k}_t T_t = \tilde{k}_t T_0 e^{gt}$ and $r_t = f'(\tilde{k}_t) - \delta$, we see that (10.12) is equivalent to

$$\lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t \left(f'(\tilde{k}_s) - \delta - g - n\right) ds} = 0. \tag{10.36}$$

In the point E, $\tilde{k}_t = \tilde{k}^*$ and $f'(\tilde{k}_t) - \delta = \rho + \theta g$ for all t. So the condition (10.36) becomes

$$\lim_{t\to\infty} \tilde{k}^* e^{-(\rho + \theta g - g - n)t} = 0. \tag{10.37}$$

This is fulfilled if and only if $\rho + \theta g > g + n$, that is,

$$\rho - n > (1 - \theta)g. \tag{A1}$$

[9] As (10.32) shows, if $\tilde{c}_t = 0$, then $\dot{\tilde{c}} = 0$. Therefore, mathematically, point B (if it exists) in Fig. 10.1 is also a stationary point of the dynamic system. And if $f(0) = 0$, then according to (10.29) and (10.30) also the point $(0, 0)$ in the figure is a stationary point. But these stationary points have zero consumption forever and are therefore not steady states of any *economic* system. From an economic point of view they are "trivial" steady states.

[10] Note that the ρ of the Ramsey model corresponds to the intergenerational discount rate R of the Barro dynasty model in Chapter 7. Indeed, in the discrete time Barro model we have $1 + r^* = (1 + R)(1 + g)^\theta$, which, by taking logs on both sides and using first-order Taylor approximations of $\ln(1+x)$ around $x = 0$ gives $r^* \approx \ln(1+r^*) = \ln(1+R) + \theta \ln(1+g) \approx R + \theta g$. Recall, however, that in view of the considerable period length (about 25-30 years) of the Barro model, this approximation may not be good.

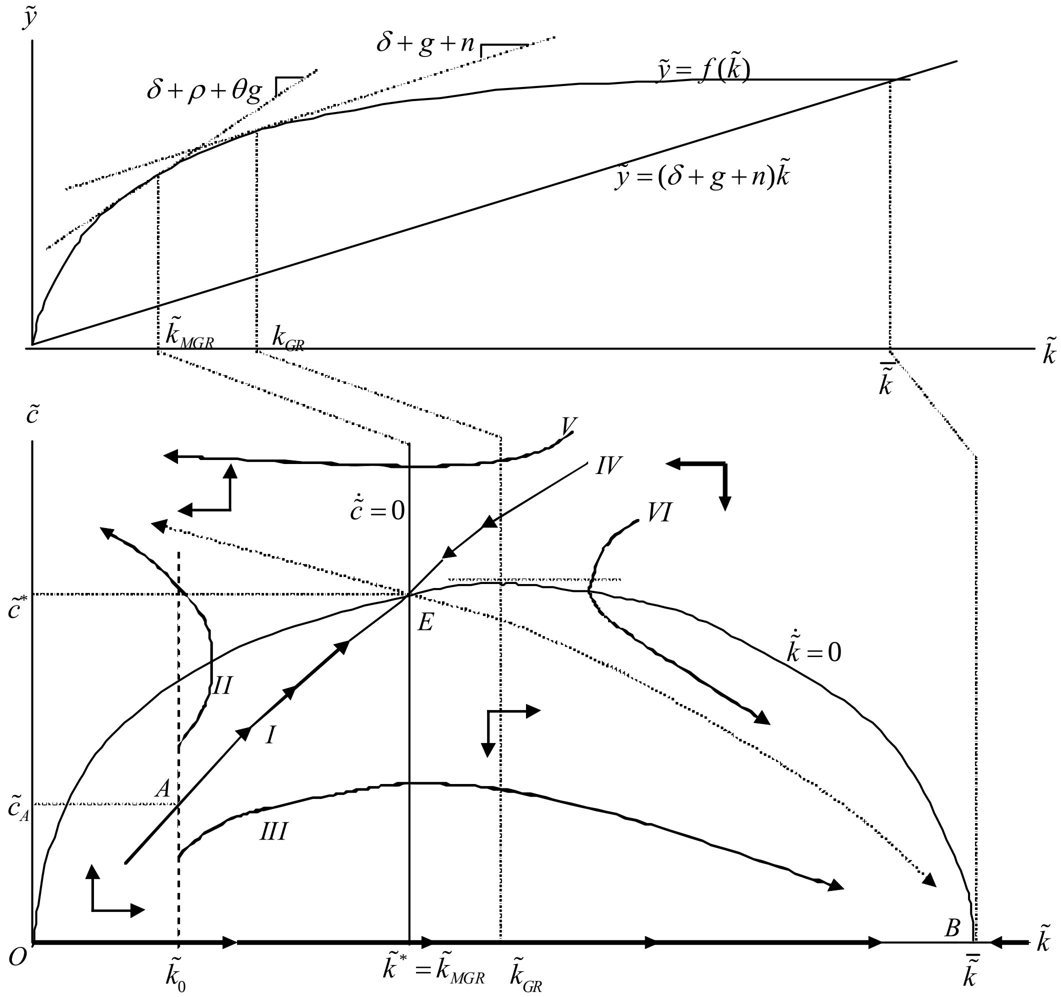

Figure 10.1: Phase portrait of the Ramsey model.

This condition ensures also that the improper integral U_0 is bounded, at least when the system is in steady state (see Appendix B). If $\theta \geq 1$, (A1) is fulfilled as soon as the effective utility discount rate, $\rho - n$, is positive; (A1) may even hold for a negative $\rho - n$ if not "too" negative. If $\theta < 1$, (A1) requires $\rho - n$ to be "sufficiently positive".

Since the parameter restriction (A1) can be written $\rho + \theta g > g + n$, it implies that the steady-state interest rate, r^*, given in (10.35), is higher than the "natural" growth rate, $g + n$. If this did not hold, the transversality condition (10.12) would fail in the steady state. Indeed, along the steady state path we have

$$a_t e^{-r^*t} = a_0 e^{(g+n)t} e^{-r^*t} = k_0 e^{(g+n-r^*)t},$$

which would take the value $k_0 > 0$ for all $t \geq 0$ if $r^* = g + n$ and would go to ∞ for $t \to \infty$ if $r^* < g + n$. The individual households would be over-saving. Each household would in this situation alter its behavior and the steady state could thus not be an equilibrium path.

Another way of seeing that $r^* \leq g + n$ can never be an equilibrium in a Ramsey model is to recognize that this condition would make the household's human wealth infinite because wage income, wL, would grow at a rate, $g+n$, at least as high as the real interest rate, r^*. This would motivate an immediate increase in consumption and so the considered steady-state path would again not be an equilibrium.

To have a model of interest, from now on we assume that the parameters satisfy the inequality (A1). As an implication, the capital intensity in steady state, $\tilde{k}^*$, is less than the golden rule value $\tilde{k}_{GR}$. Indeed, $f'(\tilde{k}^*) - \delta = \rho + \theta g > g + n = f'(\tilde{k}_{GR}) - \delta$, so that $\tilde{k}^* < \tilde{k}_{GR}$, in view of $f'' < 0$.

So far we have only ensured that if the steady state, E, exists, it is consistent with general equilibrium. Existence of both a steady state and a golden rule capital intensity requires that the marginal productivity of capital is sufficiently sensitive to variation in the capital intensity. We therefore assume that f has the properties

$$\lim_{\tilde{k} \to 0} f'(\tilde{k}) - \delta > \rho + \theta g \quad \text{and} \quad \lim_{\tilde{k} \to \infty} f'(\tilde{k}) - \delta < g + n. \tag{A2}$$

Together with (A1) this implies $\lim_{\tilde{k} \to 0} f'(\tilde{k}) - \delta > \rho + \theta g > g + n > \lim_{\tilde{k} \to \infty} f'(\tilde{k}) - \delta$. By continuity of f', these inequalities ensure the existence of both $\tilde{k}^* > 0$ and $\tilde{k}_{GR} > 0$.[11] Moreover, the inequalities ensure the existence

[11] The often presumed Inada conditions, $\lim_{\tilde{k} \to 0} f'(\tilde{k}) = \infty$ and $\lim_{\tilde{k} \to \infty} f'(\tilde{k}) = 0$, are stricter than (A2) and not necessary.

of a $\bar{\tilde{k}} > 0$ with the property that $f(\bar{\tilde{k}}) - (\delta + g + n)\tilde{k} = 0$, as in Fig. 10.1.[12] Because $f'(\tilde{k}) > 0$ for all $\tilde{k} > 0$, it is implicit in the technology assumption (A2) that $\delta + g + n > 0$. Even without deciding on the sign of n (a decreasing workforce should not be excluded in our days), this seems like a plausible presumption.

Trajectories in the phase diagram

A first condition for a path $(\tilde{k}_t, \tilde{c}_t)$, with $\tilde{k}_t > 0$ and $\tilde{c}_t > 0$ for all $t \geq 0$, to be a solution to the model is that it satisfies the system of differential equations (10.28)-(10.29). Indeed, it must satisfy (10.28) to be technically feasible and it must satisfy (10.29) to comply with the Keynes-Ramsey rule. Technical feasibility of the path also requires that its initial value for $\tilde{k}$ equals the historically given (pre-determined) value $\tilde{k}_0 \equiv K_0/(T_0 L_0)$. In contrast, for $\tilde{c}$ we have no exogenously given initial value. This is because $\tilde{c}_0$ is a so-called *jump variable* or *forward-looking variable.* By this is meant an endogenous variable which can immediately shift to another value if new information arrives so as to alter expectations about the future. We shall see that the terminal condition (10.36), reflecting the transversality condition of the households, makes up for this lack of an initial condition for c.

In Fig. 10.1 we have drawn some possible paths that could be solutions as t increases. We are especially interested in the paths which are consistent with the historically given $\tilde{k}_0$, that is, paths starting at some point on the stippled vertical line in the figure. If the economy starts out with a high value of $\tilde{c}$, it will follow a curve like II in the figure. The low level of saving implies that the capital stock goes to zero in finite time (see Appendix C). If the economy starts out with a low level of $\tilde{c}$, it will follow a curve like III in the figure. The high level of saving implies that the capital intensity converges towards $\bar{\tilde{k}}$ in the figure.

All in all this suggests the existence of an initial level of consumption somewhere in between, which gives a path like I. Indeed, since the curve II emerged with a high $\tilde{c}_0$, then by lowering this $\tilde{c}_0$ slightly, a path will emerge in which the maximal value of $\tilde{k}$ on the $\dot{\tilde{k}} = 0$ locus is greater than curve II's maximal $\tilde{k}$ value.[13] We continue lowering $\tilde{c}_0$ until the path's maximal $\tilde{k}$ value is exactly equal to $\tilde{k}^*$. The path which emerges from this, namely the

[12] We claim that $\bar{\tilde{k}} > \tilde{k}_{GR}$ *must* hold. Indeed, this inequality follows from $f'(\tilde{k}_{GR}) = \delta + n + g \equiv f(\bar{\tilde{k}})/\bar{\tilde{k}} > f'(\bar{\tilde{k}})$, the latter inequality being due to $f'' < 0$ and $f(0) \geq 0$.

[13] As an implication of the uniqueness theorem for differential equations (see Math tools), two solution paths in the phase plane cannot intersect.

path I, starting at the point A, is special in that it converges towards the steady-state point E. No other path starting at the stippled line, $\tilde{k} = \tilde{k}_0$, has this property. Paths starting above A do not, as we just saw. It is similar for a path starting below A, like path III. Either this path never reaches the consumption level $\tilde{c}_A$ and then it can not converge to E, of course. Or, after a while its consumption level reaches $\tilde{c}_A$, but at the same time it has $\tilde{k} > \tilde{k}_0$. From then on, as long as $\tilde{k} \leq \tilde{k}^*$, for every $\tilde{c}$-value that path III has in common with path I, path III has a higher $\dot{\tilde{k}}$ and a lower $\dot{\tilde{c}}$ than path I (use (10.28) and (10.29)). Hence, path III diverges from point E.

Equivalently, had we considered a value of $\tilde{k}_0 > \tilde{k}^*$, there would also be a unique value of $\tilde{c}_0$ such that the path starting from $(\tilde{k}_0, \tilde{c}_0)$ would converge to E (see path IV in Fig. 10.1).

The point E is a *saddle point*. By this is meant a steady-state point with the following property: there exists exactly two paths, one from each side of $\tilde{k}^*$, that converge towards the steady-state point; all other paths (at least starting in a neighborhood of the steady state) move away from the steady state and asymptotically approach one of the two diverging paths, the dotted North-West and South-East curves in Fig. 10.1. The two converging paths together make up what is known as the *stable branch* (or *stable arm)*; on their own they are referred to as *saddle paths* (sometimes referred to in the singular as the saddle path).[14] The dotted diverging paths in Fig. 10.1 together make up the *unstable branch* (or *unstable arm*).

The equilibrium path

A solution to the model is a path which is technically feasible and in addition satisfies a set of equilibrium conditions. In analogy with the definition in discrete time (see Chapter 3) a path $(\tilde{k}_t, \tilde{c}_t)_{t=0}^{\infty}$ is called a *technically feasible path* if (i) the path has $\tilde{k}_t \geq 0$ and $\tilde{c}_t \geq 0$ for all $t \geq 0$; (ii) it satisfies the accounting equation (10.28); and (iii) it starts out, at $t = 0$, with the historically given initial capital intensity. An *equilibrium path* with perfect foresight is then a technically feasible path $(\tilde{k}_t, \tilde{c}_t)_{t=0}^{\infty}$ with the properties that the path (a) is consistent with the households' optimization given their expectations; (b) is consistent with market clearing for all $t \geq 0$; and (c) has the property that the evolution over time of the pair (w_t, r_t), where $w_t = \tilde{w}(\tilde{k}_t)T_t$ and $r_t = f'(\tilde{k}_t) - \delta$, is as expected by the households. The condition (a) in this definition requires the transformed Keynes-Ramsey rule

[14] An algebraic definition of a saddle point, in terms of eigenvalues, is given in Appendix A. There it is also shown that if $\lim_{\tilde{k}\to 0} f(\tilde{k}) = 0$, then the saddle path on the left side of the steady state in Fig. 10.1 will start out infinitely close to the origin.

(10.29) and the transversality condition (10.36) to hold for all $t \geq 0$.

Consider the case where $0 < \tilde{k}_0 < \tilde{k}^*$, as illustrated in Fig. 10.1. Then, the path starting at point A and following the saddle path towards the steady state is an equilibrium path because, by construction, it is technically feasible and in addition has the required properties, (a), (b), and (c). More intuitively: if the households expect an evolution of w_t and r_t corresponding to this path (that is, expect a corresponding underlying movement of $\tilde{k}_t$, which we know unambiguously determines r_t and w_t), then these expectations will induce a behavior the aggregate result of which is an actual path for $(\tilde{k}_t, \tilde{c}_t)$ that confirms the expectations. And along this path the households find no reason to correct their behavior because the path allows both the Keynes-Ramsey rule and the transversality condition to be satisfied.

No other path than the saddle path can be an equilibrium. This is because no other technically feasible path is compatible with the households' individual utility maximization under perfect foresight. An initial point above point A can be excluded in that the implied path, II, does not satisfy the household's NPG condition (and, consequently, not at all the transversality condition).[15] So, if the individual household expected an evolution of r_t and w_t corresponding to path II, then the household would immediately choose a *lower* level of consumption, that is, the household would *deviate* in order not to suffer the same fate as Charles Ponzi. In fact *all* the households would react in this way. Thus path II would not be realized and the expectation that it would, can not be a rational expectation.

Likewise, an initial point below point A can be ruled out because the implied path, III, does not satisfy the household's transversality condition but implies over-saving. Indeed, at some point in the future, say at time t_1, the economy's capital intensity would pass the golden rule value so that for all $t > t_1$, $r_t < g + n$. But with a rate of interest permanently below the growth rate of wage income of the household, the present value of human wealth is *infinite.* This motivates a *higher* consumption level than that along the path. Thus, if the household expects an evolution of r_t and w_t corresponding to path III, then the household will immediately *deviate* and choose a higher initial level of consumption. But so will *all* the households react and the expectation that the economy will follow path III can not be rational.

We have presumed $0 < \tilde{k}_0 < \tilde{k}^*$. If instead $\tilde{k}_0 > \tilde{k}^*$, the economy would move along the saddle path *from above.* Paths like V and VI in Fig. 10.1 can be ruled out because they violate the NPG condition and the transversality condition, respectively. With this we have shown:

PROPOSITION 1 Assume (A1) and (A2). Let there be a given $\tilde{k}_0 > 0$.

[15] This is shown in Appendix C.

Then the Ramsey model exhibits a unique equilibrium path, characterized by $(\tilde{k}_t, \tilde{c}_t)$ converging, for $t \to \infty$, towards a unique steady state with a capital intensity $\tilde{k}^*$ satisfying $f'(\tilde{k}^*) - \delta = \rho + \theta g$. In the steady state the real interest rate is given by the modified golden rule formula, $r^* = \rho + \theta g$, the per capita consumption path is $c_t^* = \tilde{c}^* T_0 e^{gt}$, where $\tilde{c}^* = f(\tilde{k}^*) - (\delta + g + n)\tilde{k}^*$, and the real wage path is $w_t^* = \tilde{w}(\tilde{k}^*) T_0 e^{gt}$.

A numerical example based on one year as the time unit: $\theta = 2$, $g = 0.02$, $n = 0.01$ and $\rho = 0.01$. Then, $r^* = 0.05 > 0.03 = g + n$.

So output per capita, $y_t \equiv Y_t/L_t \equiv \tilde{y}_t T_t$, tends to grow at the rate of technological progress, g :

$$\frac{\dot{y}_t}{y_t} \equiv \frac{\dot{\tilde{y}}_t}{\tilde{y}_t} + \frac{\dot{T}_t}{T_t} = \frac{f'(\tilde{k}_t)\dot{\tilde{k}}_t}{f(\tilde{k}_t)} + g \to g \quad \text{for} \quad t \to \infty,$$

in view of $\dot{\tilde{k}}_t \to 0$. This is also true for the growth rate of consumption per capita and the real wage, since $c_t \equiv \tilde{c}_t T_t$ and $w_t = \tilde{w}(\tilde{k}_t) T_t$.

The intuition behind the convergence lies in the neoclassical principle that starting from a low capital intensity and therefore high marginal and average product of capital, the resulting high aggregate saving[16] will be more than enough to maintain the capital intensity which therefore increases. But when this happens, the marginal and average product of capital decreases and the resulting saving, as a proportion of the capital stock, declines until eventually it is only sufficient to replace worn-out machines and equip new "effective" workers with enough machines to just maintain the capital intensity. If instead we start from a high capital intensity a similar story can be told in reverse. Thus in the Ramsey model the long-run state is attained when the marginal saving and investment yields a return as great as the representative household's willingness to postpone the marginal unit of consumption.

The equilibrium path generated by the Ramsey model is necessarily dynamically efficient and satisfies the modified golden rule in the long run. Why this contrast to Diamonds OLG model where equilibrium paths *may* be dynamically inefficient? The reason lies in the fact that only a "single infinity", not a "double infinity", is involved in the Ramsey model. The time horizon of the economy is infinite but the number of decision makers is finite. Births (into adult life) do not reflect the emergence of new economic agents with separate interests. It is otherwise in the Diamond OLG model where births imply the entrance of new economic decision makers whose preferences

[16] Saving will be high because the negative substitution and wealth effects on current consumption of the high interest rate dominate the income effect.

no-one has cared about in advance. In that model neither is there any final date, nor any final decision maker. It is this difference that lies behind that the two models in some respects give different results. A type of equilibria, namely dynamically inefficient ones, can be realized in the Diamond model but not so in the Ramsey model. A rate of time preference low enough to generate a *tendency* to a long-run interest rate below the income growth rate is inconsistent with the conditions needed for general equilibrium in the Ramsey model. And such a low rate of time preference is in fact ruled out in the Ramsey model by the parameter restriction (A1).

The concept of saddle-point stability

The steady state of the model is globally asymptotically stable for arbitrary initial values of the capital intensity (the phase diagram only verifies local asymptotic stability, but the extension to global asymptotic stability is verified in Appendix A). If $\tilde{k}$ is hit by a shock at time 0 (say by a discrete jump in the technology level T_0), the economy will converge toward the same unique steady state as before. At first glance this might seem peculiar considering that the steady state is a saddle point. Such a steady state is unstable for arbitrary initial values of *both* variables, $\tilde{k}$ and $\tilde{c}$. But the crux of the matter is that it is only the initial $\tilde{k}$ that *is* arbitrary. The model assumes that the decision variable c_0, and therefore the value of $\tilde{c}_0 \equiv c_0/T_0$, immediately adjusts to the given circumstances and information about the future. That is, the model assumes that $\tilde{c}_0$ always takes the value needed for the household's transversality condition under perfect foresight to be satisfied. This ensures that the economy is initially on the saddle path, cf. the point A in Fig. 10.1. In the language of differential equations *conditional* asymptotic stability is present. The condition that ensures the stability in our case is the transversality condition.

We shall follow the common terminology in macroeconomics and call a steady state of a two-dimensional dynamic system (locally) *saddle-point stable* if:

1. the steady state is a saddle point;
2. there is one predetermined variable and one jump variable;
3. the saddle path is not parallel to the jump variable axis; and
4. there is a boundary condition on the system such that the diverging paths are ruled out as solutions.

Thus, to establish saddle-point stability all four properties must be verified. If for instance point 1 and 2 hold but, contrary to point 3, the saddle path is parallel to the jump variable axis, then saddle-point stability does not obtain. Indeed, given that the predetermined variable initially deviated from its steady-state value, it would not be possible to find any initial value of the jump variable such that the solution of the system would converge to the steady state for $t \rightarrow \infty$

In the present case, we have already verified point 1 and 2. And as the phase diagram indicates, the saddle path is not vertical. So also point 3 holds. The transversality condition ensures that also point 4 holds. Thus, the Ramsey model is saddle-point stable. In Appendix A it is shown that the positively-sloped saddle path in Fig. 10.1 ranges over *all* $\tilde{k} > 0$ (there is nowhere a vertical asymptote to the saddle path). Hence, the steady state is *globally* saddle point stable. All in all, these characteristics of the Ramsey model are analogue to those of Barro's dynasty model in discrete time when the bequest motive is operative.

10.4 Comparative analysis

10.4.1 The role of key parameters

The conclusion that in the long run the real interest rate is given by the modified golden rule formula, $r^* = \rho + \theta g$, tells us that only three parameters matter: the rate of time preference, the elasticity of marginal utility, and the rate of technological progress. A higher ρ, i.e., more impatience and thereby less willingness to defer consumption, implies less capital accumulation and thus smaller capital intensity and in the long run a higher interest rate and lower consumption than otherwise. The long-run growth rate is unaffected.

A higher desire for consumption smoothing, θ, will have a similar effect in that it implies that a larger part of the greater consumption opportunities in the future (reflecting a positive g) will be consumed immediately. Similarly, the long-run interest rate will depend positively on g (the growth rate of labor productivity) because the higher this is, the greater is the expected future wage income and the associated consumption possibilities even without any current saving. This discourages current saving and we end up with lower capital accumulation and lower effective capital intensity in the long run, hence higher interest rate. It is also true that the higher is g, the higher is the rate of return needed to induce the saving required for maintaining a steady state and resist the desire for more consumption smoothing.

The long-run interest rate is independent of the particular form of the aggregate production function, f. This function matters for *what* effective capital intensity and *what* consumption level per unit of effective labor are compatible with the long-run interest rate. This kind of results are specific to representative agent models. This is because only in these models will the Keynes-Ramsey rule hold not only for the individual household, but also at the aggregate level.

Unlike the Solow growth model, the Ramsey model provides a *theory* of the evolution and long-run level of the saving rate. The endogenous gross saving rate of the economy is

$$\begin{aligned} s_t &\equiv \frac{Y_t - C_t}{Y_t} = \frac{\dot{K}_t + \delta K_t}{Y_t} = \frac{\dot{K}_t/K_t + \delta}{Y_t/K_t} = \frac{\dot{\tilde{k}}_t/\tilde{k}_t + g + n + \delta}{f(\tilde{k}_t)/\tilde{k}_t} \\ &\rightarrow \frac{g+n+\delta}{f(\tilde{k}^*)/\tilde{k}^*} \equiv s^* \quad \text{for} \quad t \rightarrow \infty. \end{aligned} \tag{10.38}$$

By determining the path of $\tilde{k}_t$, the Ramsey model determines how s_t moves over time and adjusts to its constant long-run level. Indeed, for any given $\tilde{k} > 0$, the equilibrium value of $\tilde{c}_t$ is uniquely determined by the requirement that the economy must be on the saddle path. Since this defines $\tilde{c}_t$ as a function, $\tilde{c}(\tilde{k}_t)$, of $\tilde{k}_t$, there is a corresponding function for the saving rate in that $s_t = 1 - \tilde{c}(\tilde{k}_t)/f(\tilde{k}_t) \equiv s(\tilde{k}_t)$; so $s(\tilde{k}^*) = s^*$.

We note that the long-run saving rate is a decreasing function of the rate of impatience, ρ, and the desire of consumption smoothing, θ; it is an increasing function of the capital depreciation rate, δ, and the rate of population growth, n.

For an example with an explicit formula for the long-run saving rate, consider:

EXAMPLE 1 Suppose the production function is Cobb-Douglas:

$$\tilde{y} = f(\tilde{k}) = A\tilde{k}^{\alpha}, \qquad A > 0, 0 < \alpha < 1. \tag{10.39}$$

Then $f'(\tilde{k}) = A\alpha\tilde{k}^{\alpha-1} = \alpha f(\tilde{k})/\tilde{k}$. In steady state we get, by use of the steady-state result (10.33),

$$\frac{f(\tilde{k}^*)}{\tilde{k}^*} = \frac{1}{\alpha} f'(\tilde{k}^*) = \frac{\delta + \rho + \theta g}{\alpha}.$$

Substitution in (10.38) gives

$$s^* = \alpha \frac{\delta + g + n}{\delta + \rho + \theta g} < \alpha, \tag{10.40}$$

where the inequality follows from our parameter restriction (A1). Indeed, (A1) implies $\rho + \theta g > g + n$. The long-run saving rate depends positively on the following parameters: the elasticity of production w.r.t. to capital, α, the capital depreciation rate, δ, and the population growth rate, n. The long-run saving rate depends negatively on the rate of impatience, ρ, and the desire for consumption smoothing, θ. The role of the rate of technological progress is ambiguous.[17]

It can be shown (see Appendix D) that if, by coincidence, $\theta = 1/s^*$, then $s'(\tilde{k}) = 0$, that is, the saving rate s_t is also outside of steady state equal to s^*. In view of (10.40), the condition $\theta = 1/s^*$ is equivalent to the "knife-edge" condition $\theta = (\delta+\rho)/\left[\alpha(\delta + g + n) - g\right] \equiv \bar{\theta}$. More generally, assuming $\alpha(\delta + g + n) > g$ (which seems likely empirically), we have that if $\theta \lesseqqgtr 1/s^*$ (i.e., $\theta \lesseqqgtr \bar{\theta}$), then $s'(\tilde{k}) \lesseqqgtr 0$, respectively (and if instead $\alpha(\delta + g + n) \leq g$, then $s'(\tilde{k}) < 0$, unconditionally).[18] Data presented in Barro and Sala-i-Martin (2004, p. 15) indicate no trend for the US saving rate, but a positive trend for several other developed countries since 1870. One interpretation is that whereas the US has for a long time been close to its steady state, the other countries are still in the adjustment process toward the steady state. As an example, consider the parameter values $\delta = 0.05$, $\rho = 0.02$, $g = 0.02$ and $n = 0.01$. In this case we get $\bar{\theta} = 10$ if $\alpha = 0.33$; given $\theta < 10$, these other countries should then have $s'(\tilde{k}) < 0$ which, according to the model, is compatible with a rising saving rate over time only if these countries are approaching their steady state from *above* (i.e., they should have $\tilde{k}_0 > \tilde{k}^*$). It may be argued that α should also reflect the role of education and R&D in production and thus be higher; with $\alpha = 0.75$ we get $\bar{\theta} = 1.75$. Then, if $\theta > 1.75$, these countries would have $s'(\tilde{k}) > 0$ and thus approach their steady state from *below* (i.e., $\tilde{k}_0 < \tilde{k}^*$). $\square$

10.4.2 Solow's growth model as a special case

The above results give a hint that Solow's growth model, with a given constant saving rate $s \in (0, 1)$ and given δ, g, and n (with $\delta + g + n > 0$), can, under certain circumstances, be interpreted as a special case of the Ramsey model. The Solow model in continuous time is given by

$$\dot{\tilde{k}}_t = sf(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t.$$

[17] Partial differentiation w.r.t. g yields $\partial s^*/\partial g = \alpha\left[\rho - \theta n - (\theta - 1)\delta\right]/(\delta + \rho + \theta g)^2$, the sign of which cannot be determined in general.

[18] See Appendix D.

The constant saving rate implies proportionality between consumption and income. In growth-corrected terms per capita consumption is

$$\tilde{c}_t = (1 - s)f(\tilde{k}_t).$$

For the Ramsey model to yield this, the production function must be like in (10.39) (i.e., Cobb-Douglas) with $\alpha > s$. And the elasticity of marginal utility, θ, must satisfy $\theta = 1/s$. Finally, the rate of time preference, ρ, must be such that (10.40) holds with s^* replaced by s, which implies $\rho = \alpha(\delta + g + n)/s - \delta - \theta g$. It remains to show that this ρ satisfies the inequality, $\rho - n > (1 - \theta)g$, which is necessary for existence of an equilibrium in the Ramsey model. Since $\alpha/s > 1$, the chosen ρ satisfies $\rho > \delta + g + n - \delta - \theta g = n + (1 - \theta)g$, which was to be proved. Thus, in this case the Ramsey model generates an equilibrium path which implies an evolution identical to that generated by the Solow model with $s = 1/\theta$.[19]

With this foundation of the Solow model, it will always hold that $s = s^* < s_{GR}$, where s_{GR} is the golden rule saving rate. Indeed, from (10.38) and (10.31), respectively,

$$s_{GR} = \frac{(\delta + g + n)\tilde{k}_{GR}}{f(\tilde{k}_{GR})} = \frac{f'(\tilde{k}_{GR})\tilde{k}_{GR}}{f(\tilde{k}_{GR})} = \alpha > s^*,$$

from the Cobb-Douglas specification and (10.40), respectively.

A point of the Ramsey model vis-a-vis the Solow model is to replace a mechanical saving rule by maximization of discounted utility and thereby, on the one hand, open up for a wider range of possible evolutions and on the other hand at the same time narrow down the range in certain respects. The model also opens up for studying welfare consequences of alternative economic policies.

10.5 A social planner's problem

Another implication of the Ramsey setup is that the decentralized market equilibrium (within the idealized presumptions of the model) brings about the same allocation of resources as would a social planner facing the same technology and initial resources as described above and having the same criterion function as the representative household. As in Chapter 8, by a *social planner* we mean a hypothetical central authority who is "all-knowing and all-powerful". The social planner is constrained only by the limitations

[19] A more elaborate account of the Solow model as a special case of the Ramsey model is given in Appendix D.

arising from technology and initial resources. Within these confines the social planner can fully decide on the resource allocation. Since we consider a closed economy, the social planner has no access to an international loan market.

Let the economy be closed and let the social welfare function be time separable with constant elasticity, $\hat{\theta}$, of marginal utility and a pure rate of time preference $\hat{\rho}$.[20] Then the social planner's optimization problem is

$$\begin{aligned}
\max_{(c_t)_{t=0}^{\infty}} W_0 &= \int_0^{\infty} \frac{c_t^{1-\hat{\theta}}}{1-\hat{\theta}} e^{-(\hat{\rho}-n)t} dt \qquad \text{s.t.} && (10.41)\\
c_t &\geq 0, && (10.42)\\
\dot{\tilde{k}}_t &= f(\tilde{k}_t) - \frac{c_t}{T_t} - (\delta + g + n)\tilde{k}_t, && (10.43)\\
\tilde{k}_t &\geq 0 \qquad \text{for all } t \geq 0. && (10.44)
\end{aligned}$$

We assume $\hat{\theta} > 0$ and $\hat{\rho} - n > (1 - \hat{\theta})g$ in line with the assumption (A1) for the market economy above. In case $\hat{\theta} = 1$, the expression $c_t^{1-\hat{\theta}}/(1-\hat{\theta})$ should be interpreted as $\ln c_t$. The dynamic constraint (10.43) reflects the national product account. Because the economy is closed, the social planner does not have the opportunity of borrowing or lending from abroad. Hence there is no solvency requirement. Instead we just impose the definitional constraint (10.44) of non-negativity of the state variable $\tilde{k}$. The problem is the continuous time analogue of the social planner's problem in discrete time in Chapter 8. Note, however, a minor conceptual difference, namely that in continuous time there is in the short run no *upper* bound on the *flow* variable c_t, that is, no bound like $c_t \leq T_t \left[f(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t \right]$. A consumption intensity c_t which is higher than the right-hand side of this inequality will just be reflected in a negative value of the flow variable $\dot{\tilde{k}}_t$.[21]

To solve the problem we apply the Maximum Principle. The current-value Hamiltonian is

$$H(\tilde{k}, c, \lambda, t) = \frac{c^{1-\hat{\theta}}}{1-\hat{\theta}} + \lambda \left[f(\tilde{k}) - \frac{c}{T} - (\delta + g + n)\tilde{k} \right],$$

where λ is the adjoint variable associated with the dynamic constraint (10.43). An interior optimal path $(\tilde{k}_t, c_t)_{t=0}^{\infty}$ will satisfy that there exists a continuous

[20] Possible reasons for allowing these two preference parameters to deviate from the corresponding parameters in the private sector are given Chapter 8.

[21] As usual we presume that capital can be "eaten". That is, we consider the capital good to be instantaneously convertible to a consumption good. Otherwise there *would* be at any time an upper bound on c, namely $c \leq Tf(\tilde{k})$, saying that the per capita consumption flow cannot exceed the per capita output flow. The role of such constraints is discussed in Feichtinger and Hartl (1986).

function $\lambda = \lambda(t)$ such that, for all $t \geq 0$,

$$\frac{\partial H}{\partial c} = c^{-\hat{\theta}} - \frac{\lambda}{T} = 0, \text{ i.e., } c^{-\hat{\theta}} = \frac{\lambda}{T}, \quad \text{and} \tag{10.45}$$

$$\frac{\partial H}{\partial \tilde{k}} = \lambda(f'(\tilde{k}) - \delta - g - n) = (\hat{\rho} - n)\lambda - \dot{\lambda} \tag{10.46}$$

hold along the path and the transversality condition,

$$\lim_{t\to\infty} \tilde{k}_t \lambda_t e^{-(\hat{\rho}-n)t} = 0, \tag{10.47}$$

is satisfied.[22]

The condition (10.45) can be seen as a $MC = MB$ condition and illustrates that λ_t is the social planner's shadow price, measured in terms of current utility, of $\tilde{k}_t$ along the optimal path.[23] The differential equation (10.46) tells us how this shadow price evolves over time. The transversality condition, (11.51), together with (10.45), entails the condition

$$\lim_{t\to\infty} \tilde{k}_t c_t^{-\hat{\theta}} e^{gt} e^{-(\hat{\rho}-n)t} = 0,$$

where the unimportant factor T_0 has been eliminated. Imagine the opposite were true, namely that $\lim_{t\to\infty} \tilde{k}_t c_t^{-\hat{\theta}} e^{[g-(\hat{\rho}-n)]t} > 0$. Then, intuitively U_0 could be increased by reducing the long-run value of $\tilde{k}_t$, i.e., consume more and save less.

By taking logs in (10.45) and differentiating w.r.t. t, we get $-\hat{\theta}\dot{c}/c = \dot{\lambda}/\lambda - g$. Inserting (10.46) and rearranging gives the condition

$$\frac{\dot{c}}{c} = \frac{1}{\hat{\theta}}(g - \frac{\dot{\lambda}}{\lambda}) = \frac{1}{\hat{\theta}}(f'(\tilde{k}) - \delta - \hat{\rho}). \tag{10.48}$$

This is the social planner's Keynes-Ramsey rule. If the rate of time preference, $\hat{\rho}$, is lower than the net marginal product of capital, $f'(\tilde{k}) - \delta$, the social planner will let per capita consumption be relatively low in the beginning in order to attain greater per capita consumption later. The lower the impatience relative to the return on capital, the more favorable it becomes to defer consumption.

[22] The infinite-horizon Maximum Principle itself does not guarantee validity of such a straightforward extension of a necessary transversality condition from a finite horizon to an infinite horizon. Yet, this extension *is* valid for the present problem when $\hat{\rho} - n > (1 - \hat{\theta})g$, cf. Appendix E.

[23] Decreasing c_t by one unit, increases $\tilde{k}_t$ by $1/T_t$ units, each of which are worth λ_t utility units to the social planner.

Because $\tilde{c} \equiv c/T$, we get from (11.52) qualitatively the same differential equation for $\tilde{c}$ as we obtained in the decentralized market economy. And the dynamic resource constraint (10.43) is of course identical to that of the decentralized market economy. Thus, the dynamics are in principle unaltered and the phase diagram in Fig. 10.1 is still valid. The solution of the social planner implies that the economy will move along the saddle path towards the steady state. This trajectory, path I in the diagram, satisfies both the first-order conditions and the transversality condition. However, paths such as III in the figure do not satisfy the transversality condition of the social planner but imply permanent over-saving. And paths such as II in the figure will experience a sudden end when all the capital has been used up. Intuitively, they cannot be optimal. A rigorous argument is given in Appendix E, based on the fact that the Hamiltonian is *strictly concave* in $(\tilde{k}, \tilde{c})$. Thence, not only is the saddle path an optimal solution, it is the *unique* optimal solution.

Comparing with the market solution of the previous section, we have established:

PROPOSITION 2 *(equivalence theorem)* Consider an economy with neoclassical CRS technology as described above and a representative infinitely-lived household with preferences as in (11.1) with $u(c) = c^{1-\theta}/(1-\theta)$. Assume (A1) and (A2). Let there be a given $\tilde{k}_0 > 0$. Then perfectly competitive markets bring about the same resource allocation as that brought about by a social planner with the same criterion function as the representative household, i.e., with $\hat{\theta} = \theta$ and $\hat{\rho} = \rho$.

This is a continuous time analogue to the discrete time equivalence theorem of Chapter 8.

The capital intensity $\tilde{k}$ in the social planner's solution will not converge towards the golden rule level, $\tilde{k}_{GR}$, but towards a level whose distance to the golden rule level depends on how much $\hat{\rho} + \hat{\theta}g$ exceeds the natural growth rate, $g + n$. Even if society would be able to consume more in the long term if it aimed for the golden rule level, this would not compensate for the reduction in current consumption which would be necessary to achieve it. This consumption is relatively more valuable, the greater is the social planner's effective rate of time preference, $\hat{\rho} - n$. In line with the market economy, the social planner's solution ends up in a *modified golden rule.* In the long term, net marginal productivity of capital is determined by preference parameters and productivity growth and equals $\hat{\rho} + \hat{\theta}g > g + n$. Hereafter, given the net marginal productivity of capital, the capital intensity and the level of the consumption path is determined by the production function.

Classical versus average utilitarianism

In the above analysis the social planner maximizes the sum of discounted per capita utilities *weighed* by generation size. This is known as discounted *classical utilitarianism.* As an implication, the *effective* utility discount rate, $\rho - n$, varies negatively (one to one) with the population growth rate. Since this corresponds to how the per capita rate of return on saving, $r - n$, is "diluted" by population growth, the net marginal product of capital in steady state becomes independent of n, namely equal to $\hat{\rho} + \hat{\theta}g$.

An alternative to discounted classical utilitarianism is to maximize discounted *per capita* utility. This accords with the principle of discounted *average utilitarianism.* Here the social planner maximizes the sum of discounted per capita utilities without weighing by generation size. Then the effective utility discount rate is independent of the population growth rate, n. With $\hat{\rho}$ still denoting the pure rate of time preference, the criterion function becomes

$$W_0 = \int_0^\infty \frac{{c_t}^{1-\hat{\theta}}}{1-\hat{\theta}} e^{-\hat{\rho}t} dt.$$

The social planner's solution then converges towards a steady state with the net marginal product of capital

$$f'(\tilde{k}^*) - \delta = \hat{\rho} + n + \hat{\theta}g. \tag{10.49}$$

Here, an increase in n will imply higher long-run net marginal product of capital and lower capital intensity, everything else equal.

The representative household in the Ramsey model may of course also have a criterion function in line with discounted average utilitarianism, that is, $U_0 = \int_0^\infty u(c_t)e^{-\rho t}dt$. Then, the interest rate in the economy will in the long run be $r^* = \rho + n + \theta g$ and so an increase in n will increase r^* and decrease $\tilde{k}^*$.

Ramsey's original zero discount rate and the overtaking criterion

It was mostly the perspective of a social planner, rather than the market mechanism, which was at the center of Ramsey's original analysis. The case considered by Ramsey has $g = n = 0$. Ramsey maintained that the social planner should "not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of the imagination" (Ramsey 1928). So Ramsey has $\rho - n = \rho = 0$. Given the instantaneous utility function, u, where $u' > 0, u'' < 0$, and given $\rho = 0$, Ramsey's original problem was: choose $(c_t)_{t=0}^\infty$ so as to optimize (in some sense, see below)

$$\begin{aligned} W_0 &= \int_0^\infty u(c_t)dt \qquad \text{s.t.} \\ c_t &\geq 0, \\ \dot{k}_t &= f(k_t) - c_t - \delta k_t, \\ k_t &\geq 0 \qquad \text{for all } t \geq 0. \end{aligned}$$

A condition corresponding to our assumption (A1) above does not apply. So the improper integral W_0 will generally not be bounded[24] and Ramsey can not use maximization of W_0 as an optimality criterion. Instead he considers a criterion akin to the overtaking criterion we considered in a discrete time context in Chapter 8. We only have to reformulate this criterion for a continuous time setting.

Let $(c_t)_{t=0}^\infty$ be the consumption path associated with an arbitrary technically feasible path and let $(\hat{c}_t)$ be the consumption path associated with our candidate as an optimal path, that is, the path we wish to test for optimality. Define

$$D_T \equiv \int_0^T u(\hat{c}_t)dt - \int_0^T u(c_t)dt. \tag{10.50}$$

Then the feasible path $(\hat{c}_t)_{t=0}^\infty$ is *overtaking optimal*, if for any feasible path, $(c_t)_{t=0}^\infty$, there exists a number $T' \geq 0$ such that $D_T \geq 0$ for all $T \geq T'$. That is, if for every alternative feasible path, the candidate path has from some date on, cumulative utility up to *all* later dates at least as great as that of the alternative feasible path, then the candidate path is overtaking optimal.

We say that the candidate path is *weakly preferred* in case we just know that $D_T \geq 0$ for all $T \geq T'$. If $D_T \geq 0$ can be replaced by $D_T > 0$, we say it is *strictly preferred.*[25]

Optimal control theory is also applicable with this criterion. The current-value HamiItonian is

$$H(k, c, \lambda, t) = u(c) + \lambda \left[f(k) - c - \delta k \right].$$

The Maximum Principle states that an interior overtaking-optimal path will satisfy that there exists an adjoint variable λ such that for all $t \geq 0$ it holds

[24] Suppose for instance that $c_t \to \bar{c}$ for $t \to \infty$. Then $\int_0^\infty u(c_t)dt = \pm\infty$ for $u(\bar{c}) \gtrless 0$, respectively.

[25] A slightly more generally applicable optimality criterion is the *catching-up* criterion. The meaning of this criterion in continuous time is analogue to its meaning in discrete time, cf. Chapter 8.3 (where also a limitation of the overtaking and catching-up criteria is dealt with).

along this path that

$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0, \text{ and} \tag{10.51}$$

$$\frac{\partial H}{\partial k} = \lambda(f'(k) - \delta) = -\dot{\lambda}. \tag{10.52}$$

Since $\rho = 0$, the Keynes-Ramsey rule reduces to

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta(c_t)}(f'(k_t) - \delta), \qquad \text{where } \theta(c) \equiv -\frac{c}{u'(c)}u''(c).$$

One might conjecture that also the transversality condition,

$$\lim_{t\to\infty} k_t \lambda_t = 0, \tag{10.53}$$

is necessary for optimality but, as we will see below, this turns out to be wrong in this case with no discounting.

Our assumption (A2) here reduces to $\lim_{k\to 0} f'(k) > \delta > \lim_{k\to\infty} f'(k)$ (which requires $\delta > 0$). Apart from this, the phase diagram is fully analogue to that in Fig. 10.1, except that the steady state, E, is now at the top of the $\dot{k} = 0$ curve. This is because in steady state, $f'(k^*) - \delta = 0$, and this equation also defines k_{GR} in this case. It can be shown that the saddle path is again the unique solution to the optimization problem (by essentially the same method as in the discrete time case of Chapter 8).

A noteworthy feature is that in this case the Ramsey model constitutes a counterexample to the widespread presumption that an optimal plan with infinite horizon *must* satisfy a transversality condition like (10.53). Indeed, by (10.51), $\lambda_t = u'(c_t) \to u'(c^*)$ for $t \to \infty$ along the overtaking-optimal path (the saddle path). Thus, instead of (10.53), we get

$$\lim_{t\to\infty} k_t \lambda_t = k^* u'(c^*) > 0.$$

With CRRA utility it is straightforward to generalize these results to the case $g \geq 0, n \geq 0$ and $\hat{\rho} - n = (1-\hat{\theta})g$. The social planner's overtaking-optimal solution is still the saddle path approaching the golden rule steady state; and this solution violates the seemingly "natural" transversality condition.

Note also that with zero effective utility discounting, there can not be equilibrium in the *market* economy version of this story. The real interest rate would in the long run be zero and thus the human wealth of the infinitely-lived household would be infinite. But then the demand for consumption goods would be unbounded and equilibrium thus be impossible.

10.6 Concluding remarks

The Ramsey model has played an important role as a way of structuring economists' thoughts about many macrodynamic phenomena including economic growth. The model constitutes an examination of a benchmark case. As noted in the introduction to this chapter, this case is in some sense the opposite of the Diamond OLG model. Both models build on very idealized assumptions. Whereas the Diamond model ignores any bequest motive and emphasizes life-cycle behavior and at least some heterogeneity in the population, the Ramsey model implicitly assumes an altruistic bequest motive which is always operative and which turns households into homogeneous, infinitely-lived agents. In this way the Ramsey model ends up as an easy-to-apply framework, implying, among other things, a clear-cut theory of the level of the real interest rate in the long run. The model's usefulness lies in allowing general equilibrium analysis of an array of problems in a "vacuum". The fact that the assumption of perfect foresight in the Ramsey model always generates a *unique* equilibrium path makes the strong assumption of perfect foresight slightly less problematic than in the Diamond OLG model. In the latter model, as we saw in Chapter 3, multiple equilibria paths could arise under certain circumstances (and with this multiplicity the mystery about how coordination of expectations was brought about).

The next chapter discusses different applications of the Ramsey model. Because of the model's simplicity, one should always be aware of the risk of non-robust conclusions. The assumption of a representative household is a main limitation of the Ramsey model. It is not easy to endow the dynasty portrait of households with plausibility. One of the problems is, as argued by Bernheim and Bagwell (1988), that this portrait does not comply with the fact that families are interconnected in a complex way via marriage of partners coming from different parent families. And the lack of heterogeneity in the model's population of households implies a danger that important interdependencies between different classes of agents are unduly neglected. For some problems these interdependencies may be of only secondary importance, but for others (for instance, issues concerning public debt or interaction between private debtors and creditors) they are crucial.

Another critical limitation of the model comes from its reliance on saddle-point stability with the associated presumption of perfect foresight infinitely far out in the future. There can be good reasons for bearing in mind the following warning (by Solow, 1990, p. 221) against overly reliance on the Ramsey framework in the analysis of a market economy:

> "The problem is not just that perfect foresight into the indefinite

> future is so implausible away from steady states. The deeper problem is that in practice – if there is any practice – miscalculations about the equilibrium path may not reveal themselves for a long time. The mistaken path gives no signal that it will be "ultimately" infeasible. It is natural to comfort oneself: whenever the error is perceived there will be a jump to a better approximation to the converging arm. But a large jump may be required. In a decentralized economy it will not be clear who knows what, or where the true converging arm is, or, for that matter, exactly where we are now, given that some agents (speculators) will already have perceived the need for a mid-course correction while others have not. This thought makes it hard even to imagine what a long-run path would look like. It strikes me as more or less devastating for the interpretation of quarterly data as the solution of an infinite time optimization problem."

Along the line segment where the Ramsey model and the Diamond OLG model are polar cases, less abstract models are scattered, some being closer to the one pole and others closer to the other. A given model may open up for different *regimes*, one close to Ramsey's pole, another close to Diamond's. An example is Robert Barro's model with parental altruism presented in Chapter 7. When the bequest motive in the Barro model is operative, the model coincides with a Ramsey model (in discrete time) as was shown in Chapter 8. But when the bequest motive is not operative, the Barro model coincides with a Diamond OLG model. The Blanchard (1985) OLG model in continuous time (analyzed in chapters 12, 13, and 15) also belongs to the interior of this line segment, but is closer to the Diamond pole than the Ramsey pole.

10.7 Literature notes

1. Frank Ramsey (1903-1930) died at the age of 26 but he managed to publish several important articles. Ramsey discussed economic issues with, among others, John Maynard Keynes. In an obituary published in the Economic Journal (March 1932) some months after Ramsey's death, Keynes described Ramsey's article about the optimal savings as "one of the most remarkable contributions to mathematical economics ever made, both in respect of the intrinsic importance and difficulty of its subject, the power and elegance of the technical methods employed, and the clear purity of illumination with which the writer's mind is felt by the reader to play about its subject".

2. The version of the Ramsey model we have considered is in accordance with the general tenet of neoclassical preference theory: saving is motivated

only by higher consumption in the future. Extended versions assume that accumulation of wealth is to some extend an end in itself or perhaps motivated by a desire for social prestige and economic and political power rather than consumption. In Kurz (1968b) an extended Ramsey model is studied where wealth is an independent argument in the instantaneous utility function.

3. The equivalence in the Ramsey model between the decentralized market equilibrium and the social planner's solution can be seen as an extension of the first welfare theorem as it is known from elementary textbooks, to the case where the market structure stretches infinitely far out in time, and the finite number of economic agents (family dynasties) face an infinite time horizon: in the absence of externalities etc., the allocation of resources under perfect competition will lead to a Pareto optimal allocation. The Ramsey model is indeed a special case in that all households are identical. But the result can be shown in a far more general setup, cf. Debreu (1954). The result, however, does not hold in overlapping generations models where an unbounded sequence of new generations enter and the "interests" of the new households have not been accounted for in advance.

4. Cho and Graham (1996) consider the empirical question whether countries tend to be above or below their steady state. Based on the Penn World Table they find that *on average*, countries with a relatively low income per adult are above their steady state and that countries with a higher income are below.

10.8 Appendix

A. Algebraic analysis of the dynamics around the steady state

To supplement the graphical approach of Section 10.3 with an exact analysis of the adjustment dynamics of the model, we compute the Jacobian matrix for the system of differential equations (10.28) - (10.29):

$$J(\tilde{k},\tilde{c}) = \begin{bmatrix} \partial\dot{\tilde{k}}/\partial\tilde{k} & \partial\dot{\tilde{k}}/\partial\tilde{c} \\ \partial\dot{\tilde{c}}/\partial\tilde{k} & \partial\dot{\tilde{c}}/\partial\tilde{c} \end{bmatrix} = \begin{bmatrix} f'(\tilde{k}) - (\delta+g+n) & -1 \\ \frac{1}{\theta}f''(\tilde{k})\tilde{c} & \frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho + \theta g) \end{bmatrix}.$$

Evaluated in the steady state this reduces to

$$J(\tilde{k}^*,\tilde{c}^*) = \begin{bmatrix} \rho - n - (1-\theta)g & -1 \\ \frac{1}{\theta}f''(\tilde{k}^*)\tilde{c}^* & 0 \end{bmatrix}$$

This matrix has the determinant

$$\frac{1}{\theta}f''(\tilde{k}^*)\tilde{c}^* < 0.$$

Since the product of the eigenvalues of the matrix equals the determinant, the eigenvalues are real and opposite in sign.

In standard math terminology a steady-state point in a two dimensional continuous-time dynamic system is called a *saddle point* if the associated eigenvalues are opposite in sign.[26] For the present case we conclude that the steady state is a saddle point. This mathematical definition of a saddle point is equivalent to that given in the text of Section 10.3. Indeed, with two eigenvalues of opposite sign, there exists, in a small neighborhood of the steady state, a stable arm consisting of two saddle paths which point in opposite directions. From the phase diagram in Fig. 10.1 we know that the stable arm has a positive slope. At least for $\tilde{k}_0$ sufficiently close to $\tilde{k}^*$ it is thus possible to start out on a saddle path. Consequently, there is a (unique) value of $\tilde{c}_0$ such that $(\tilde{k}_t, \tilde{c}_t) \rightarrow (\tilde{k}^*, \tilde{c}^*)$ for $t \rightarrow \infty$. Finally, the dynamic system has exactly one jump variable, $\tilde{c}$, and one predetermined variable, $\tilde{k}$. It follows that the steady state is (locally) *saddle-point stable.*

We claim that for the present model this can be strengthened to *global* saddle-point stability. Indeed, for *any* $\tilde{k}_0 > 0$, it is possible to start out on the saddle path. For $0 < \tilde{k}_0 \leq \tilde{k}^*$, this is obvious in that the extension of the saddle path towards the left reaches the y-axis at a non-negative value of $\tilde{c}^*$. That is to say that the extension of the saddle path cannot, according to the uniqueness theorem for differential equations, intersect the $\tilde{k}$-axis for $\tilde{k} > 0$ in that the positive part of the $\tilde{k}$-axis is a solution of (10.28) - (10.29).[27]

For $\tilde{k}_0 > \tilde{k}^*$, our claim can be verified in the following way: suppose, contrary to our claim, that there exists a $\tilde{k}_1 > \tilde{k}^*$ such that the saddle path does not intersect that region of the positive quadrant where $\tilde{k} \geq \tilde{k}_1$. Let $\tilde{k}_1$ be chosen as the smallest possible value with this property. The slope, $d\tilde{c}/d\tilde{k}$, of the saddle path will then have no upper bound when $\tilde{k}$ approaches $\tilde{k}_1$ from the left. Instead $\tilde{c}$ will approach ∞ along the saddle path. But then $\ln \tilde{c}$ will also approach ∞ along the saddle path for $\tilde{k} \rightarrow \tilde{k}_1$ $(\tilde{k} < \tilde{k}_1)$. It follows that $d \ln \tilde{c}/d\tilde{k} = (d\tilde{c}/d\tilde{k})/\tilde{c}$, computed along the saddle path, will have no upper bound. Nevertheless, we have

$$\frac{d \ln \tilde{c}}{d\tilde{k}} = \frac{d \ln \tilde{c}/dt}{d\tilde{k}/dt} = \frac{\dot{\tilde{c}}/\tilde{c}}{\dot{\tilde{k}}} = \frac{\frac{1}{\theta}(f'(\tilde{k}) - \delta - \rho - \theta g)}{f(\tilde{k}) - \tilde{c} - (\delta + g + n)\tilde{k}}.$$

[26] Note the difference compared to a discrete time system, cf. Appendix D of Chapter 8. In the discrete time system we have next period's $\tilde{k}$ and $\tilde{c}$ on the left-hand side of the dynamic equations, not the increase in $\tilde{k}$ and $\tilde{c}$, respectively. Therefore, the criterion for a saddle point is different in discrete time.

[27] Because the extension of the saddle path towards the left in Fig. 10.1 can not intersect the $\tilde{c}$-axis at a value of $\tilde{c} > f(0)$, it follows that if $f(0) = 0$, the extension of the saddle path ends up in the origin.

When $\tilde{k} \to \tilde{k}_1$ and $\tilde{c} \to \infty$, the numerator in this expression is bounded, while the denominator will approach $-\infty$. Consequently, $d \ln \tilde{c}/d\tilde{k}$ will approach zero from above, as $\tilde{k} \to \tilde{k}_1$. But this contradicts that $d \ln \tilde{c}/d\tilde{k}$ has no upper bound, when $\tilde{k} \to \tilde{k}_1$. Thus, the assumption that such a $\tilde{k}_1$ exists is false and our original hypothesis holds true.

B. Boundedness of the utility integral

We claimed in Section 10.3 that if the parameter restriction

$$\rho - n > (1-\theta)g \tag{A1}$$

holds, then the utility integral, $U_0 = \int_0^\infty \frac{c^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt$, is bounded along the steady-state path, $c_t = \tilde{c}^* T_t$. The proof is as follows. For $\theta \neq 1$,

$$\begin{aligned}(1-\theta)U_0 &= \int_0^\infty c_t^{1-\theta} e^{-(\rho-n)t} dt = \int_0^\infty (c_0 e^{gt})^{1-\theta} e^{-(\rho-n)t} dt \\ &= c_0 \int_0^\infty e^{[(1-\theta)g-(\rho-n)]t} dt = \frac{c_0}{\rho - n - (1-\theta)g},\end{aligned} \tag{10.54}$$

by (A1). If $\theta = 1$, we get

$$U_0 = \int_0^\infty (\ln c_0 + gt) e^{-(\rho-n)t} dt,$$

which is also finite, in view of (A1) implying $\rho - n > 0$ in *this* case. It follows that also any path converging to the steady state will entail bounded utility, when (A1) holds.

On the other hand, suppose that (A1) does *not* hold, i.e., $\rho - n \leq (1-\theta)g$. Then by the third equality in (10.54) and $c_0 > 0$ follows that $U_0 = \infty$.

C. The diverging paths

In Section 10.3 we stated that paths of types *II* and *III* in the phase diagram in Fig. 10.1 can not be equilibria with perfect foresight. Given the expectation corresponding to any of these paths, every single household will choose to *deviate* from the expected path (i.e., deviate from the expected "average behavior" in the economy). We will now show this formally.

We first consider a path of type *III*. A path of this type will not be able to *reach* the horizontal axis in Fig. 10.1. It will only *converge* towards the point $(\bar{\tilde{k}}, 0)$ for $t \to \infty$. This claim follows from the uniqueness theorem for differential equations with continuously differentiable right-hand sides.

The uniqueness implies that two solution curves cannot intersect. And we see from (10.29) that the positive part of the x-axis is from a mathematical point of view a solution curve (and the point $(\bar{\tilde{k}}, 0)$ is a trivial steady state). This rules out another solution curve hitting the x-axis.

The convergence of $\tilde{k}$ towards $\bar{\tilde{k}}$ implies $\lim_{t\to\infty} r_t = f'(\bar{\tilde{k}}) - \delta < g + n$, where the inequality follows from $\bar{\tilde{k}} > \tilde{k}_{GR}$. So,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} = \lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (r_s - g - n)ds} = \lim_{t\to\infty} \tilde{k}_t e^{-\int_0^t (f'(\tilde{k}_s) - \delta - g - n)ds} = \bar{\tilde{k}} e^{\infty} > 0. \tag{10.55}$$

Hence the transversality condition of the households is violated. Consequently, the household will choose higher consumption than along this path and can do so without violating the NPG condition.

Consider now instead a path of type *II*. We shall first show that if the economy follows such a path, then depletion of all capital occurs in finite time. Indeed, in the text it was shown that any path of type *II* will pass the $\dot{\tilde{k}} = 0$ locus in Fig. 10.1. Let t_0 be the point in time where this occurs. If path *II* lies above the $\dot{\tilde{k}} = 0$ locus for all $t \geq 0$, then we set $t_0 = 0$. For $t > t_0$, we have

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t < 0.$$

By differentiation w.r.t. t we get

$$\ddot{\tilde{k}}_t = f'(\tilde{k}_t)\dot{\tilde{k}}_t - \dot{c}_t - (\delta + g + n)\dot{\tilde{k}}_t = [f'(\tilde{k}_t) - \delta - g - n]\dot{\tilde{k}}_t - \dot{c}_t < 0,$$

where the inequality comes from $\dot{\tilde{k}}_t < 0$ combined with the fact that $\tilde{k}_t < \tilde{k}_{GR}$ implies $f'(\tilde{k}_t) - \delta > f'(\tilde{k}_{GR}) - \delta = g + n$. Therefore, there exists a $t_1 > t_0 \geq 0$ such that

$$\tilde{k}_{t_1} = \tilde{k}_{t_0} + \int_{t_0}^{t_1} \dot{\tilde{k}}_t dt = 0,$$

as was to be shown. At time t_1, $\tilde{k}$ cannot fall any further and $\tilde{c}_t$ immediately drops to $f(0)$ and stay there hereafter.

Yet, this result does not in itself explain why the individual household will deviate from such a path. The individual household has a negligible impact on the movement of $\tilde{k}_t$ in society and correctly perceives r_t and w_t as essentially independent of its own consumption behavior. Indeed, the economy-wide $\tilde{k}$ is not the household's concern. What the household cares about is its own financial wealth and budget constraint. In the perspective of the household nothing prevents it from planning a negative financial wealth, a, and possibly a continuously declining financial wealth, if only the NPG condition,

$$\lim_{t\to\infty} a_t e^{-\int_0^t (r_s - n)ds} \geq 0,$$

is satisfied.

But we can show that paths of type *II* will *violate* the NPG condition. The reasoning is as follows. The household plans to follow the Keynes-Ramsey rule. Given an expected evolution of r_t and w_t corresponding to path *II*, this will imply a planned gradual transition from positive financial wealth to debt. The transition to positive net debt, $\tilde{d}_t \equiv -\tilde{a}_t \equiv -a_t/T_t > 0$, takes place at time t_1 defined above.

The continued growth in the debt will meanwhile be so fast that the NPG condition is violated. To see this, note that the NPG condition implies the requirement

$$\lim_{t\to\infty} \tilde{d}_t e^{-\int_0^t (r_s - g - n)ds} \leq 0, \tag{NPG}$$

that is, the productivity-corrected debt, $\tilde{d}_t$, is allowed to grow in the long run only at a rate *less* than the growth-corrected real interest rate. For $t > t_1$ we get from the accounting equation $\dot{a}_t = (r_t - n)a_t + w_t - c_t$ that

$$\dot{\tilde{d}}_t = (r_t - g - n)\tilde{d}_t + \tilde{c}_t - \tilde{w}_t > 0,$$

where $\tilde{d}_t > 0$, $r_t > \rho + \theta g > g + n$, and where $\tilde{c}_t$ grows exponentially according to the Keynes-Ramsey rule, while $\tilde{w}_t$ is non-increasing in that $\tilde{k}_t$ does not grow. This implies

$$\lim_{t\to\infty} \frac{\dot{\tilde{d}}_t}{\tilde{d}_t} \geq \lim_{t\to\infty} (r_t - g - n),$$

which is in conflict with (NPG).

Consequently, the household will choose a lower consumption path and thus *deviate* from the reference path considered. Every household will do this and the evolution of r_t and w_t corresponding to path *II* is thus *not* an equilibrium with perfect foresight.

The conclusion is that all individual households understand that the only evolution which can be expected rationally is the one corresponding to the saddle path.

D. A constant saving rate as a special case

As we noted in Section 10.4, Solow's growth model can be seen as a special case of the Ramsey model. Indeed, a constant saving rate may, under certain conditions, emerge as an endogenous result in the Ramsey model.

Let the rate of saving, $(Y_t - C_t)/Y_t$, be s_t. We have generally

$$\tilde{c}_t = (1 - s_t)f(\tilde{k}_t), \qquad \text{and so} \tag{10.56}$$

$$\dot{\tilde{k}}_t = f(\tilde{k}_t) - \tilde{c}_t - (\delta + g + n)\tilde{k}_t = s_t f(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t. \tag{10.57}$$

In the Solow model the rate of saving is a constant, s, and we then get, by differentiating with respect to t in (10.56) and using (10.57),

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = f'(\tilde{k}_t)[s - \frac{(\delta + g + n)\tilde{k}_t}{f(\tilde{k}_t)}]. \tag{10.58}$$

By maximization of discounted utility in the Ramsey model, given a rate of time preference ρ and an elasticity of marginal utility θ, we get in equilibrium

$$\frac{\dot{\tilde{c}}_t}{\tilde{c}_t} = \frac{1}{\theta}(f'(\tilde{k}_t) - \delta - \rho - \theta g). \tag{10.59}$$

There will not generally exist a constant, s, such that the right-hand sides of (10.58) and (10.59), respectively, are the same for varying $\tilde{k}$ (that is, outside steady state). But Kurz (1968a) showed the following:

CLAIM Let δ, g, n, α, and θ be given. If the elasticity of marginal utility θ is greater than 1 and the production function is $\tilde{y} = A\tilde{k}^{\alpha}$ with $\alpha \in (1/\theta, 1)$, then a Ramsey model with $\rho = \theta\alpha(\delta+g+n) - \delta - \theta g$ will generate a constant saving rate $s = 1/\theta$. Thereby the same resource allocation and transitional dynamics arise as in the corresponding Solow model with $s = 1/\theta$.

Proof. Let $1/\theta < \alpha < 1$ and $f(\tilde{k}) = A\tilde{k}^{\alpha}$. Then $f'(\tilde{k}) = A\alpha\tilde{k}^{\alpha-1}$. The right-hand-side of the Solow equation, (10.58), becomes

$$A\alpha\tilde{k}^{\alpha-1}[s - \frac{(\delta + g + n)\tilde{k}_t}{A\tilde{k}^{\alpha}}] = sA\alpha\tilde{k}^{\alpha-1} - \alpha(\delta + g + n). \tag{10.60}$$

The right-hand-side of the Ramsey equation, (10.59), becomes

$$\frac{1}{\theta}A\alpha\tilde{k}^{\alpha-1} - \frac{\delta + \rho + \theta g}{\theta}.$$

By inserting $\rho = \theta\alpha(\delta + g + n) - \delta - \theta g$, this becomes

$$\begin{aligned} & \frac{1}{\theta}A\alpha\tilde{k}^{\alpha-1} - \frac{\delta + \theta\alpha(\delta + g + n) - \delta - \theta g + \theta g}{\theta} \\ = \; & \frac{1}{\theta}A\alpha\tilde{k}^{\alpha-1} - \alpha(\delta + g + n). \end{aligned} \tag{10.61}$$

For the chosen ρ we have $\rho = \theta\alpha(\delta+g+n) - \delta - \theta g > n + (1-\theta)g$, because $\theta\alpha > 1$ and $\delta+g+n > 0$. Thus, $\rho - n > (1-\theta)g$ and existence of equilibrium in the Ramsey model with this ρ is ensured. We can now make (10.60) and (10.61) the same by inserting $s = 1/\theta$. This also ensures that the two models require the same $\tilde{k}^*$ to obtain a *constant* $\tilde{c} > 0$. With this $\tilde{k}^*$, the requirement $\dot{\tilde{k}}_t = 0$ gives the same steady-state value of $\tilde{c}$ in both models, in view of (10.57). It follows that $(\tilde{k}_t, \tilde{c}_t)$ is the same in the two models for all $t \geq 0$. $\square$

On the other hand, maintaining $\tilde{y} = A\tilde{k}^\alpha$, but allowing $\rho \neq \theta\alpha(\delta+g+n) -\delta - \theta g$, so that $\theta \neq 1/s^*$, then $s'(\tilde{k}) \neq 0$, i.e., the Ramsey model does not generate a constant saving rate except in steady state. Defining s^* as in (10.40) and $\bar{\theta} \equiv (\delta+\rho)/[\alpha(\delta+g+n)-g]$, we have: When $\alpha(\delta+g+n) > g$ (which seems likely empirically), it holds that if $\theta \lesseqgtr 1/s^*$ (i.e., if $\theta \lesseqgtr \bar{\theta}$), then $s'(\tilde{k}) \lesseqgtr 0$, respectively; if instead $\alpha(\delta+g+n) \leq g$, then $\theta < 1/s^*$ and $s'(\tilde{k}) < 0$, unconditionally. These results follow by considering the slope of the saddle path in a phase diagram in the $(\tilde{k}, \tilde{c}/f(\tilde{k}))$ plane and using that $s(\tilde{k}) = 1 - \tilde{c}/f(\tilde{k})$, cf. Exercise 10.?? The intuition is that when $\tilde{k}$ is rising over time (i.e., society is becoming wealthier), then, when the desire for consumption smoothing is "high" (θ "high"), the prospect of high consumption in the future is partly taken out as high consumption already today, implying that saving is initially low, but rising over time until it eventually settles down in the steady state. But if the desire for consumption smoothing is "low" (θ "low"), saving will initially be high and then gradually fall in the process towards the steady state. The case where $\tilde{k}$ is falling over time gives symmetric results.

E. The social planner's solution

In the text of Section 10.5 we postponed some of the more technical details. First, by (A2), the existence of the steady state, E, and the saddle path in Fig. 10.1 is ensured. Solving the linear differential equation (10.46) gives $\lambda_t = \lambda_0 e^{-\int_0^t (f'(\tilde{k}_s)-\delta-\hat{\rho}-g)ds}$. Substituting this into the transversality condition (11.51) gives

$$\lim_{t\to\infty} e^{-\int_0^t (f'(\tilde{k}_s)-\delta-g-n)ds}\tilde{k}_t = 0, \tag{10.62}$$

where we have eliminated the unimportant positive factor $\lambda_0 = c_0^{-\hat{\theta}} T_0$.

This condition is essentially the same as the transversality condition (10.36) for the market economy and holds in the steady state, given the parameter restriction $\hat{\rho} - n > (1-\hat{\theta})g$, which is satisfied in view of (A1). Thus, (10.62) also holds along the saddle path. Since we must have $\tilde{k} \geq 0$ for all $t \geq 0$, (10.62) has the form required by Mangasarian's sufficiency theorem. Thus, if we can show that the Hamiltonian is jointly concave in $(\tilde{k}, c)$ for all $t \geq 0$, then the saddle path *is* a solution to the social planner's problem. And if we can show strict concavity, the saddle path is the *unique* solution. We have:

$$\begin{aligned}
\frac{\partial H}{\partial \tilde{k}} &= \lambda(f'(\tilde{k}) - (\delta + g + n)), \quad \frac{\partial H}{\partial c} = c^{-\hat{\theta}} - \frac{\lambda}{T}, \\
\frac{\partial^2 H}{\partial \tilde{k}^2} &= \lambda f''(\tilde{k}) < 0 \quad (\text{by } \lambda = c^{-\hat{\theta}} T > 0), \quad \frac{\partial^2 H}{\partial c^2} = -\hat{\theta} c^{-\hat{\theta}-1} < 0, \\
\frac{\partial^2 H}{\partial \tilde{k} \partial c} &= 0.
\end{aligned}$$

Thus, the leading principal minors of the Hessian matrix of H are

$$D_1 = -\frac{\partial^2 H}{\partial \tilde{k}^2} > 0, \quad D_2 = \frac{\partial^2 H}{\partial \tilde{k}^2} \frac{\partial^2 H}{\partial c^2} - \left(\frac{\partial^2 H}{\partial \tilde{k} \partial c} \right)^2 > 0.$$

Hence, H is strictly concave in $(\tilde{k}, c)$ and the saddle path is the unique optimal solution.

It also follows that the transversality condition (11.51) is a *necessary* optimality condition. Note that we have had to derive this conclusion in a different way than when solving the household's consumption/saving problem in Section 10.2. There we could appeal to a link between the No-Ponzi-Game condition (with strict equality) and the transversality condition to verify necessity of the transversality condition. But that proposition does not cover the social planner's problem where there is no NPG condition.

As to the diverging paths in Fig. 10.1, note that paths of type II (those paths which, as shown in Appendix C, in finite time deplete all capital) can not be optimal, in spite of the temporarily high consumption level. This follows from the fact that the saddle path is the unique solution. Finally, paths of type III in Fig. 10.1 behave as in (10.55) and thus violate the transversality condition (11.51), as claimed in the text.

10.9 Exercises