Too little or too much R&D?
Mathematical Supplement.

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This is a mathematical supplement to the paper:

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March 28, 2003

by the same authors.

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March 28, 2003

In Section 1 below it is proved that the social planner's problem has a unique solution which converges towards the steady state considered in Section 3 of the paper. In view of the parameter separations made (allowing more "non-convexities" than usual), this result is not obvious beforehand, and we are not aware of any proof in the literature covering this case.

In Section 2.1 it is proved that the transversality condition of the household in the market economy is equivalent to the assumption A3 in Section 4.2 of the paper.

In Section 2.2 it is proved that the market equilibrium is unique and that it converges towards the steady state considered in Section 4.2 of the paper.

1 The solution to the social planner's problem

This refers to Section 3 of the paper. In order to permit existence of an optimal solution we require the parameter restriction

$$\rho > (1-\theta)\frac{\eta}{1-\alpha}\gamma L. \tag{A1}$$

To obtain concavity of the maximized Hamiltonian and to check sufficient conditions for optimality it is convenient to rewrite the maximization problem. Introducing the transformations

$$\tilde{A} \equiv A^{\frac{\eta}{1-\alpha}} \text{ and } \tilde{\gamma} \equiv \frac{\eta}{1-\alpha}\gamma, \tag{1}$$

we have $Y = K^{\alpha}(\tilde{A}N_Y)^{1-\alpha}$, and $\dot{\tilde{A}} = \tilde{\gamma}(L-N_Y)\tilde{A}$. Then the current value Hamiltonian for the social planning problem becomes

$$H = \frac{c^{1-\theta}-1}{1-\theta} + \lambda_1[K^{\alpha}(\tilde{A}N_Y)^{1-\alpha} - \delta K - cL] + \lambda_2\tilde{\gamma}(L-N_Y)\tilde{A},$$

where λ_1 and λ_2 are the shadow prices of the state variables K and $\tilde{A}$, respectively.[1] Necessary conditions for an interior solution are that for all $t \geq 0$:

$$c^{-\theta} = \lambda_1 L, \tag{2}$$

$$\lambda_1(1-\alpha)\frac{Y}{\tilde{A}N_Y} = \lambda_2\tilde{\gamma}, \tag{3}$$

$$\dot{\lambda}_1 = \rho\lambda_1 - \lambda_1(\frac{\partial Y}{\partial K} - \delta), \tag{4}$$

$$\dot{\lambda}_2 = \rho\lambda_2 - \lambda_1\frac{\partial Y}{\partial \tilde{A}} - \lambda_2\tilde{\gamma}(L-N_Y), \tag{5}$$

$$\lim_{t\to\infty}\lambda_1 e^{-\rho t}K = 0, \lim_{t\to\infty}\lambda_2 e^{-\rho t}\tilde{A} = 0. \tag{6}$$

[1] Observe that, in view of the transformation (1), λ_2 here is not the same as λ_2 in the paper.

Log-differentiating (2) wrt. t and using (4) gives

$$g_c \equiv \frac{\dot{c}}{c} = \frac{1}{\theta}(\frac{\partial Y}{\partial K} - \delta - \rho) = \frac{1}{\theta}(\alpha \tilde{k}^{\alpha-1} - \delta - \rho), \tag{7}$$

where $\tilde{k} \equiv k/\tilde{A}$. Since in a steady state, by definition, g_c is constant, $\tilde{k}$ is also constant in view of (7).

Define $\tilde{q} \equiv \lambda_2/\lambda_1$. From (3)

$$\tilde{q} = \frac{(1-\alpha)Y}{\tilde{\gamma}\tilde{A}N_Y} = \frac{1-\alpha}{\tilde{\gamma}}\tilde{k}^{\alpha}. \tag{8}$$

Hence, in steady state

$$\frac{\dot{\tilde{q}}}{\tilde{q}} = 0. \tag{9}$$

But, by definition of $\tilde{q}$, $\dot{\tilde{q}}/\tilde{q} = (\dot{\lambda}_2/\lambda_2) - (\dot{\lambda}_1/\lambda_1)$, hence, from (4), and (5),

$$\frac{\dot{\tilde{q}}}{\tilde{q}} = \frac{\partial Y}{\partial K} - \delta - \tilde{\gamma}L.$$

This, together with (9) and the definition of $\tilde{\gamma}$, imply

$$\frac{\partial Y}{\partial K} = \frac{\eta}{1-\alpha}\gamma L + \delta. \tag{10}$$

Inserting this into (7) gives

$$g_c^* = \frac{1}{\theta}(\frac{\eta}{1-\alpha}\gamma L - \rho), \tag{11}$$

which is (3.7) in the paper.

To show that the steady state is saddle-point stable, let $z \equiv Y/K$ and $\bar{c} \equiv cL/K$. Then, from the first order conditions (2), (3), (4), (5), and the dynamic constraints of the social planner's problem we get the differential equations

$$\begin{aligned}
\dot{z} &= \left[(\alpha-1)z + \frac{1-\alpha}{\alpha}(\tilde{\gamma}L + \delta)\right] z, \\
\dot{\bar{c}} &= \left[\frac{\alpha-\theta}{\theta}z + \bar{c} + \frac{\theta-1}{\theta}\delta - \frac{\rho}{\theta}\right] \bar{c}, \\
\dot{u} &= \left[-\hat{c} + \tilde{\gamma}Lu + \frac{1-\alpha}{\alpha}(\tilde{\gamma}L + \delta)\right] u.
\end{aligned}$$

The Jacobian evaluated in the steady state[2] is triangular and has the eigenvalues $\rho_1 = (\partial\dot{z}/\partial z)^* = (\alpha-1)z^* < 0$, $\rho_2 = (\partial\dot{\bar{c}}/\partial\bar{c})^* = \bar{c}^* > 0$, and $\rho_3 = (\partial\dot{u}/\partial u)^* = \tilde{\gamma}Lu^* > 0$. The three variables, $z, \bar{c}$, and u, are jump variables, but the initial conditions for z

[2] Steady state values referring to the social planner's solution are marked by *.

and u are not independent. Given the predetermined variable $\bar{k} \equiv K/(\tilde{A}L)$, we have $u = \bar{k}z^{\frac{1}{1-\alpha}}$, from the production function $Y = K^{\alpha}(\tilde{A}N_Y)^{1-\alpha}$. Hence, with two free initial conditions and exactly two positive eigenvalues, existence and uniqueness of a convergent solution (that is, saddle-point stability) holds generically.

A path $(c, N_Y, K, \tilde{A})_{t=0}^{\infty}$, which satisfies the first order conditions (2), (3), (4), and (5), and which approaches the steady state for $t \to \infty$, is our candidate for an optimal solution. To make sure that the path *is* an optimal solution, we note, *first*, that it satisfies the necessary transversality conditions in (6). Indeed, along the path we have

$$\begin{aligned} \lim_{t\to\infty}\left(\frac{\dot{\lambda}_1}{\lambda_1} - \rho + \frac{\dot{K}}{K}\right) &= \lim_{t\to\infty}\left(\frac{\dot{\lambda}_2}{\lambda_2} - \rho + \frac{\dot{\tilde{A}}}{\tilde{A}}\right) \\ &= -\left(\frac{\partial Y}{\partial K} - \delta\right) + g_c^* \qquad \text{(from (4) and (5))} \\ &= -\frac{\eta}{1-\alpha}\gamma L + g_c^*, \quad \text{(from (10))} \end{aligned}$$

which is negative in view of (11) and A1. Hence, (6) holds.

Second, we check whether the Hamiltonian is jointly concave in the state variables K and $\tilde{A}$ after the controls c and N_Y have been substituted by their maximizing values from (2) and (3), respectively. We find $H = B + \lambda_1\tilde{\gamma}LK + \lambda_2\tilde{\gamma}L\tilde{A}$, where the term B does not depend on K or $\tilde{A}$. Clearly this function is concave.

The *third* thing to check is whether our candidate path $(c, N_Y, K, \tilde{A})_{t=0}^{\infty}$ satisfies a *sufficient* transversality condition. In view of (2) and (3), $\lambda_1 > 0$ and $\lambda_2 > 0$ for all $t \geq 0$. Hence, by (6), the path $(c, N_Y, K, \tilde{A})_{t=0}^{\infty}$ satisfies

$$\lim_{t\to\infty}\left[\lambda_1 e^{-\rho t}(\hat{K} - K) + \lambda_2 e^{-\rho t}(\hat{\tilde{A}} - \tilde{A})\right] \geq 0,$$

for all feasible paths $(\hat{c}, \hat{N}_Y, \hat{K}, \hat{\tilde{A}})_{t=0}^{\infty}$. The conclusion is that, by Arrow's sufficiency theorem (see Seierstad and Sydsæter, 1987, p. 236), our candidate path $(c, N_Y, K, \tilde{A})_{t=0}^{\infty}$ *is* an optimal solution.

The above applies to the case where the steady state is interior, i.e., the case where

$$\rho < \frac{\eta}{1-\alpha}\gamma L \tag{A2}$$

holds. If A2 is violated there can be no R&D and no growth in a steady state; instead the steady state solution is like that of a standard one-sector Ramsey model without technical progress.

2 The market economy

2.1 The transversality condition is equivalent to A3

In Section 4.2 of the paper we claimed that assuming the transversality condition of the household to hold in a steady state is equivalent to assuming

$$\rho > \frac{1-\theta}{1-\varepsilon\alpha}\eta\frac{1-\varepsilon}{1-\alpha}\alpha\gamma L. \tag{A3}$$

Here we give the argument in detail. For the steady state to be an equilibrium, the transversality condition of the household,

$$\lim_{t\to\infty} ve^{-\int_0^t rds} = 0, \tag{12}$$

must be satisfied. We have $v \equiv \frac{K+pA}{L}$, and in steady state $g_K = g_c = \frac{\eta}{1-\alpha} g_A = g_p + g_A$, from (4.12), since $g_{\tilde{k}} = 0$; hence $g_v = g_c$. It follows that the steady state satisfies (12) if and only if the steady state has

$$r > g_c. \tag{13}$$

By the Keynes-Ramsey rule, $r = \theta g_c + \rho$, hence, if $\theta \geq 1$, (13) holds automatically, since $\rho > 0$. Suppose, on the contrary, $\theta < 1$. The value of η for which $\theta g_c + \rho = g_c$ is, using (4.15),

$$\bar{\eta} \equiv \frac{(1-\varepsilon\alpha)(1-\alpha)\rho}{(1-\theta)(1-\varepsilon)\alpha\gamma L}. \tag{14}$$

Now, $0 < \frac{\partial r}{\partial \eta} = \theta \frac{\partial g_c}{\partial \eta} < \frac{\partial g_c}{\partial \eta}$ in this case, since, by (4.16), $\frac{\partial g_c}{\partial \eta} > 0$. Therefore, (13) holds if and only if $\eta < \bar{\eta}$. And, by (14), $\eta < \bar{\eta}$ if and only if

$$\rho > \frac{1-\theta}{1-\varepsilon\alpha}\eta\frac{1-\varepsilon}{1-\alpha}\alpha\gamma L,$$

which is A3.

2.2 Dynamics

Here we prove the claim in Section 4.2 that given A3 and A4 the market equilibrium is described by a unique convergent solution, at least as long as either $\eta \leq (1-\varepsilon\alpha)/(1-\alpha)$ or $\theta = 1$. That is, saddle-point stability holds, at least within the empirically relevant domain of the parameter space.

Assume A3 and

$$\rho < \frac{1-\varepsilon}{1-\alpha}\alpha\gamma L. \tag{A4}$$

The equations describing the dynamics of the market equilibrium can be reduced to three differential equations in $z \equiv Y/K$, $\bar{c} \equiv cL/K$, and u. Indeed, by (4.11)

$$z = \tilde{k}^{\alpha-1}. \tag{15}$$

Then, from (4.12) and (2.5),

$$\dot{z} = \frac{\alpha-1}{\alpha}\alpha g_{\tilde{k}} = \frac{\alpha-1}{\alpha}\left[\frac{\dot{p}}{p} - \frac{\eta-(1-\alpha)}{1-\alpha}\gamma(1-u)L\right]z.$$

Insert $\dot{p}/p$ from (4.6), using (4.12) and (5.4), to get

$$\dot{z} = \left[\varepsilon(\alpha-1)z + (1-\varepsilon\alpha-\eta)\frac{\gamma}{\alpha}Lu + \frac{1}{\alpha}\left[(\eta+\alpha-1)\gamma L + (1-\alpha)\delta\right]\right]z \tag{16}$$

with steady state value

$$z = (\frac{1-\varepsilon\alpha}{1-\alpha} - \frac{\eta}{1-\alpha})\frac{\gamma L}{\varepsilon\alpha}u + \frac{[\eta-(1-\alpha)]\,\gamma L + (1-\alpha)\delta}{(1-\alpha)\varepsilon\alpha}. \tag{17}$$

By (4.8), (4.12), and (2.4) we get

$$\dot{\bar{c}} = \left[\frac{\varepsilon\alpha-\theta}{\theta}z + \bar{c} + \frac{\theta-1}{\theta}\delta - \frac{\rho}{\theta}\right]\bar{c}. \tag{18}$$

The steady state value of $\bar{c}$ is

$$\bar{c} = \frac{1}{\theta}\left[(\theta-\varepsilon\alpha)z + (1-\theta)\delta + \rho\right]. \tag{19}$$

Finally, from the definition $\tilde{k} = K/(A^{\frac{\eta}{1-\alpha}}uL)$ we find, using (15),

$$u = z^{1/(1-\alpha)}K/(A^{\frac{\eta}{1-\alpha}}L). \tag{20}$$

Differentiating wrt. time t gives

$$\dot{u} = \left[(1-\varepsilon)z - \bar{c} + (\frac{1-\varepsilon\alpha}{1-\alpha} - \eta)\frac{\gamma}{\alpha}Lu + \frac{1-\alpha}{\alpha}\delta + \frac{\eta-1}{\alpha}\gamma L\right]u, \tag{21}$$

by (16), (2.4), and (2.5). Using (21), (17), and (19), we find the steady state value of u as

$$u = \frac{[1-\alpha-(1-\theta)\eta]\,\gamma L + (1-\alpha)\rho}{[1-\varepsilon\alpha-(1-\theta)\eta]\,\gamma L}, \tag{22}$$

confirming (8.2).

The equations (16), (18), and (21) constitute a three-dimensional dynamical system. The Jacobian evaluated at the steady state point is

$$J = \begin{bmatrix} \varepsilon(\alpha-1)z & 0 & [1-\varepsilon\alpha-\eta]\,\frac{\gamma L}{\alpha}z \\ \frac{\varepsilon\alpha-\theta}{\theta}\bar{c} & \bar{c} & 0 \\ (1-\varepsilon)u & -u & (\frac{1-\varepsilon\alpha}{1-\alpha}-\eta)\frac{\gamma L}{\alpha}u \end{bmatrix}. \tag{23}$$

The characteristic polynomial for J is

$$P(\lambda) = -\lambda^3 + b_2\lambda^2 - b_1\lambda + b_0,$$

where $b_0 \equiv \det(J)$, $b_1 \equiv \sum_{j>i}\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj}\end{vmatrix}$, $b_2 \equiv \operatorname{tr}(J)$, a_{ij} being the element in row i and column j of J. Let λ_1, λ_2, and λ_3 be the roots of J. Then

$$\begin{aligned} b_0 &= \lambda_1\lambda_2\lambda_3, \\ b_1 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \\ b_2 &= \lambda_1 + \lambda_2 + \lambda_3. \end{aligned} \tag{24}$$

We get, from (23), $b_0 = -\left[1-\varepsilon\alpha-(1-\theta)\eta\right]\frac{\varepsilon\gamma}{\theta}Lzcu < 0$, by A3 and A4 in view of (i) of Lemma 2. Hence, so far there are two possibilities: (i) All three roots are negative or have negative real part (multiple convergent solutions); (ii) one root is negative, and two roots have non-negative real part (unique convergent solution, in view of the boundary condition (20)). A sufficient (though not necessary) condition for excluding possibility (i) is that $b_1 \leq 0$ or $b_2 \geq 0$. Now, from (23),

$$\begin{aligned} b_2 &= \varepsilon(\alpha-1)z + c + (\frac{1-\varepsilon\alpha}{1-\alpha} - \eta)\frac{\gamma L}{\alpha}u \\ &= \frac{\theta-1}{\theta}(\varepsilon\alpha z - \delta) + (1-\varepsilon)z + \frac{\rho}{\theta} + (\frac{1-\varepsilon\alpha}{1-\alpha} - \eta)\frac{\gamma L}{\alpha}u \qquad \text{(by (19))} \\ &= r - g_c + (1-\varepsilon)z + (\frac{1-\varepsilon\alpha}{1-\alpha} - \eta)\frac{\gamma L}{\alpha}u \qquad \text{(by (4.12) and (4.8)).} \end{aligned}$$

As shown above, in view of the transversality condition of the household, $r > g_c$; hence, it follows that a sufficient (but not necessary) condition for $b_2 > 0$ is

$$\eta \leq \frac{1-\varepsilon\alpha}{1-\alpha}. \tag{25}$$

On the other hand, it is easy to construct cases where, for η sufficiently large, $b_2 < 0$. As an example, for $\theta = 1$ we get

$$\begin{aligned} b_2 &= \rho + (1-\varepsilon)z + (\frac{1-\varepsilon\alpha}{1-\alpha} - \eta)\frac{\gamma L}{\alpha}u \qquad (26) \\ &= \frac{1}{\varepsilon\alpha}\left[(1+\varepsilon\alpha)\rho \ + \varepsilon\gamma L \ + (1-\varepsilon)\delta \ + (\frac{\alpha-\varepsilon}{1-\alpha}\gamma L - \rho)\eta\right], \end{aligned}$$

which, for $\varepsilon \geq \alpha$, can always be made negative by choosing η very large.

Fortunately, however, we can show that at least when $\theta = 1$, $b_2 \leq 0$ implies $b_1 < 0$. Indeed, the general formula for b_1 is

$$\begin{aligned} b_1 &= \varepsilon(\alpha-1)z\bar{c} + (\frac{1-\varepsilon\alpha}{1-\alpha}(\theta-\varepsilon) - (1-\theta)\varepsilon\eta)\frac{\gamma Lu}{\theta}z \\ &\quad + (\frac{1-\varepsilon\alpha}{1-\alpha} - \eta)\left[\rho + (1-\theta)\delta\right]\frac{\gamma L}{\alpha\theta}u, \end{aligned}$$

which, for $\theta = 1$, reduces to

$$b_1 = \varepsilon(\alpha-1)z\bar{c} + \frac{1-\varepsilon\alpha}{1-\alpha}(1-\varepsilon)\gamma Luz + (\frac{1-\varepsilon\alpha}{1-\alpha} - \eta)\rho\frac{\gamma L}{\alpha}u, \tag{27}$$

where

$$z = \frac{1}{\varepsilon\alpha}(\rho + \delta + \frac{(1-\varepsilon)\alpha\gamma L - (1-\alpha)\rho}{(1-\varepsilon\alpha)(1-\alpha)}\eta), \tag{28}$$

from (17) and (22) when $\theta = 1$.

For $\theta = 1$, by (26), we have that $b_2 \leq 0 \Rightarrow$

$$\begin{aligned} (\frac{1-\varepsilon\alpha}{1-\alpha} - \eta)\frac{\gamma L}{\alpha}u &\leq -\left[\rho + (1-\varepsilon)z\right] \qquad (29) \\ &\Rightarrow \ \eta \geq \frac{1-\varepsilon\alpha}{1-\alpha} + \frac{\alpha\left[\rho + (1-\varepsilon)z\right]}{\gamma Lu} \\ &> \ \frac{1-\varepsilon\alpha}{1-\alpha} = \frac{(1-\varepsilon)\alpha}{1-\alpha} + 1 > 1. \qquad (30) \end{aligned}$$

From (27) and (29),

$$\begin{aligned}
b_1 &\leq \varepsilon(\alpha-1)z\bar{c} + \frac{1-\varepsilon\alpha}{1-\alpha}(1-\varepsilon)\gamma Luz - \rho\left[\rho + (1-\varepsilon)z\right] \\
&= \left[(1-\varepsilon)\gamma L - \varepsilon(1-\alpha)\rho - \varepsilon(1-\alpha)(1-\varepsilon\alpha)z\right] z - \rho^2 \text{ (by (19) and (22) with } \theta = 1) \\
&= \{(1-\varepsilon)\gamma L - \varepsilon(1-\alpha)\rho \\
&\quad -\frac{1-\alpha}{\alpha}\left[(1-\varepsilon\alpha)(\rho+\delta) + ((1-\varepsilon)\alpha\gamma L - (1-\alpha)\rho)\frac{\eta}{1-\alpha}\right]\Big\} z - \rho \qquad \text{(by (28))} \\
&= \left\{\left[(1-\varepsilon)\gamma L - \frac{1-\alpha}{\alpha}\rho\right](1-\eta) - \frac{1-\alpha}{\alpha}(1-\varepsilon\alpha)\delta\right\} z - \rho^2 < 0,
\end{aligned}$$

by A4 and (30).

Hence, given A3 and A4, each of the conditions (25) and $\theta = 1$ are, separately, sufficient for the existence of a unique convergent solution. With $\alpha = .4$ and $\varepsilon < .95$, we have $(1-\varepsilon\alpha)/(1-\alpha) > 1.02$, a number that seems beyond realistic values of η. In this way we have established uniqueness of the convergent solution (that is, saddle-point stability) at least within the empirically relevant domain of the parameter space.