Advanced Macroeconomics 2 Short Note 2

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A refresher about continuous time analysis

In the next months we will consider models in continuous time. This short note is a refresher of how to understand and analyze models in continuous time.

We start from a discrete time framework: the run of time is divided into successive periods of constant length, taken as the time-unit. Let financial wealth at the beginning of period i be denoted a_i, $i = 0, 1, 2, ...$ Then wealth accumulation in discrete time can be written

$$a_{i+1} - a_i = s_i, \qquad a_0 \text{ given},$$

where s_i is (net) saving in period i.

1 Transition to continuous time analysis

With time flowing continuously, we let $a(t)$ refer to financial wealth at time t. Similarly, $a(t+\Delta t)$ refers to financial wealth at time $t+\Delta t$. To begin with, let Δt be equal to one time unit. Then $a(i\Delta t) = a_i$. Consider the forward first difference in a, $\Delta a(t) \equiv a(t+\Delta t) - a(t)$. It makes sense to consider this change in a in relation to the length of the time interval involved, that is, the *ratio* $\Delta a(t)/\Delta t$. As long as $\Delta t = 1$, with $t = i\Delta t$ we have $\Delta a(t)/\Delta t = (a_{i+1} - a_i)/1 = a_{i+1} - a_i$. Now, keep the time unit unchanged, but let the length of the time interval $[t, t + \Delta t)$ approach zero, i.e., let $\Delta t \to 0$. Assuming $a(\cdot)$ is a continuous and differentiable function, then $\lim_{\Delta t \to 0} \Delta a(t)/\Delta t$ exists and is denoted the *derivative of* $a(\cdot)$ at t, usually written $da(t)/dt$ or just $\dot{a}(t)$. That is,

$$\dot{a}(t) = \frac{da(t)}{dt} = \lim_{\Delta t \to 0} \frac{a(t + \Delta t) - a(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta a(t)}{\Delta t}.$$

By implication, wealth accumulation in continuous time is written

$$\dot{a}(t) = s(t), \qquad a(0) = a_0 \text{ given}, \tag{1}$$

where $s(t)$ is the saving at time t. For Δt "small" we have the approximation $\Delta a(t) \approx \dot{a}(t)\Delta t = s(t)\Delta t$. In particular, for $\Delta t = 1$ we have $\Delta a(t) = a(t + 1) - a(t) \approx s(t)$.

As time unit let us choose one year. Going back to discrete time, if wealth grows at the constant rate $g > 0$ per year, then after i periods of length one year (with annual compounding)

$$a_i = a_0(1+g)^i, \quad i = 0, 1, 2, \ldots . \tag{2}$$

When compounding is n times a year, corresponding to a period length of $1/n$ year, then after i such periods:

$$a_i = a_0(1+\frac{g}{n})^i. \tag{3}$$

With t still denoting time (measured in years) passed since the initial date (here date 0), we have $i = nt$ periods. Substituting into (3) gives

$$a(t) = a_{nt} = a_0(1+\frac{g}{n})^{nt} = a_0\left[(1+\frac{1}{m})^m\right]^{gt}, \qquad \text{where } m \equiv \frac{n}{g}.$$

We keep g and t fixed, but let n (and so m) $\to \infty$. Then, in the limit there is continuous compounding and

$$a(t) = a_0 e^{gt}, \tag{4}$$

where e is the "exponential" defined as $e \equiv \lim_{m\to\infty}(1+1/m)^m \simeq 2.718281828....$ The formula (4) is the analogue in continuous time (with continuous compounding) to the discrete time formula (2) with annual compounding. Thus, a geometric growth factor is replaced by an exponential growth factor.

We can also view these two formulas as the solutions to a difference equation and a differential equation, respectively. Thus, (2) is the solution to the simple linear difference equation $a_{i+1} = (1+g)a_i$, given the initial value a_0. And (4) is the solution to the simple linear differential equation $\dot{a}(t) = ga(t)$, given the initial condition $a(0) = a_0$. With a time dependent growth rate, $g(t)$, the corresponding differential equation is $\dot{a}(t) = g(t)a(t)$ with solution

$$a(t) = a_0 e^{\int_0^t g(\tau)d\tau}, \tag{5}$$

where the exponent, $\int_0^t g(\tau)d\tau$, is the definite integral of the function $g(\tau)$ from 0 to t. The result (5) is called the *basic growth formula* in continuous time analysis and the factor $e^{\int_0^t g(\tau)d\tau}$ is called the *growth factor* or the *accumulation factor*.

Notice that the allowed range for parameters may change when we go from discrete time to continuous time with continuous compounding. For example, the usual equation for aggregate capital accumulation in continuous time is

$$\dot{K}(t) = I(t) - \delta K(t), \qquad K(0) = K_0 \text{ given}, \tag{6}$$

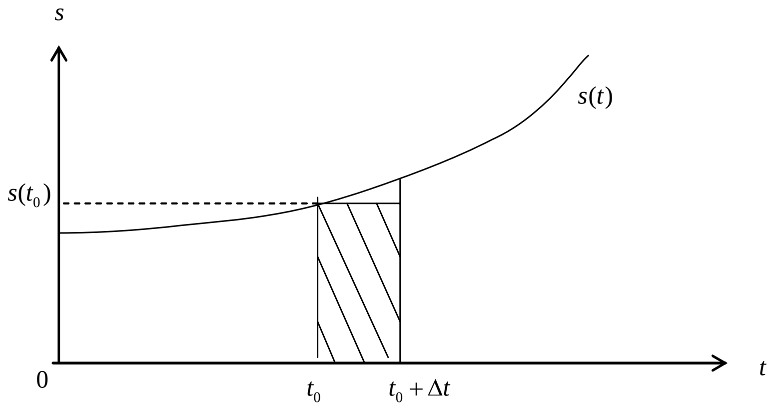
where $K(t)$ is the capital stock, $I(t)$ is the gross investment at time t and $\delta \geq 0$ is the (physical) capital depreciation rate. Unlike in discrete time, here $\delta > 1$ is conceptually allowed. Indeed, suppose for simplicity that $I(t) = 0$ for all $t \geq 0$; then (6) gives $K(t) = K_0 e^{-\delta t}$ (exponential decay). This formula is meaningful for any $\delta \geq 0$. Usually, the time unit used in continuous time macro models is one year (or a quarter of a year) and then a realistic value of δ is of course < 1 (say, between 0.05 and 0.10). However, if the time unit applied to the model is large (think of a Diamond-style OLG model converted into continuous time), say 30 years, then $\delta > 1$ may fit better, empirically. Suppose, for example, that physical capital has a half-life of 10 years. Then with 30 years as our time unit, inserting into the formula $1/2 = e^{-\delta/3}$ gives $\delta = (\ln 2) \cdot 3 \simeq 2$.

2 Stocks and flows

An advantage of continuous time analysis is that it forces one to make a clear distinction between *stocks* (say wealth) and *flows* (say consumption and saving). A *stock* variable is a variable measured as just a quantity at a given point in time. The variables $a(t)$ and $K(t)$ considered above are stock variables. A *flow* variable is a variable measured as quantity *per time unit* at a given point in time. The variables $s(t)$, $\dot{K}(t)$ and $I(t)$ above are flow variables.

One can not add a stock and a flow, because they have *different denomination*. What exactly is meant by this? The elementary measurement units in economics are *quantity units* (so and so many machines of a certain kind or so and so many litres of oil or so and so many units of payment) and *time units* (months, quarters, years). On the basis of these we can form *composite measurement units*. Thus, the capital stock K has the denomination "quantity of machines". In contrast, investment I has the denomination "quantity of machines per time unit" or, shorter, "quantity/time". If we change our time unit, say from quarters to years, the value of a flow variable is quadrupled (presupposing annual compounding). A growth rate or interest rate has the denomination "(quantity/time)/quantity" = "time^{-1}".

Thus, in continuous time analysis expressions like $K(t) + I(t)$ or $K(t) + \dot{K}(t)$ are illegitimate. But one can write $K(t + \Delta t) \approx K(t) + I(t)\Delta t$ and $\dot{K}(t) = I(t)$ (if $\delta = 0$). In the same way, if a bath tub contains 50 litres of water and the tap pours $\frac{1}{2}$ litre per second into the tub, a sum like $50\ \ell + \frac{1}{2}\ (\ell/\text{sec})$ does not make sense. But the *amount* of water in the tub after one minute is meaningful. This amount would be $50\ \ell + \frac{1}{2} \cdot 60$



Figur 1: With Δt "small" the integral of $s(t)$ from t_0 to $t_0 + \Delta t$ is $\approx$ the hatched area.

$((\ell/\text{sec})\times\text{sec}) = 90\ \ell$. In analogy, economic flow variables in continuous time should be seen as *intensities* defined for every t in the time interval considered, say the time interval $[0, T)$ or perhaps $[0, \infty)$. For example, when we say that $I(t)$ is "investment" at time t, this is really a short-hand for "investment intensity" at time t. The actual investment in a time interval $[t_0, t_0 + \Delta t)$, i.e., the invested amount *during* this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} I(t)dt \approx I(t)\Delta t$. Similarly, $s(t)$ (the flow of individual saving) should be interpreted as the saving *intensity* at time t. The actual saving in a time interval $[t_0, t_0 + \Delta t)$, i.e., the saved (or accumulated) amount during this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} s(t)dt$. If Δt is "small", this integral is approximately equal to the product $s(t_0)\cdot\Delta t$, cf. the hatched area in Fig. 1.

The notation commonly used in discrete time analysis blurs the distinction between stocks and flows. Expressions like $a_{i+1} = a_i + s_i$, without further comment, are usual. Seemingly, here a stock, wealth, and a flow, saving, are added. But, it is really wealth at the beginning of period i and the saved *amount during* period i that are added: $a_{i+1} = a_i + s_i \cdot \Delta t$. The tacit condition is that the period length, Δt, is the time unit. But suppose that, for example in a business cycle model, the period length is one quarter, but the time unit is one year. Then saving in quarter i is $s_i = (a_{i+1} - a_i) \cdot 4$ per year.

In empirical economics data typically come in discrete time form and data for flow variables typically refer to periods of constant length. One could argue that this discrete form of the data speaks for discrete time rather than continuous time modelling. And the fact that economic actors often think and plan in period terms, may be a good reason for putting at least microeconomic analysis in period terms. Yet, it can hardly be said that the *mass* of economic actors think and plan with one and the same period. In macroeconomics we consider the *sum* of the actions and then a formulation in continuous time may be preferable. This also allows variation *within* the usually artificial periods in which the data are chopped up.[1] For example, stock markets (markets for bonds and shares) are more naturally modelled in continuous time because such markets equilibrate almost instantaneously; they respond immediately to new information.

In his discussion of this modelling issue, Allen (1967) concluded that from the point of view of the economic contents, the choice between discrete time or continuous time analysis may be a matter of taste. But from the point of view of mathematical convenience, the continuous time formulation, which has worked so well in the natural sciences, is preferable.[2]

3 References

Allen, R.G.D., 1967, *Macro-economic Theory. A mathematical Treatment*, Macmillan, London.

[1] Allowing for such variations may be necessary to avoid the *artificial* oscillations which sometimes arise in a discrete time model due to a too large period length.

[2] At least this is so in the absence of uncertainty. For problems where uncertainty is important, discrete time formulations are easier if one is not familiar with stochastic calculus.