Economic Growth    Lecture Note 3

05.02.2013.    Christian Groth

# Balanced growth theorems

In this lecture note I shall discuss three fundamental propositions about balanced growth. In view of the generality of the propositions, they have a broad field of application.

The note covers the stuff in Acemoglu's §2.7.3. Our propositions 1 and 2 are slight extensions of part 1 and 2, respectively, of what Acemoglu calls Uzawa's Theorem I (Acemoglu, 2009, p. 60). Proposition 3 essentially corresponds to what Acemoglu calls Uzawa's Theorem II (Acemoglu, 2009, p. 63).

## 1 Balanced growth and constancy of key ratios

First we shall define the terms "steady state" and "balanced growth" as they are usually defined in growth theory. With respect to "balanced growth" this implies a minor deviation from the way Acemoglu briefly defines it informally on his page 57. The main purpose of the present note is to lay bare the connections between these two concepts as well as their relation to the hypothesis of Harrod-neutral technical progress and Kaldor's stylized facts.

### 1.1 The concepts of steady state and balanced growth

A basic equation from many one-sector growth models for a closed economy in continuous time is

$$\dot{K} = I - \delta K = Y - C - \delta K \equiv S - \delta K, \tag{1}$$

where $K$ is aggregate capital, $I$ aggregate gross investment, $Y$ aggregate output, $C$ aggregate consumption, $S$ aggregate gross saving ($\equiv Y - C$), and $\delta \geq 0$ is a constant physical capital depreciation rate.

Usually, in the theoretical literature on dynamic models, a *steady state* is defined in the following way:

**Definition 1** *A steady state of a dynamic model is a stationary solution to the fundamental differential equation(s) of the model.*

Or briefly: a steady state is a stationary point of a dynamic process.

Let us take the Solow growth model as an example. Here gross saving equals $sY$, where $s$ is a constant, $0 < s < 1$. Aggregate output is given by a neoclassical production function, $F$, with CRS and Harrod-neutral technical progress: $Y = F(K, AL) = ALF(\tilde{k}, 1) \equiv TLf(\tilde{k})$, where $L$ is the labor force, $A$ is the level of technology, and $\tilde{k} \equiv K/(AL)$ is the (effective) capital intensity. Moreover, $f' > 0$ and $f'' < 0$. Solow assumes $L(t) = L(0)e^{nt}$ and $A(t) = A(0)e^{gt}$, where $n \geq 0$ and $g \geq 0$ are the constant growth rates of the labor force and technology, respectively. By log-differentiating $\tilde{k}$ w.r.t. $t$,[1] we end up with the *fundamental differential equation* ("law of motion") of the Solow model:

$$\dot{\tilde{k}} = sf(\tilde{k}) - (\delta + g + n)\tilde{k}. \tag{2}$$

Thus, in the Solow model, a (non-trivial) steady state is a $\tilde{k}^* > 0$ such that, if $\tilde{k} = \tilde{k}^*$, then $\dot{\tilde{k}} = 0$.

The most common definition in the literature of balanced growth for an aggregate economy is the following:

**Definition 2** *A balanced growth path is a path $(Y, K, C)_{t=0}^{\infty}$ along which the quantities $Y, K,$ and $C$ are positive and grow at constant rates (not necessarily positive and not necessarily the same).*

Acemoglu, however, defines (p. 57) balanced growth in the following way: "balanced growth refers to an allocation where output grows at a constant rate and capital-output ratio, the interest rate, and factor shares remain constant". My problem with this definition is that it mixes growth of quantities with distribution aspects (interest rate and factor income shares). And it is not made clear what is meant by the output-capital ratio if the relative price of capital goods is changing over time. So I stick to the standard definition above which is known to function well in many different contexts.

[1] Or by directly using the fraction rule, see Appendix to Lecture Note 2.

## 1.2 A general result about balanced growth

We now leave the specific Solow model. The interesting fact is that, given the dynamic resource constraint (1), we have *always* that if there is balanced growth with positive gross saving, then the ratios $Y/K$ and $C/Y$ are constant (by "*always*" is meant: independently of how saving is determined and of how the labor force and technology change). And also the other way round: as long as gross saving is positive, constancy of the $Y/K$ and $C/Y$ ratios is enough to ensure balanced growth. So balanced growth and constancy of key ratios are essentially equivalent.

This is a very practical general observation. And since Acemoglu does not state any balanced growth theorem at this general level, we shall do it, in a precise way, here, together with a proof. Letting $g_x$ denote the growth rate of the (positively valued) variable $x$, i.e., $g_x \equiv \dot{x}/x$, we claim:

**Proposition 1** *(the balanced growth equivalence theorem). Let $(Y, K, C)_{t=0}^{\infty}$ be a path along which $Y, K$, $C$, and $S \equiv Y - C$ are positive for all $t \geq 0$. Then, given the accumulation equation (1), the following holds:*

*(i) if there is balanced growth, then $g_Y = g_K = g_C$, and the ratios $Y/K$, $C/K$, and $C/Y$ are constant;*

*(ii) if $Y/K$ and $C/Y$ are constant, then $Y, K$, and $C$ grow at the same constant rate, i.e., not only is there balanced growth, but the growth rates of $Y, K$, and $C$ are the same.*

*Proof* Consider a path $(Y, K, C)_{t=0}^{\infty}$ along which $Y, K$, $C$, and $S \equiv Y - C$ are positive for all $t \geq 0$. (i) Assume there is balanced growth. Then, by definition, $g_Y, g_K$, and $g_C$ are constant. Hence, by (1), we have that $S/K = g_K + \delta$ is constant, implying

$$g_S = g_K. \tag{3}$$

Further, since $Y = C + S$,

$$\begin{aligned} g_Y &= \frac{\dot{Y}}{Y} = \frac{\dot{C}}{Y} + \frac{\dot{S}}{Y} = g_C\frac{C}{Y} + g_S\frac{S}{Y} = g_C\frac{C}{Y} + g_K\frac{S}{Y} \qquad \text{(by (3))} \\ &= g_C\frac{C}{Y} + g_K\frac{Y - C}{Y} = \frac{C}{Y}(g_C - g_K) + g_K. \end{aligned} \tag{4}$$

Now, let us provisionally assume that $g_K \neq g_C$. Then (4) gives

$$\frac{C}{Y} = \frac{g_Y - g_K}{g_C - g_K}, \tag{5}$$

a constant, so that $g_Y = g_C$. But this result implies, by (5), that $C/Y = 1$, i.e., $C = Y$. In view of (1), however, this outcome contradicts the given condition that $S > 0$. Hence, our provisional assumption is wrong, and we have $g_K = g_C$. By (4), this implies $g_Y = g_K = g_C$, but now without the condition $C/Y = 1$ being implied. It follows that $Y/K$ and $C/K$ are constant. Then, also $C/Y = (Y/K)/(C/K)$ is constant.

(ii) Suppose $Y/K$ and $C/Y$ are constant. Then $g_Y = g_K = g_C$, so that $C/K$ is a constant. We now show that this implies that $g_K$ is constant. Indeed, from (1), $S/Y = 1 - C/Y$, so that also $S/Y$ is constant. It follows that $g_S = g_Y = g_K$, so that $S/K$ is constant. By (1),

$$\frac{S}{K} = \frac{\dot{K} + \delta K}{K} = g_K + \delta,$$

so that $g_K$ is constant. This, together with constancy of $Y/K$ and $C/Y$, implies that also $g_Y$ and $g_C$ are constant. □

*Remark.* It is part (i) of the proposition which requires the assumption $S > 0$ for all $t \geq 0$. If $S = 0$ for all $t \geq 0$, we would have $g_K = -\delta$ and $C \equiv Y - S = Y$, hence $g_C = g_Y$ for all $t \geq 0$. Then there would be balanced growth if the common value of $g_C$ and $g_Y$ had a constant growth rate. This growth rate, however, could easily differ from that of $K$. Suppose $Y = AK^\alpha L^{1-\alpha}$, $g_A = \gamma$ and $g_L = n$ ($\gamma$ and $n$ constants). Then we would have $g_C = g_Y = \gamma - \alpha\delta + (1-\alpha)n$, which could easily be strictly positive and thereby different from $g_K = -\delta \leq 0$ so that (i) no longer holds. □

The nice feature is that this proposition holds for *any* model for which the simple dynamic resource constraint (1) is valid. No assumptions about for example CRS and other technology aspects or about market form are involved. Further, the proposition suggests that if one accepts Kaldor's stylized facts as a description of the past century's growth experience, and if one wants a model consistent with them, one should construct the model such that it can generate balanced growth. For a model to be capable of generating balanced growth, however, technological progress must be of the Harrod-neutral type (i.e., be labor-augmenting), at least in a neighborhood of the balanced growth path. For a fairly general context (but of course not as general as that of Proposition 1), this was shown already by Uzawa (1961). The next section presents a modernized version of Uzawa's contribution.

## 2 The crucial role of Harrod-neutrality

Let the aggregate production function be

$$Y(t) = \tilde{F}(K(t), L(t); t). \tag{6}$$

The only technology assumption needed is that $\tilde{F}$ has CRS w.r.t. the first two arguments ($\tilde{F}$ need not be neoclassical for example). As a representation of technical progress, we assume $\partial \tilde{F}/\partial t > 0$ for all $t \geq 0$ (i.e., as time proceeds, unchanged inputs result in more and more output). We also assume that the labor force evolves according to

$$L(t) = L(0)e^{nt}, \tag{7}$$

where $n$ is a constant. Further, non-consumed output is invested and so (1) is the dynamic resource constraint of the economy.

**Proposition 2** *(Uzawa's balanced growth theorem) Let $(Y(t), K(t), C(t))_{t=0}^{\infty}$, where $0 < C(t) < Y(t)$ for all $t \geq 0$, be a path satisfying the capital accumulation equation (1), given the CRS-production function (6) and the labor force path in (7). Then:*

*(i) a necessary condition for this path to be a balanced growth path is that along the path it holds that*

$$Y(t) = \tilde{F}(K(t), L(t); t) = \tilde{F}(K(t), A(t)L(t); 0), \tag{8}$$

*where $A(t) = e^{gt}$ with $g \equiv g_Y - n$;*

*(ii) for any $g > 0$ such that there is a $q > g+n+\delta$ with the property that $\tilde{F}(1, k^{-1}; 0) = q$ for some $k > 0$ (i.e., at any $t$, hence also $t = 0$, the production function $\tilde{F}$ in (6) allows an output-capital ratio equal to $q$), a sufficient condition for the existence of a balanced growth path with output-capital ratio $q$, is that the technology can be written as in (8) with $A(t) = e^{gt}$.*

*Proof* (i)[2] Suppose the path $(Y(t), K(t), C(t))_{t=0}^{\infty}$ is a balanced growth path. By definition, $g_K$ and $g_Y$ are then constant, so that $K(t) = K(0)e^{g_K t}$ and $Y(t) = Y(0)e^{g_Y t}$. We then have

$$Y(t)e^{-g_Y t} = Y(0) = \tilde{F}(K(0), L(0); 0) = \tilde{F}(K(t)e^{-g_K t}, L(t)e^{-nt}; 0), \tag{9}$$

[2]This part draws upon Schlicht (2006), who generalized a proof in Wan (1971, p. 59) for the special case of a constant saving rate.

where we have used (6) with $t = 0$. In view of the precondition that $S(t) \equiv Y(t) - C(t) > 0$, we know from (i) of Proposition 1, that $Y/K$ is constant so that $g_Y = g_K$. By CRS, (9) then implies

$$Y(t) = \tilde{F}(K(t)e^{g_Y t}e^{-g_K t}, L(t)e^{g_Y t}e^{-nt}; 0) = \tilde{F}(K(t), e^{(g_Y - n)t}L(t); 0).$$

We see that (8) holds for $A(t) = e^{gt}$ with $g \equiv g_Y - n$.

(ii) Suppose (8) holds with $A(t) = e^{gt}$. Let $g > 0$ be given such that there is a $q > g + n + \delta$ with the property that $\tilde{F}(1, k^{-1}; 0) = q$ for some $k > 0$. Then our first claim is that with $K(0) = kL(0)$, $s \equiv (g + n + \delta)/q$, and $S(t) = sY(t)$, (1), (7), and (8) imply $Y(t)/K(t) = q$ for all $t \geq 0$. Indeed, by construction

$$\frac{Y(0)}{K(0)} = \frac{\tilde{F}(K(0), L(0); 0)}{K(0)} = \tilde{F}(1, k^{-1}; 0) = q = \frac{\delta + g + n}{s}. \tag{10}$$

It follows that $sY(0)/K(0) - \delta = g + n$. So, by (1), we have $\dot{K}(0)/K(0) = sY(0)/K(0) - \delta = g + n$, implying that $K$ initially grows at the same rate as effective labor input, $A(t)L(t)$. Then, in view of $\tilde{F}$ being homogeneous of degree one w.r.t. its first two arguments, also $Y$ grows initially at this rate. As an implication, the ratio $Y/K$ does not change, but remains equal to the right-hand side of (10) for all $t \geq 0$. Consequently, $K$ and $Y$ continue to grow at the same constant rate, $g + n$. As $C = (1 - s)Y$, $C$ grows forever also at this constant rate. Hence, the path $(Y(t), K(t), C(t))_{t=0}^{\infty}$ is a balanced growth path, as was to be proved. □

The form (8) indicates that along a balanced growth path, technical progress must, at least in a neighborhood of the path, be purely "labor augmenting", that is, Harrod-neutral. It is in this case convenient to define a new CRS function, $F$, by $F(K(t), A(t)L(t)) \equiv \tilde{F}(K(t), A(t)L(t); 0)$. Then (i) of the proposition implies that at least in a neighborhood of the balanced growth path, we can rewrite the production function this way:

$$Y(t) = \tilde{F}(K(t), L(t); t) = F(K(t), A(t)L(t)). \tag{11}$$

It is important to recognize that the occurrence of Harrod-neutrality says nothing about what the *source* of technological progress is. Harrod-neutrality should not be interpreted as indicating that the technological progess emanates specifically from the labor input. Harrod-neutrality only means that technical innovations predominantly are such that not only do labor and capital in combination become more productive, but this happens to *manifest itself* at the aggregate level in the form (8).[3]

What is the intuition behind the Uzawa result that for balanced growth to be possible, technical progress must have the purely labor-augmenting form? First, notice that there is an asymmetry between capital and labor. Capital is an accumulated amount of non-consumed output. In contrast, in simple macro models labor is a non-produced production factor which (at least in this context) grows in an exogenous way. Second, because of CRS, the aggregate production function (6) implies that

$$1 = \tilde{F}(\frac{K(t)}{Y(t)}, \frac{L(t)}{Y(t)}; t). \tag{12}$$

Now, since capital is accumulated non-consumed output, it inherits the trend in output such that $K(t)/Y(t)$ must be constant along a balanced growth path (this is what Proposition 1 is about). Labor does not inherit the trend in output; indeed, the ratio $L(t)/Y(t)$ is free to adjust as time proceeds. When there is technical progress ($\partial\tilde{F}/\partial t > 0$) along a balanced growth path, this progress must manifest itself in the form of a changing $L(t)/Y(t)$ in (12) as $t$ proceeds, precisely because $K(t)/Y(t)$ *must* be constant along the path. In the "normal" case where $\partial\tilde{F}/\partial L > 0$, the needed change in $L(t)/Y(t)$ is a *fall* (i.e., a rise in $Y(t)/L(t)$). This is what (12) shows. Indeed, the fall in $L(t)/Y(t)$ must exactly offset the effect on $\tilde{F}$ of the rising $t$, for a fixed capital-output ratio.[4] It follows that along the balanced growth path, $Y(t)/L(t)$ is an increasing implicit function of $t$. If we denote this function $A(t)$, we end up with (11).

The generality of Uzawa's theorem is noteworthy. The theorem assumes CRS, but does not presuppose that the technology is neoclassical, not to speak of satisfying the Inada conditions.[5] And the theorem holds for exogenous as well as endogenous technological progress. It is also worth mentioning that the proof of the sufficiency part of the theorem is *constructive*. It provides a method to construct a hypothetical balanced growth path (BGP from now).[6]

A simple implication of the Uzawa theorem is the following. Interpreting the $A(t)$ in (8) as the "level of technology", we have:

[3] For a CRS Cobb-Douglas production function with technological progress, Harrod-neutrality is present whenever the output elasticity w.r.t capital (often denoted $\alpha$) is constant over time.

[4] This way of presenting the intuition behind the Uzawa result draws on the nice exposition in Jones and Scrimgeour (2008).

[5] Many accounts of the Uzawa theorem, including Jones and Scrimgeour (2008), presume a neoclassical production function, but the theorem is much more general.

[6] Part (ii) of Proposition 2 is ignored in Acemoglu's book.

COROLLARY Along a BGP with positive gross saving and the technology level, $A(t)$, growing at the rate $g$, output grows at the rate $g + n$ while labor productivity, $y \equiv Y/L$, and consumption per unit of labor, $c \equiv C/L$, grow at the rate $g$.

*Proof* That $g_Y = g + n$ follows from (i) of Proposition 2. As to the growth rate of labor productivity we have

$$y_t = \frac{Y(0)e^{g_Y t}}{L(0)e^{nt}} = y(0)e^{(g_Y - n)t} = y(0)e^{gt}.$$

Finally, by Proposition 1, along a BGP with $S > 0$, $c$ must grow at the same rate as $y$. □

We shall now consider the implication of Harrod-neutrality for the income shares of capital and labor when the technology is neoclassical and markets are perfectly competitive.

# 3 Harrod-neutrality and the functional income distribution

There is one facet of Kaldor's stylized facts we have so far not related to Harrod-neutral technical progress, namely the long-run "approximate" constancy of both the income share of labor, $wL/Y$, and the rate of return to capital. At least with neoclassical technology, profit maximizing firms, and perfect competition in the output and factor markets, these properties are inherent in the combination of constant returns to scale, balanced growth, and the assumption that the relative price of capital goods (relative to consumption goods) equals one. The latter condition holds in models where the capital good is nothing but non-consumed output, cf. (1).[7]

To see this, we start out from a neoclassical CRS production function with Harrod-neutral technological progress,

$$Y(t) = F(K(t), A(t)L(t)). \tag{13}$$

With $w(t)$ denoting the real wage at time $t$, in equilibrium under perfect competition the labor income share will be

$$\frac{w(t)L(t)}{Y(t)} = \frac{\frac{\partial Y(t)}{\partial L(t)}L(t)}{Y(t)} = \frac{F_2(K(t), A(t)L(t))A(t)L(t)}{Y(t)}. \tag{14}$$

[7] The reader may think of the "corn economy" example in Acemoglu, p. 28.

In this simple model, without natural resources, capital (gross) income equals non-labor income, $Y(t) - w(t)L(t)$. Hence, if $r(t)$ denotes the (net) rate of return to capital at time $t$, then

$$r(t) = \frac{Y(t) - w(t)L(t) - \delta K(t)}{K(t)}. \tag{15}$$

Denoting the capital (gross) income share by $\alpha(t)$, we can write this $\alpha(t)$ (in equilibrium) in three ways:

$$\begin{aligned}
\alpha(t) &\equiv \frac{Y(t) - w(t)L(t)}{Y(t)} = \frac{(r(t)+\delta)K(t)}{Y(t)}, \\
\alpha(t) &= \frac{F(K(t), A(t)L(t)) - F_2(K(t), A(t)L(t))A(t)L(t)}{Y(t)} = \frac{F_1(K(t), A(t)L(t))K(t)}{Y(t)}, \\
\alpha(t) &= \frac{\frac{\partial Y(t)}{\partial K(t)}K(t)}{Y(t)},
\end{aligned} \tag{16}$$

where the first row comes from (15), the second from (13) and (14), the third from the second together with Euler's theorem.[8] Comparing the first and the last row, we see that in equilibrium

$$\frac{\partial Y(t)}{\partial K(t)} = r(t) + \delta.$$

In this condition we recognize one of the first-order conditions in the representative firm's profit maximization problem under perfect competition, since $r(t) + \delta$ can be seen as the firm's required gross rate of return.[9]

In the absence of uncertainty, the equilibrium real interest rate in the bond market must equal the rate of return on capital, $r(t)$. And $r(t) + \delta$ can then be seen as the firm's cost of disposal over capital per unit of capital per time unit, consisting of interest cost plus capital depreciation.

**Proposition 3** *(factor income shares and rate of return under balanced growth) Let the path $(K(t), Y(t), C(t))_{t=0}^{\infty}$ be a BGP in a competitive economy with the production function (13) and with positive saving. Then, along the BGP, the $\alpha(t)$ in (16) is a constant, $\alpha \in (0, 1)$. The labor income share will be $1 - \alpha$ and the (net) rate of return on capital will be $r = \alpha q - \delta$, where $q$ is the constant output-capital ratio along the BGP.*

[8] From Euler's theorem, $F_1 K + F_2 AL = F(K, AL)$, when $F$ is homogeneous of degree one.

[9] With natural resources, say land, entering the set of production factors, the formula, (15), for the rate of return to capital should be modified by subtracting rents from the numerator.

*Proof* By CRS we have $Y(t) = F(K(t), A(t)L(t)) = A(t)L(t)F(\tilde{k}(t), 1) \equiv A(t)L(t)f(\tilde{k}(t))$. In view of part (i) of Proposition 2, by balanced growth, $Y(t)/K(t)$ is some constant, $q$. Since $Y(t)/K(t) = f(\tilde{k}(t))/\tilde{k}(t)$ and $f'' < 0$, this implies $\tilde{k}(t)$ constant, say equal to $\tilde{k}^*$. But $\partial Y(t)/\partial K(t) = f'(\tilde{k}(t))$, which then equals the constant $f'(\tilde{k}^*)$ along the BGP. It then follows from (16) that $\alpha(t) = f'(\tilde{k}^*)/q \equiv \alpha$. Moreover, $0 < \alpha < 1$, where $0 < \alpha$ follows from $f' > 0$ and $\alpha < 1$ from the fact that $q = Y/K = f(\tilde{k}^*)/\tilde{k}^* > f'(\tilde{k}^*)$, in view of $f'' < 0$ and $f(0) \geq 0$. Then, by the first equality in (16), $w(t)L(t)/Y(t) = 1-\alpha(t) = 1-\alpha$. Finally, by (15), the (net) rate of return on capital is $r = (1-w(t)L(t)/Y(t))Y(t)/K(t)-\delta$ $= \alpha q - \delta$. $\square$

This proposition is of interest by displaying a link from balanced growth to constancy of factor income shares and the rate of return, that is, some of the "stylized facts" claimed by Kaldor. Note, however, that although the proposition implies constancy of the income shares and the rate of return, it does not *determine* them, except in terms of $\alpha$ and $q$. But both $q$ and, generally, $\alpha$ are endogenous and depend on $\tilde{k}^*$,[10] which will generally be unknown as long as we have not specified a theory of saving. This takes us to theories of aggregate saving, for example the simple Ramsey model, cf. Chapter 8 in Acemoglu's book.

# 4 Disembodied vs. embodied technological change

In our presentation of technological progress above we have implicitly assumed that all technological change is *disembodied*. And the way the propositions 1, 2, and 3, are formulated assume this.

*Disembodied technological change* occurs when new technical knowledge advances the combined productivity of capital and labor independently of whether the workers operate old or new machines. Consider again the aggregate dynamic resource constraint (1) and the production function (6):

$$\dot{K}(t) = Y(t) - C(t) - \delta K(t), \qquad (*)$$

$$Y(t) = \tilde{F}(K(t), L(t); t), \qquad \partial \tilde{F}/\partial t > 0. \qquad (**)$$

Here $Y(t)-C(t)$ is aggregate gross investment, $I(t)$. For a given level of $I(t)$, the resulting amount of new capital goods per time unit ($\dot{K}(t) + \delta K(t)$), measured in efficiency units,

[10] As to $\alpha$, there is of course a trivial exception, namely the case where the production function is Cobb-Douglas and $\alpha$ therefore is a given parameter.

is independent of *when* this investment occurs. It is thereby not affected by technological progress. Similarly, the interpretation of $\partial \tilde{F}/\partial t > 0$ in (**) is that the higher technology level obtained as time proceeds results in higher productivity of *all* capital and labor. Thus also firms that have only old capital equipment benefit from recent advances in technical knowledge. No new investment is needed to take advantage of the recent technological and organizational developments.[11]

In contrast, we say that technological change is *embodied*, if taking advantage of new technical knowledge requires construction of new investment goods. The newest technology is incorporated in the design of newly produced equipment; and this equipment will not participate in subsequent technological progress. An example: only the most recent vintage of a computer series incorporates the most recent advance in information technology. Then investment goods produced later (investment goods of a later "vintage") have higher productivity than investment goods produced earlier at the same resource cost. Whatever the source of new technical knowledge, investment becomes an important bearer of the productivity increases which this new knowledge makes possible. Without new investment, the potential productivity increases remain potential instead of being realized.

An alternative name for embodied technological change is investment-specific technological change.

One way to formally represent embodied technological progress is to write capital accumulation in the following way,

$$\dot{K}(t) = q(t)I(t) - \delta K(t), \tag{17}$$

where $I(t)$ is gross investment at time $t$ and $q(t)$ measures the "quality" (productivity) of newly produced investment goods. The increasing level of technology implies increasing $q(t)$ so that a given level of investment gives rise to a greater and greater additions to the capital stock, $K$, measured in efficiency units. Even if technological change does not directly appear in the production function, that is, even if for instance (**) is replaced by $Y(t) = F(K(t), L(t))$, the economy may in this manner still experience a rising standard of living.

Embodied technological progress is likely to result in a steady decline in the price of capital equipment relative to the price of consumption goods. This prediction is confirmed

[11] In the standard versions of the Solow model and the Ramsey model it is assumed that all technological progress has this form - for no other reason than that this is by far the simplest case to analyze.

by the data. Greenwood et al. (1997) find for the U.S. that the relative price, $p \approx 1/q$, of capital equipment has been declining at an average rate of 0.03 per year in the period 1950-1990.[12] They also show that over the same period there has been a secular rise in the ratio of new equipment investment, measured in efficiency units, to GDP. The authors estimate that embodied technical change explains 60% of the growth in output per man hour and thus is the dominating form of technological progress.

This raises the question how the propositions 1, 2, and 3 fare in the case of embodied technological progress. The answer is that a generalized version of Proposition 1 goes through. Essentially, we only need to replace (1) by (17) and interpret $K$ in Proposition 1 as the *value* of the capital stock, i.e., we have to replace $K$ by $\tilde{K} = pK$.

But the concept of Harrod-neutrality no longer fits the situation without further elaboration. Hence to obtain analogies to Proposition 2 and Proposition 3 is a more complicated matter. Suffice it to say that with embodied technological progress, the class of production functions that are consistent with balanced growth is smaller than with disembodied technological progress.

# 5 Concluding remarks

In the Solow model as well as in many other models with disembodied technological progress, a steady state and a balanced growth path imply each other. Indeed, they are two sides of the same process. There *exist* cases, however, where this equivalence does not hold (some open economy models and some models with *embodied* technical change). Therefore, it is recommendable always to maintain a terminological distinction between the two concepts, steady state and balanced growth.[13]

Note that the definition of balanced growth refers to *aggregate* variables. At the same time as there is balanced growth at the aggregate level, *structural change* may occur. That is, a changing sectorial composition of the economy is under certain conditions compatible with balanced growth (in a generalized sense) at the aggregate level, cf. the "Kuznets facts" (see Kongsamut et al., 2001, and Acemoglu, 2009, Chapter 20).

[12] The relative price index applied by Greenwood et al. (1997) is based on the book by R. Gordon (1990), which is an attempt to correct previous price indices for equipment by better taking into account quality improvements in new equipment.

[13] Here we depart from Acemoglu, p. 65, where he says that he will use the two terms "interchangingly". We also depart from Barro and Sala-i-Martin (2004, pp. 33-34) who *define* a steady state as synonymous with a balanced growth path as the latter was defined above.

In view of the key importance of Harrod-neutrality, a natural question is: has growth theory uncovered any *endogenous* tendency for technical progress to converge to Harrod-neutrality? Fortunately, in his Chapter 15 Acemoglu outlines a theory about a mechanism entailing such a tendency, the theory of "directed technical change". Jones (2005) suggests an alternative mechanism.

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