

Can non-renewable resources alleviate the knife-edge character of endogenous growth?

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Standard endogenous growth models rely on the arbitrary assumption that the technology has exactly constant returns with respect to producible inputs. Can this knife-edge restriction be relaxed by including non-renewable resources as necessary inputs in production? In a one-sector optimal growth model, we find that the strain on the economy imposed by the need to extract successively smaller amounts of the non-renewable resource can offset the potentially explosive effects of allowing for increasing returns to producible inputs. However, growth in *per capita* consumption will be unstable unless there is population growth. Thus, the knife-edge problem of (strictly) endogenous growth reappears as an instability problem. But a 'semi-endogenous' growth framework turns out to be an attractive alternative, relying on less restrictive parameter values, maintaining stability, and allowing a rich set of determinants of long-run growth.

1. Introduction

The purpose of the present paper is twofold. First, the purpose is to study the properties of an endogenous growth model where the 'growth engine' depends on a non-renewable natural resource. Non-renewable resources are clearly an important element in the technologies of present-day economies, and their special properties make relevant an examination of the consequences for endogenous growth of introducing these resources. Indeed, while in recent years an increasing number of countries have managed to achieve a growing *per capita* output, the sustainability of such growth is bound to depend on the ability of the economic system, through substitution and technical progress, to overcome the constraints implied by scarce natural resources.

The second purpose of the paper is related to the non-robustness problem of standard endogenous growth models (Lucas, 1988; Romer, 1990; Rebelo, 1991). This problem arises from the fact that these models have a knife-edge character because they rely on the seemingly arbitrary assumption that there is exactly constant returns with respect to the producible inputs (at least asymptotically). We want to investigate whether introducing non-renewable natural resources into these models will make them more robust.

As to the first mentioned, more general purpose of the paper, the aim is to fill a hole in the literature on endogenous growth. Stiglitz (1974) presented a basic one-sector optimal growth model with non-renewable resources which relies on exogenous technical progress to generate growth. A number of later papers like Robson (1980), Takayama (1980), Jones and Manuelli (1997), Aghion and Howitt (1998, pp. 163–4), Scholz and Ziemes (1999), and Schou (2000) have examined the implications of the presence of non-renewable resources in various endogenous growth models. However, common to these papers is the fact that natural resources do not appear in the core sector (the growth engine)¹ of the model (not even indirectly in the sense of resources being a necessary ingredient in the production of physical capital goods which are then used in the growth-creating sector, e.g. a research sector). This is a crucial feature compared with the model of the present paper and clearly an unrealistic one. It seems unlikely that the growth-creating sector should be completely independent of physical capital and thus ultimately of the non-renewable resource. After all, most production sectors, including educational institutions and research labs, use fossil fuels for heating and transportation purposes, or minerals and oil products for machinery, computers, etc.²

In contrast to the ‘Schumpeterian’ model of the above mentioned section of the 1998 book by Aghion and Howitt, another section of that book presents a growth model with non-renewable resources entering the growth engine (Aghion and Howitt, 1998, pp. 162–3). The authors consider a one-sector AK-model with the resource added in a Cobb-Douglas fashion. In that model, however, it turns out that sustained growth is impossible without exogenous technical progress. The present paper shows how this result may be reversed (without violating stability) by introducing population growth and an explicit productive role for labour. Also, in Rebelo (1991) there is a short section on a kind of AK-model with natural resources. As we shall see (Section 2 below), some of Rebelo’s observations on this model need qualification if the natural resource is a non-renewable resource.

While our model will be presented and discussed as if the natural resource involved is a non-renewable resource in the usual meaning (i.e. fossil fuels, minerals), there is an alternative interpretation of the model. Instead of thinking of the reserves of, say, oil in the ground, one could think of the stock of environmental quality. When pollution is an inevitable by-product of economic activity, and when abatement possibilities are insufficient, the environment represented as a stock of ‘natural capital’ will be gradually exhausted in more or less the same way as a traditional non-renewable resource stock. In this spirit, Stokey (1998) and Aghion and Howitt (1998, pp. 157–60) present growth models where the modelling

¹ We define the growth engine of a model as the set of capital-producing sectors or activities using their own output as an input; of course, this set may consist of only one sector such as the educational sector in Lucas (1988) producing human capital or the R&D sector in Romer (1990) producing ‘knowledge capital’.

² An early contribution that indeed does take account of this fact is Chiarella (1980). In Section 4 below we comment on the ‘paradoxical’ comparative statics implied by the Chiarella model.

of pollution is quite similar to usual representations of a non-renewable resource. In these models, sustained growth is either not possible or at least non-optimal when there is no exogenous technical progress. Our paper indicates that these results may not necessarily hold when population growth and increasing returns with respect to producible inputs are allowed.

As to the second purpose of our paper, the non-robustness problem of standard endogenous growth models resides in the fact that exactly constant returns to scale with respect to the producible factors in the growth generating sector of the economy are required (at least asymptotically) for sustained *per capita* growth. Slightly increasing returns would lead to explosive growth (infinite output in finite time), whereas slightly decreasing returns lead to growth petering out unless some exogenous factor (e.g. population) grows. This is what we label the 'knife-edge problem' of standard endogenous growth theory. It is a serious challenge to the theory, as it implies lack of robustness to even slight parameter changes, as pointed out by, e.g. Solow (1994a).

Now, intuition suggests that the introduction of non-renewable resources could alleviate this knife-edge problem because the strain on the economy imposed by the need to extract successively smaller amounts of the resource might offset the potentially explosive effects of increasing returns to scale with respect to the reproducible factor(s) of the model. And even if this intuition may be valid as far as the pure technical feasibility of growth is concerned, the problem of examining the restrictions required for optimality and stability still remains.

Thus, this paper examines whether robust results can be obtained in a model where non-renewable natural resources are a necessary input into the growth engine and where there is utility discounting and consumption smoothing. Our approach to this problem is based on an extension of the model by Stiglitz (1974)³. Stiglitz concentrates on the case of constant returns with respect to capital, labour, and the resource taken together. We extend this analysis, characterizing steady growth paths for a broader range of parameter values. In particular, in line with endogenous growth theory, we allow for a constant or increasing marginal product of capital.

Indeed, a constant marginal product with respect to (some kind of) capital is essential to standard endogenous growth models (those that display balanced growth). These models may be divided into two sub-classes: models relying on pure capital accumulation of some sort, and models describing the process generating technical progress. However, a decisive feature common to both types of models is that steady growth relies on the above-mentioned knife-edge property.⁴

³ Dasgupta and Heal (1974, Section 1) analyzes a rather similar model. For convenience, in the following we refer generally only to Stiglitz (1974).

⁴ This is also true for the so-called quality ladder models of Aghion and Howitt (1998) and Grossmann and Helpman (1991), although in a more disguised form. Indeed, in these models it is assumed that every new innovation raises quality (or productivity) by a constant factor $\lambda > 1$. Given a constant cost of innovation, this assumption provides automatically for exponential growth.

Because of this, we believe that the distinction between models stressing capital accumulation and models stressing technical innovation is not important for the questions discussed in the present paper, even though the distinction may have serious implications for other questions such as market failures or consequences of various government policies.

It is often argued that a non-decreasing marginal product of capital is not so unrealistic when one recognizes the existence of externalities or if 'capital' is interpreted broadly as a combination of physical capital, technical knowledge, and human capital. In our view, if a constant marginal product of capital is accepted (at least asymptotically), a slightly increasing marginal product should be allowed as well. Obviously, such conditions may not be consistent with a *laissez-faire* market economy with perfect competition. However, in the growth literature imperfect competition and/or tax and subsidy interventions à la Pigou are often viewed as institutions that bring about an allocation approximating the social optimum.

It should be recognized, of course, that there exists a class of endogenous growth models that generate balanced growth only asymptotically when time approaches infinity. These 'convex models of endogenous growth' rely on constant returns to scale with respect to producible inputs to hold only asymptotically for the capital input approaching infinity (Jones and Manuelli, 1990, 1997). This assumption, though somewhat more robust, is still debatable. When extended to cover the case of non-renewable resources entering the growth engine, this approach requires the non-renewable resource to be non-essential (not necessary for production). In contrast, we want to study instances where the non-renewable resource enters the growth engine in an essential way. In this case, as we shall see, either (a) increasing returns to capital (read 'broad capital'), or (b) increasing returns to capital and labour taken together combined with population growth, is needed for sustained *per capita* growth to be technically feasible. On top of either of these conditions further parameter restrictions are required for sustained growth to be optimal. Finally, it turns out that stability is possible only if condition (b) holds. This implies that in the absence of population growth, the knife-edge problem of endogenous growth theory reappears as an instability problem.

By emphasizing the importance of increasing returns and population growth for stable positive *per capita* growth (when there is no exogenous technology growth), our analysis has affinity with what has been called semi-endogenous growth as distinct from (strictly) endogenous growth. The defining characteristic of (strictly) endogenous growth models (surveyed in Barro and Sala-i-Martin, 1995) is that *per capita* consumption in the long run grows at a constant positive rate, even in the absence of any exogenously growing factor. It is this type of model that suffers from the non-robustness problem described above.⁵ In contrast, weakly endogenous

⁵ And it is this non-robustness which cannot be rescued by merely including non-renewable resources this only transforms it into an instability problem as long as population growth is absent.

growth (Groth 1992) or *semi-endogenous* growth (Jones 1995) is defined as growth in *per capita* consumption in the long run at a constant positive rate, even without any exogenously given technology growth. An early example of a semi-endogenous growth model in this sense is the famous learning-by-doing paper by Arrow (1962); other examples are the modifications of the Romer 1990 R&D model suggested by Groth (1992) and Jones (1995). These conventional semi-endogenous growth models (surveyed in Eicher and Turnovsky 1999) differ from their (strictly) endogenous-growth relatives in two important ways: (a) They are less demanding with respect to returns to producible inputs; and (b) they imply long-run growth rates that are independent of preference parameters. As we shall see, however, letting non-renewable resources enter the growth engine brings good news: Property (a) of the semi-endogenous framework is preserved, while property (b) is circumvented. Indeed, semi-endogenous growth now assigns a role to preference parameters—hence also a role to incentives and, e.g. fiscal policy—as a determinant of the long-run growth rate.

The organization of the paper is as follows. The next section presents the technological aspects of the extended Stiglitz model and describes aspects of technically feasible paths. In Section 3, an intertemporal utility function is added in order to study aspects of optimal growth; existence and properties of steady states are analysed and the transitional dynamics described. Section 4 studies the special case of no population growth where the knife-edge reappears in the form of instability. A summary of the conclusions is given in the final section.⁶

2. Technically feasible paths

To ensure that the non-renewable resource is necessary for production, but does not *a priori* rule out sustainable (non-decreasing) consumption in the long run, we follow Stiglitz and assume an aggregate production function, F , of Cobb-Douglas form

$$Y(t) = F(K(t), N(t), R(t)) = AK(t)^\alpha N(t)^\beta R(t)^\gamma, A, \alpha > 0, 0 < \beta < 1, 0 < \gamma < 1 \quad (1)$$

where $Y(t)$ is output, $K(t)$ is the capital stock, and $R(t)$ is input of the non-renewable resource (henceforth, simply called the resource) at time t . In contrast to Stiglitz (1974), we ignore exogenous technical progress. More importantly, we impose no upper bound on α . While Stiglitz (1974) and others⁷ focused on $\alpha + \beta + \gamma = 1$, a case can be made for the parameters α , β , and γ summing to

⁶Some of the more technical parts of the analysis (proofs of lemmas etc.) are given in Supplementary material (2002) which can be found at <http://www.oep.oupjournals.org>

⁷Dasgupta and Heal, in the relevant chapters of their 1979 book, concentrate on the case $\alpha + \beta + \gamma = 1$. In their 1974 paper, however, Dasgupta and Heal consider the case $\alpha + \gamma = 1$. Still, decreasing returns to capital ($\alpha < 1$) is implied and a further difference with the present paper is that the roles of β and population growth are ignored.

some larger value.⁸ A constant marginal product with respect to some kind of capital (i.e. $\alpha = 1$) in the sector(s) constituting the growth engine is indeed standard, and essential to all basic endogenous growth models displaying balanced growth. As emphasized in the introduction, though we name K 'capital', one may interpret K as 'broad capital' including technical knowledge and human capital. The separate argument N in the production function should then be interpreted as representing the role of raw physical labour. The reasoning of Mankiw (1995) suggests that the output elasticity, β , with respect to this factor is at least 0.2. In any case, we emphasize that if $\alpha = 1$ is accepted, values slightly above unity should be allowed as well.

As to the output elasticity with respect to the natural resource, empirical examinations of resources such as minerals and fossil fuels usually consider γ to be relatively low, say less than 0.05 (cf. Nordhaus and Tobin, 1972, p. 419 ff., and Neumayer, 2000, p. 322). Nevertheless, as mentioned in the introduction, an alternative interpretation of the resource makes the potential value range of γ larger: R could be taken to represent a pollution flow considered as an inevitable by-product of production which may imply a value of γ somewhat above 0.05. In a situation with insufficient abatement possibilities, all pollution detracts from the quality of the environment, considered to be part of the stock of 'natural capital'. If cumulative pollution becomes high enough, then at a certain point a critical ecological threshold will be reached, below which environmental quality cannot fall further without entailing a prohibitive cost in the form of an ecological catastrophe.⁹ Though this interpretation is possible, in the remainder of this paper we will refer to R as an input of a non-renewable resource. In any case, it seems reasonable to assume $\gamma < 1$ as in (1).

Whatever the interpretation of R , output is used for consumption and for investment in capital goods, so that¹⁰

$$\dot{K} = Y - C - \delta K \equiv I - \delta K, \quad K(0) = K_0 > 0, \quad \delta \geq 0 \quad (2)$$

where $C \equiv cN$ is total consumption, I is gross investment, δ is the capital depreciation rate (disregarded in Stiglitz, 1974), and K_0 is given. Labour N is supplied inelastically, and the labour force grows at a constant exogenous rate $n \geq 0$, i.e. $N = N(0)e^{nt}$, $N(0) = N_0 > 0$, where N_0 is given. The resource stock S diminishes with resource extraction

$$\dot{S} = -R, \quad S(0) = S_0 > 0 \quad (3)$$

where S_0 is given. Like Stiglitz we abstract from extraction costs as well as uncertainty.

⁸ Empirical evidence furnished by, e.g. Hall (1990) and Caballero and Lyons (1992), suggests that there are quantitatively significant increasing returns to scale or external effects in U.S. and European manufacturing.

⁹ This is similar to the way in which Aghion and Howitt (1998, pp.157–60), inspired by Stokey (1998), present a growth model with pollution.

¹⁰ From now on we will suppress the time argument when not needed for clarity.

A path $(C, Y, K, R, S)_{t=0}^{\infty}$ is called feasible if: (a) K and S are continuous functions of t ; (b) $C, Y,$ and R are piecewise continuous functions of t ; (c) the path satisfies (1) for all $t \geq 0$, and it satisfies (2) and (3) for all $t \geq 0$, except at points of discontinuity of C and R ; and (d) the path satisfies the non-negativity constraints

$$C, R \geq 0 \quad \text{for all } t \geq 0 \quad (4)$$

(for the directly controllable variables) and

$$K, S \geq 0 \quad \text{for all } t \geq 0 \quad (5)$$

(for the state variables). The conditions (3), (4), and (5) on a feasible path imply the restriction

$$\int_0^{\infty} R(t) dt \leq S(0) \quad (6)$$

showing the finite upper bound on cumulative extraction of the resource over the infinite future. Obviously, from this restriction it follows that resource use must approach zero for $t \rightarrow \infty$.

A feasible path $(C, Y, K, R, S)_{t=0}^{\infty}$ is called a balanced growth path (henceforth abbreviated BGP) if $C, Y, K, R,$ and S are (strictly) positive for all $t \geq 0$ and change with constant relative rates (some or all of these rates may be negative). Let g_x denote the growth rate of the variable $x > 0$, that is $g_x \equiv \dot{x}/x$.

Lemma 1 In a BGP the following holds: (a) $g_S = g_R < 0$; (b) $R(0) = -g_R S(0)$, and

$$\lim_{t \rightarrow \infty} S = 0 \quad (7)$$

Proof See Supplementary material. \square

It is because our definition of a BGP includes the requirement that g_S is constant, that a BGP implies that the resource is exhausted in the limit.

The output-capital ratio, the consumption-capital ratio, and the resource extraction rate will be called $z, x,$ and u , respectively, i.e.

$$z \equiv \frac{Y}{K}, \quad x \equiv \frac{C}{K}, \quad u \equiv \frac{R}{S} \quad (8)$$

These ratios turn out to be central to the analysis. We may write (2) as

$$g_K = z - x - \delta \quad (9)$$

Similarly, by (3)

$$g_S = -u \quad (10)$$

In a BGP, by definition, $z, x,$ and u are always positive. *A priori* $x > z$ is not excluded; that is, we allow gross investment $I (\equiv Y - C)$ to be negative (capital can be 'eaten').

Lemma 2 In a BGP, $g_Y = g_C \equiv g$, and u is constant. If, in addition, $g_K = g$, then also z and x are constant. A sufficient condition for $g_K = g$ in a BGP is $I \neq 0$ in some non-degenerate time interval (gross investment non-vanishing).

Proof See Supplementary material. □

Differentiating (1) logarithmically with respect to time, we get

$$g_Y = \alpha g_K + \beta n + \gamma g_R \tag{11}$$

We will define a steady state as a feasible path where $C, Y, K, R,$ and S are (strictly) positive for all $t \geq 0$ and change with constant (though not necessarily positive) rates, say $g_C, g_Y, g_K, g_R,$ and $g_S,$ and where z and x are constant. That is, a steady state is a BGP such that z and x are constant (in addition to u constant as in any BGP, cf. Lemma 2).¹¹ By definition of z and $x,$ a steady state has $g_C = g_Y = g_K = g,$ a constant. A steady state is therefore well described by the constant values of $g, g_R, z, x,$ and $u,$ i.e. a quintuple $(g^*, g_R^*, z^*, x^*, u^*)$ where $u^* = -g_S^* = -g_R^* > 0$ (from (10) and Lemma 1) and where also z^* and x^* are positive (by definition of a BGP).

The growth rate g_c of *per capita* consumption satisfies $g_c = g_C - n,$ and in a BGP, still denoting the common growth rate for C and Y by $g,$ we have

$$g_c = g - n \tag{12}$$

Intuitively, to avoid decreasing *per capita* consumption, some positive net investment $(I - \delta K)$ is always needed to offset the diminishing resource use over time. Indeed:

Lemma 3 In a BGP with $g_c \geq 0,$ net investment, $I - \delta K,$ must be positive for all $t.$

Proof See Supplementary material. □

We now pose the question: Is it possible, from a purely technical point of view, that capital accumulation can support steady *per capita* growth? Indeed, intuition suggests that the answer is yes since two counterbalancing forces are at hand. First, one should expect that the strain on the economy imposed by the need to extract successively smaller amounts of the resource can be offset by increasing returns to capital. Second, the potentially explosive effects of this type of increasing returns might be exactly counterbalanced by the diminishing resource use. The following proposition shows that this intuition is correct.

Proposition 1 There exists a BGP with $g_c > 0$ if and only if

$$\alpha > 1 \text{ or } (\alpha + \beta - 1)n > 0 \tag{13}$$

Proof ‘If’: See Groth (2001) where a slightly stronger statement is proved. ‘Only if’: Consider a BGP with $g_c > 0.$ Then, by Lemma 3, $I - \delta K \geq 0$ for all $t.$ Now, from Lemma 2, $g_K = g_Y = g,$ and (11) reduces to $(1 - \alpha)g - \gamma g_R = \beta n.$ By Lemma 1, $g_R < 0,$ and therefore $(1 - \alpha)g < \beta n.$ By (12) this implies $(1 - \alpha)g_c < (\alpha + \beta - 1)n.$ Now, since $g_c > 0, \alpha \leq 1$ is seen to imply $(\alpha + \beta - 1)n > 0.$ □

¹¹ Observe that in contrast to many papers, a BGP and a steady state are not, here, entirely synonymous.

The proposition tells us that for the technology to allow steady *per capita* growth (with an indispensable resource and without exogenous technical progress), either increasing returns to capital and labour together combined with population growth or increasing returns to capital itself is needed. At least one of these conditions is required in order that capital accumulation can offset the effects of the necessarily diminishing resource use over time.¹² That is why our study need taking population growth into account and has to consider a larger range of values for α than in Stiglitz (1974) and Dasgupta and Heal (1974, 1979).¹³ Though one might find $n > 0$ (in the very long run on a limited earth) as well as $\alpha > 1$ debatable assumptions,¹⁴ an analysis of these cases is of theoretical interest for an evaluation of the robustness problem of endogenous growth theory.

3. Optimal paths

We now add preferences to the model. Assume utilitarian preferences with a constant rate of time preference ρ . Let instantaneous utility of the representative infinitely-lived household be $U(c) = (c^{1-\varepsilon} - 1)/(1 - \varepsilon)$ where $\varepsilon > 0$ is a constant, the numerical value of the elasticity of marginal utility. The intertemporal utility function then is

$$\int_0^{\infty} \frac{c(t)^{1-\varepsilon} - 1}{1 - \varepsilon} N(t) e^{-\rho t} dt, \quad \rho > n \geq 0 \quad (14)$$

where N is now interpreted as household (or family) size which grows at the rate n (the population growth rate); we assume $\rho > n$ to ensure convergence of the integral. The social planner wants to maximize (14) subject to the conditions (1)–(5).¹⁵

It can be shown that along an optimal path, if one exists, $c, R, K, S, Y > 0$ for all $t \geq 0$ (see Supplementary material). Hence, we may safely concentrate on interior solutions. The current-value Hamiltonian for the social planner's problem is

¹² Of course this presupposes an elasticity of substitution between the resource and the other inputs not larger than one as implied by the Cobb-Douglas specification (1). Historical evidence for the US may indicate otherwise (Nordhaus, 1992). In any event, it is difficult to predict the technological substitution possibilities one century ahead.

Proposition 1 sharpens a remark made by Rebelo (1991, pp.518–19) on growth with a non-reproducible factor (in a Cobb-Douglas technology); indeed, when the non-reproducible factor is a non-renewable resource and $n = 0$, to offset the diminishing resource use over time, $\alpha > 1$ is needed, not just $\alpha \geq 1$ (as claimed by Rebelo).

¹³ Interestingly, in Stiglitz (1974) there is, on p.131, a short remark on feasible growth paths in the case of increasing returns ($\alpha + \beta + \gamma > 1$). That remark is close to the 'only if' part of our Proposition 1, but ignores implicitly the possibility of $\alpha > 1$; hence, (strictly) endogenous growth is precluded. The same applies to Withagen (1990, p.391).

¹⁴ Indeed, for $\alpha > 1$ it turns out that any positive constant consumption growth rate is technically feasible (Groth, 2001); though not implying infinite output in finite time (as would be the case if, in addition to $\alpha > 1$, $\gamma = 0$), this still sounds too good to be true (to paraphrase Solow, 1994b, p.377).

¹⁵ If either $\varepsilon > 1$ or $U(c) = \ln c$ (corresponding to $\varepsilon = 1$), the restriction on $C \equiv cN$ in (4) should read $C > 0$.

$$H = \frac{c^{1-\varepsilon} - 1}{1 - \varepsilon} N + \mu_1(F(K, N, R) - cN - \delta K) - \mu_2 R \tag{15}$$

where μ_1 and μ_2 are the co-state variables associated with physical capital and the resource stock, respectively. Necessary conditions for an interior solution are given by the following first order and transversality conditions

$$c^{-\varepsilon} = \mu_1 \tag{16}$$

$$\mu_1 F_R = \mu_2 \tag{17}$$

$$\dot{\mu}_1 = -(F_K - \delta)\mu_1 + \rho\mu_1 \tag{18}$$

$$\dot{\mu}_2 = \rho\mu_2 \tag{19}$$

$$\lim_{t \rightarrow \infty} e^{-\rho t} \mu_1 K = 0 \tag{20}$$

$$\lim_{t \rightarrow \infty} e^{-\rho t} \mu_2 S = 0 \tag{21}$$

We observe that (19) implies $\mu_2 = \mu_2(0)e^{\rho t}$. Inserting this into (21) gives $\lim_{t \rightarrow \infty} \mu_2(0)S = 0$, which, since $\mu_2(0) > 0$, from (17) and (16), is equivalent to

$$\lim_{t \rightarrow \infty} S = 0 \tag{22}$$

saying that optimality requires, of course, that no finite part of the resource stock will be left unused forever.

Differentiating (16) logarithmically with respect to time and combining with (18) gives the Keynes-Ramsey Rule

$$g_c = \frac{1}{\varepsilon}(F_K - \delta - \rho) = \frac{1}{\varepsilon}(\alpha z - \delta - \rho) \tag{23}$$

where the last equality comes from using the Cobb-Douglas specification of F and the definition $z \equiv Y/K$. Similarly, differentiating (17) logarithmically with respect to time and combining with (18) and (19) gives the Hotelling Rule for optimal extraction of a non-renewable resource

$$\frac{\dot{F}_R}{F_R} = F_K - \delta \quad \text{or} \quad g_Y - g_R = \alpha z - \delta \tag{24}$$

using again the Cobb-Douglas specification of F .

3.1 Optimal steady states

Because of the wide parameter range and the drag on growth implied by the diminishing resource use, the analysis is somewhat more complicated than usual steady state analyses. In particular, we must pay attention to the definitional strict positivity of certain key variables in a steady state (the output-capital ratio z , the consumption-capital ratio x , and the extraction rate u). To put it differently, we must keep track of the difference between a (true) steady state and the conceivable asymptotic stationary states where z , x , and/or u approach zero.

We shall proceed step by step. First, a feasible path satisfying the first order conditions (16)–(19) will be called an optimal growth candidate. An optimal

growth candidate must satisfy the transversality conditions (20) and (21) to be an optimal growth path.

Lemma 4 Let $(C, Y, K, R, S)_{t=0}^{\infty}$ be an optimal growth candidate such that $\lim_{t \rightarrow \infty} g_C = \lim_{t \rightarrow \infty} g_Y = \lim_{t \rightarrow \infty} g_K = \bar{g}$. Then: (a) for some $\bar{g}_R \leq 0$, $\lim_{t \rightarrow \infty} g_R = \bar{g}_R$, and \bar{g} and \bar{g}_R satisfy

$$(1 - \alpha)\bar{g} - \gamma\bar{g}_R = \beta n \quad (25)$$

$$(\varepsilon - 1)\bar{g} + \bar{g}_R = \varepsilon n - \rho \quad (26)$$

and (b) if, in addition, the transversality conditions (20) and (21) hold, then $\lim_{t \rightarrow \infty} g_S = \bar{g}_R < 0$.

Proof See Supplementary material. \square

In Section 2 we defined a BGP as a feasible path along which Y, K, C, R , and S are positive and change with constant (though not necessarily positive) rates. It is useful to introduce the slightly more general concept of an asymptotic path. A feasible path $(C, Y, K, R, S)_{t=0}^{\infty}$ is called an asymptotic path if there exists constants $\bar{g}_C, \bar{g}_Y, \bar{g}_K, \bar{g}_R$, and \bar{g}_S , such that along the path $(C, Y, K, R, S)_{t=0}^{\infty}$, $(g_C, g_Y, g_K, g_R, g_S) \rightarrow (\bar{g}_C, \bar{g}_Y, \bar{g}_K, \bar{g}_R, \bar{g}_S)$ for $t \rightarrow \infty$. Similarly, an optimal growth candidate which is an asymptotic path will be called an asymptotic optimal growth candidate. The limiting growth rates $\bar{g}_C, \bar{g}_Y, \bar{g}_K, \bar{g}_R$, and \bar{g}_S are called asymptotic growth rates.

Lemma 5 Let $(C, Y, K, R, S)_{t=0}^{\infty}$ be an asymptotic optimal growth candidate with asymptotic growth rates $\bar{g}_C, \bar{g}_Y, \bar{g}_K, \bar{g}_R$, and \bar{g}_S . Assume $\bar{g}_C = \bar{g}_Y = \bar{g}_K = \bar{g}$. Then: (a) $\lim_{t \rightarrow \infty} (z, x, u) = (\bar{z}, \bar{x}, \bar{u})$, where

$$\bar{z} = \frac{1}{\alpha} (\bar{g} - \bar{g}_R + \delta) \quad (27)$$

$$\bar{x} = \frac{1}{\alpha} [-(1 - \gamma)\bar{g}_R + \beta n + (1 - \alpha)\delta] \quad (28)$$

$$\bar{u} = -\bar{g}_S \quad (29)$$

(b) $\bar{g}_R \leq 0$. If $\bar{g}_R < 0$, then the transversality condition (20) is satisfied; and (c) $\bar{g}_S \leq 0$. If $\bar{g}_S < 0$, then the transversality condition (21) is satisfied.

Proof See Supplementary material. \square

Corollary Under the conditions of (a) of Lemma 5, the following holds:

- (i) $\bar{g} \geq \bar{g}_R - \delta$;
- (ii) $\bar{z} > 0$ if and only if $\bar{g} > \bar{g}_R - \delta$;
- (iii) If $\bar{g}_R < 0$, then $\bar{x} > 0$ if and only if either $\alpha \leq 1$ or $\alpha > 1 \wedge \delta < -(1 - \gamma)\bar{g}_R / (\alpha - 1) + \beta n / (\alpha - 1)$.

Proof (i) and (ii) By definition, $\bar{z} \geq 0$; hence, by (27), $\bar{g} \geq \bar{g}_R - \delta$ and $\bar{z} > 0$ if and only if $\bar{g} > \bar{g}_R - \delta$. (iii) When $\bar{g}_R < 0$, (28) implies that if and only if either $\alpha \leq 1$ or $\alpha > 1 \wedge \delta < (1 - \gamma)\bar{g}_R/(1 - \alpha) - \beta n/(1 - \alpha)$, then $\bar{x} > 0$. \square

In Section 2 we noticed that a steady state, defined as a BGP such that z and x are constant (in addition to u constant which is satisfied by any BGP), has $g_C = g_Y = g_K = g$, a constant. Therefore, a steady state is well described by its constant values of g, g_R, z, x , and u , i.e. the quintuple $(g^*, g_R^*, z^*, x^*, u^*)$. Similarly, Lemma 5 implies that certain asymptotic paths with C, Y , and K having the same asymptotic growth rate \bar{g} imply asymptotically constant z and x (and also asymptotically constant u , trivially). In this case, the limiting state of the economy is well described by a quintuple $(\bar{g}, \bar{g}_R, \bar{z}, \bar{x}, \bar{u})$. The limiting state, $(\bar{g}, \bar{g}_R, \bar{z}, \bar{x}, \bar{u})$, of an asymptotic path such that: (i) C, Y , and K have the same asymptotic growth rate \bar{g} ; and (ii) z and x are asymptotically constant, will be called an asymptotic steady state. In contrast to a (true) steady state, the asymptotic steady state associated with a given asymptotic path need not itself be a feasible path (since $\bar{g}_R, \bar{z}, \bar{x}$, and/or \bar{u} could be zero); in addition, it need not have $\bar{u} = -\bar{g}_R$ (though this equality must hold for an optimal asymptotic path, cf. (29) and (b) of Lemma 4).

Lemma 5 and its corollary provide an easy way to check whether an asymptotic steady state satisfies the strict positivity conditions required by a (true) steady state and the transversality conditions required by an optimal steady state. The usefulness of the criteria in the corollary derives from the fact that \bar{g}_R and \bar{g} , as determined by (25) and (26), are independent of the rate of capital depreciation δ . Further, the ‘only if’ part in (iii) of the corollary points to the fact that if and only if $\alpha > 1$, then it is possible to choose δ large enough such that an otherwise valid asymptotic optimal growth candidate ends up having $\bar{x} < 0$. On the other hand, if δ is ‘small’, then this problem cannot arise. (Since Stiglitz, 1974, and many other papers assume $\alpha < 1$ and/or $\delta = 0$, the problem never arises there.) The upper bound for δ reflects the fact that, given g_R , sustaining any given output growth rate g_Y requires a certain amount of capital accumulation (net investment), as shown in (11). Now, the replacement part of gross investment is an increasing linear function of δ and an arbitrarily large δ would engender a risk that gross investment absorbed total output, leaving no room for consumption, as shown in (9); however, that risk cannot materialize as long as $\alpha < 1$ since in that case, the output-capital ratio z adjusts more than one to one to an increase in δ , by the Hotelling Rule (24).

With respect to (27) of Lemma 5, and (i) and (ii) of its corollary, given g_R , consistency with the Hotelling Rule implies that for the return on capital $(\alpha z - \delta)$ and the return on leaving the marginal unit of the resource in the ground to be simultaneously positive (indeed equal to each other), a not too negative rate of output growth is required.

Explicit solutions for the growth rates can be derived from (25) and (26). Let D be the determinant of that linear system

$$D \equiv 1 - \alpha + (\varepsilon - 1)\gamma \quad (30)$$

Observe that the determinant does not depend on ρ , β , n , and δ . Now, let us concentrate on the generic case $D \neq 0$. Indeed, $D > 0$ seems to be the most realistic case since $D \leq 0$ requires a considerable amount of increasing returns ($\alpha + \gamma \geq 1 + \varepsilon\gamma$). But we will not *a priori* exclude the case $D < 0$ which, at least from a theoretical point of view, deserves some interest because it implies very different dynamics (as shown in the next section).¹⁶ Given $D \neq 0$, solving the system (25) and (26) gives the following growth rates in an asymptotic steady state

$$\bar{g} = \frac{(\beta + \gamma\varepsilon)n - \gamma\rho}{D} \quad (31)$$

$$\bar{g}_R = \frac{[\varepsilon(1 - \alpha - \beta) + \beta]n - (1 - \alpha)\rho}{D} \quad (32)$$

Using these solutions we easily find the values of the output-capital ratio, the consumption-capital ratio, and the resource extraction rate in an asymptotic steady state, when $D \neq 0$. Inserting (31) and (32) into (27) gives

$$\bar{z} = \frac{(\alpha + \beta + \gamma - 1)\varepsilon n + (1 - \alpha - \gamma)\rho}{\alpha D} + \frac{\delta}{\alpha} \quad (33)$$

By (32) and (28)

$$\bar{x} = \frac{[\varepsilon(\alpha + \beta + \gamma - 1 - \alpha\gamma) - \alpha\beta]n + (1 - \alpha)(1 - \gamma)\rho}{\alpha D} + \frac{1 - \alpha}{\alpha} \delta \quad (34)$$

Finally, in an asymptotic steady state with $\bar{g}_S = \bar{g}_R$, we have, from (10) and (32)

$$\bar{u} = -\bar{g}_R = \frac{[\varepsilon(\alpha + \beta - 1) - \beta]n + (1 - \alpha)\rho}{D} \quad (35)$$

The expressions (31), (32), (33), (34), and (35) are straightforward generalizations of corresponding expressions in Stiglitz (1974). The interesting question is how the larger range allowed for the parameters affects the existence and character of optimal steady states, remembering that a candidate path, to be a (true) steady state, must have the output-capital ratio, the consumption-capital ratio, and the resource extraction rate strictly positive. We may summarize the answer to the existence question in the following proposition.

Proposition 2 Given the model (1)–(5) and (14), assume $D \equiv 1 - \alpha + (\varepsilon - 1)\gamma \neq 0$. Let \bar{g} , \bar{g}_R , \bar{z} , \bar{x} , and \bar{u} be defined as in (31), (32), (33), (34), and (35), respectively. Then: (a) there exists a steady state $(g^*, g_R^*, z^*, x^*, u^*)$ fulfilling the first-order and transversality conditions above if and only if the parameters $\alpha, \beta, \gamma, \varepsilon, \rho, n$, and δ take values such that: (i) $\bar{g}_R < 0$; (ii) $\bar{z} > 0$; and (iii) either $\alpha \leq 1$ or $\alpha > 1 \wedge \delta < -(1 - \gamma)\bar{g}_R/(\alpha - 1) + \beta n/(\alpha - 1)$. This set of parameter

¹⁶When $D = 0$, it turns out that an optimal steady state can only exist in the knife-edge case $\beta n = -\gamma(\varepsilon n - \rho)$. In most of this paper we will assume $D \neq 0$. In Stiglitz (1974), since $\alpha + \beta + \gamma = 1$, $D > 0$ always.

values has a non-empty interior; and (b) the steady state $(g^*, g_R^*, z^*, x^*, u^*)$ is unique and equal to $(\bar{g}, \bar{g}_R, \bar{z}, \bar{x}, \bar{u})$.

Proof See Appendix.

Remark 1 In the special case, $\alpha + \beta + \gamma = 1$, analysed by Stiglitz (1974), (a.ii) and (a.iii) are automatically satisfied. In this case the only thing to check is (a.i). Evidently, a case violating (a.i) for any $\gamma \in (0, 1)$ is: $\varepsilon n > \rho > n > 0$, $0 < \alpha < 1$, $0 < \beta \leq (\varepsilon n - \rho)(1 - \alpha)/[(\varepsilon - 1)n]$. A case violating (a.ii) for any $\beta \in (0, 1)$ and any $\rho > 0$ is: $n = \delta = 0$, $\alpha + \gamma > 1$, $\varepsilon > (\alpha + \gamma - 1)/\gamma$.

Remark 2 Proposition 2 speaks only of ‘a steady state fulfilling the first order and transversality conditions’ and not of an optimal steady state. This is so because outside the case $\alpha + \gamma \leq 1$ we have not been able to prove that the steady state in question is optimal (using Arrow’s sufficiency condition, see Supplementary material). We conjecture that even for $\alpha + \gamma > 1$ (at least as long as $D > 0$, i.e., $\alpha + \gamma < 1 + \varepsilon\gamma$), a steady state growth path fulfilling the first-order and transversality conditions above is indeed optimal; alternatively, no optimal steady state exists.

Our focus will be on economic growth (or at least non-contraction). It turns out in this case to be no serious limitation to restrict the attention to steady states (and their transitional dynamics which is the topic of the next section) instead of asymptotic steady states in general. Indeed, the following lemma shows that, given an optimal asymptotic path with asymptotic steady state $(\bar{g}, \bar{g}_R, \bar{z}, \bar{x}, \bar{u})$ such that $\bar{g} \geq 0$, then, generically, $(\bar{g}, \bar{g}_R, \bar{z}, \bar{x}, \bar{u})$ is a (true) steady state.¹⁷ Similarly, because we focus on optimal growth, it is no limitation to restrict our attention to steady states instead of balanced growth paths in general. In fact, any optimal BGP is an optimal steady state:

Lemma 6 (a) given an optimal asymptotic path $(C, Y, K, R, S)_{t=0}^\infty$ with asymptotic steady state $(\bar{g}, \bar{g}_R, \bar{z}, \bar{x}, \bar{u})$, then: (i) $\bar{u} = -\bar{g}_R > 0$; (ii) $\bar{z} > 0$ when $\bar{g} \geq 0$; and (iii) $\bar{x} > 0$, except in the knife-edge case where simultaneously $\alpha > 1$ and $\delta = -(1 - \gamma)\bar{g}_R/(\alpha - 1) + (\beta n/\alpha - 1)$; and (b) given an optimal BGP $(C, Y, K, R, S)_{t=0}^\infty$, then z, x , and u are constant and positive.

Proof See Supplementary material. □

In contrast to (a) of Lemma 6, when $\rho = 0$ (zero rate of time preference) there may exist asymptotic optimal paths with $\bar{g} = \bar{z} = \bar{x} = \bar{u} = 0$. For the case $\alpha + \beta + \gamma = 1$, $\delta = 0$, no population growth, and no technical progress, this was

¹⁷ By (31), when $D > 0$, $g^* \geq 0$ requires $n \geq \gamma\rho/(\beta + \varepsilon\gamma)$. In view of our general restriction $n < \rho$, $n \geq \gamma\rho/(\beta + \varepsilon\gamma)$ is allowed if and only if $(1 - \varepsilon)\gamma < \beta$. From an empirical point of view, this restriction is hardly problematic.

shown by Solow (1974) and Dasgupta and Heal (1979, p. 303 ff.) under the proviso that $\alpha > \gamma$, and $\varepsilon > (1 - \gamma)/(\alpha - \gamma)$.

Now, inserting (31) into (12) gives the *per capita* growth rate in a steady state which satisfies the first order and transversality conditions

$$g_c^* = g^* - n = \frac{(\alpha + \beta + \gamma - 1)n - \gamma\rho}{D} \quad (36)$$

Proposition 3 Given the model (1)–(5) and (14), assume $D \equiv 1 - \alpha + (\varepsilon - 1)\gamma \neq 0$. Then there exists a steady state with $g_c^* > 0$, satisfying the first order and transversality conditions, if and only if, in addition to (a.i) and (a.iii) of Proposition 2, the parameters satisfy

$$\frac{(\alpha + \beta + \gamma - 1)n - \gamma\rho}{D} > 0 \quad (37)$$

Proof See Appendix. \square

Corollary If $D > 0$, then a steady state, fulfilling the first order and transversality conditions, has $g_c^* > 0$ if and only if

$$\alpha + \beta > 1 \quad \text{and} \quad n > \frac{\gamma\rho}{\alpha + \beta + \gamma - 1} \quad (38)$$

Proof Assume $D > 0$. Then, by (36), $g_c^* > 0 \Leftrightarrow (\alpha + \beta - 1)n > \gamma(\rho - n)$. Since $\rho > n$, the last inequality is equivalent to (38). \square

The implication of the corollary to Proposition 3 is that when $D > 0$, optimal sustained *per capita* growth requires two things: increasing returns to capital and labour together *and* a sufficient amount of population growth. From Proposition 1 we know that only either $\alpha > 1$ or $\alpha + \beta > 1$ and $n > 0$ is needed to offset the effects of the decreasing input of the natural resource and make sustained *per capita* growth technically feasible. We conclude that there exist parameter constellations such that sustained *per capita* growth is technically feasible, but not viable when preferences involve utility discounting and consumption smoothing, both of which bias the ‘growth incentive’ downwards. We defer an examination of an example of this until after a presentation of the transitional dynamics.

3.2 Transitional dynamics

From (9) follows, using the identities $x \equiv C/K$ and $z \equiv Y/K$

$$\dot{x} = (g_C - z + x + \delta)x \quad (39)$$

$$\dot{z} = (g_Y - z + x + \delta)z \quad (40)$$

Inserting (9) and the Hotelling Rule, (24), into (11) yields

$$g_Y = \alpha z - \frac{\alpha}{1 - \gamma}x + \frac{\beta n + (\gamma - \alpha)\delta}{1 - \gamma} \quad (41)$$

Using this expression in (40), we find

$$\dot{z} = \left[(\alpha - 1)z + \frac{1 - \alpha - \gamma}{1 - \gamma}x + \frac{\beta n + (1 - \alpha)\delta}{1 - \gamma} \right] z \tag{42}$$

Inserting $g_C = g_c + n$ and the Keynes-Ramsey Rule, (23), into (39) gives

$$\dot{x} = \left[\left(\frac{\alpha}{\varepsilon} - 1 \right) z + x - \frac{\delta + \rho}{\varepsilon} + n + \delta \right] x \tag{43}$$

Differentiating the identity $u \equiv R/S$ with respect to time and using (3), we get

$$\dot{u} = (g_R + u)u \tag{44}$$

By (24) and (41), this gives

$$\dot{u} = \left(\frac{-\alpha}{1 - \gamma} x + \frac{\beta n + (1 - \alpha)\delta}{1 - \gamma} + u \right) u \tag{45}$$

The dynamics of $z, x,$ and u are completely described by the system (42), (43), and (45). To get an overview, we will utilize the fact that the system is decomposable: (42) and (43) constitute a dynamical subsystem for z and x alone. In a steady state, $\dot{z} = \dot{x} = 0$. We form the Jacobian evaluated in a steady state¹⁸

$$J = \begin{bmatrix} \frac{\partial \dot{z}}{\partial z} & \frac{\partial \dot{z}}{\partial x} \\ \frac{\partial \dot{x}}{\partial z} & \frac{\partial \dot{x}}{\partial x} \end{bmatrix} = \begin{bmatrix} (\alpha - 1)z^* & \frac{1 - \alpha - \gamma}{1 - \gamma} z^* \\ \left(\frac{\alpha}{\varepsilon} - 1 \right) x^* & x^* \end{bmatrix}$$

The trace of J is $(\alpha - 1)z^* + x^* = \alpha z^* - g^* - \delta = -g_R^* > 0$, by (9), (27), and (a.i) of Proposition 2. Hence, at least one eigenvalue is positive (or has positive real part). The determinant, Δ , of J is

$$\Delta = \alpha \frac{D}{(\gamma - 1)\varepsilon} z^* x^* \tag{46}$$

where D is given by (30). Δ is negative (hence, implying real eigenvalues of opposite sign) if and only if $D > 0$. In the case of constant returns to scale ($\alpha + \beta + \gamma = 1$), the condition $D > 0$ is automatically satisfied. But as mentioned in the previous section, if there is a sufficient amount of increasing returns with respect to K and R , then D could be negative.

In Fig. 1, a phase diagram is drawn for a particular constellation of the parameters satisfying *inter alia* $n > 0$, $\alpha > 1$, and $D > 0$ (the last condition obtains when $\varepsilon > \alpha \gtrsim 1$). Since the growth rates of z and x are linear functions of z and x , the isoclines $\dot{z} = 0$ and $\dot{x} = 0$ are straight lines.

In a neighbourhood of the steady state, along the saddle path (the stable manifold) we have approximately

$$\dot{z} = \lambda(z - z^*) \tag{47}$$

$$x = f(z) \tag{48}$$

¹⁸In what follows, when speaking of a steady state it is understood to be a steady state fulfilling the first order and transversality conditions unless otherwise stated.

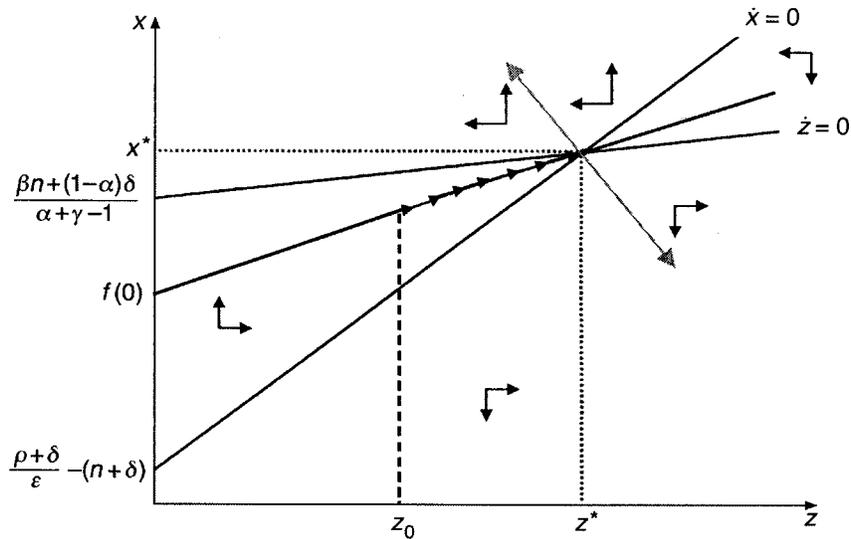


Fig. 1. Dynamics of z and x (when $n > 0$, $\alpha > 1$, and $D > 0$).

where the constant λ is the negative eigenvalue of J , and f is a linear function that can be determined by standard methods.

From this fact, we can construct a phase diagram in z and u (Fig. 2), using (47) and (45), where x is replaced by $f(z)$ from (48). The Jacobian of this system, evaluated in a steady state, has eigenvalues λ and u^* ($\lambda < 0 < u^*$). The isoclines $\dot{z} = 0$ and $\dot{u} = 0$ are straight lines.

Even though z, x , and u are all ‘jump variables’, by substituting uS for R in the production function (1) we get

$$z = AK^{\alpha-1}N^\beta u^\gamma S^\gamma \tag{49}$$

showing that, given K, N , and S , the values of z and u are not independent; indeed, initially, with $K(0) = K_0, N(0) = N_0$, and $S(0) = S_0$, we have $u_0 = A^{-1/\gamma} K_0^{(1-\alpha)/\gamma} L_0^{-\beta/\gamma} S_0^{-1} z_0^{1/\gamma} \equiv B(0)z_0^{1/\gamma}$. This boundary value condition is represented by the stippled curve in Fig. 2. The intersection between this curve and the saddle-path determines uniquely (at least locally) the initial values, z_0 and u_0 , required for convergence to the steady state. The unique intersection between the line $z = z_0$ and the saddle-path in Fig. 1 gives the required x_0 . Then the movement of the economy over time is along the saddle-paths of Figs 1 and 2 towards the steady state.¹⁹

¹⁹ All other paths are either infeasible or sub-optimal. Indeed, these paths have the unstable manifold as their asymptote. Therefore, either $x \rightarrow \infty$ (in fact, $x/z \rightarrow \infty$) for $t \rightarrow \infty$ and then K becomes negative in finite time, by (9). Or $x \rightarrow 0$ for $t \rightarrow \infty$ in which case, by (45), $\dot{u} \rightarrow [(\beta n + (1 - \alpha)\delta)/(1 - \gamma) + u]u$ for $t \rightarrow \infty$; it follows (at least for δ ‘small enough’) that either $u \rightarrow \infty$ and the resource stock is exhausted in finite time or $u \rightarrow 0$ and some of the resource is left totally unused for ever.

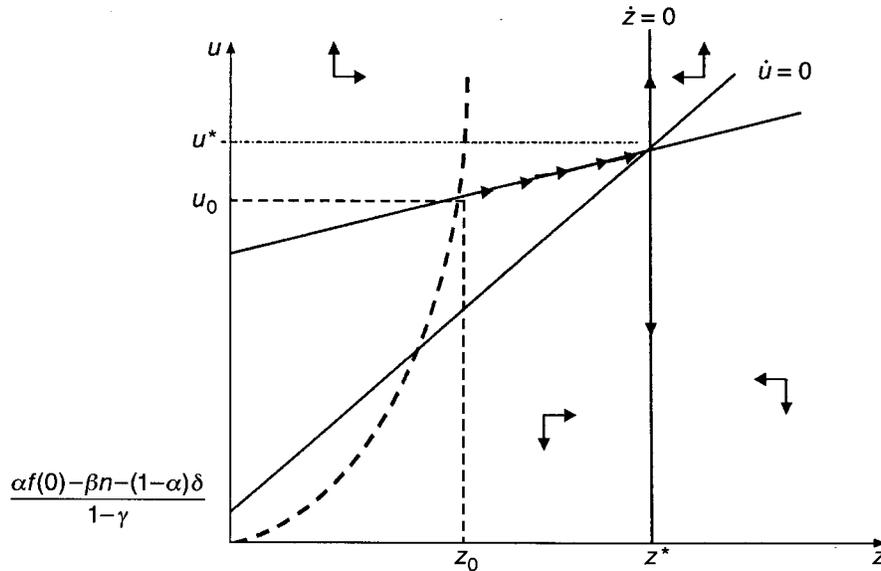


Fig. 2. Dynamics of z and u .

The slopes of the $\dot{x} = 0$ and the $\dot{z} = 0$ lines in Fig. 1 may be either positive or negative, depending on the parameters; in the case shown, α is (slightly) above 1 so that the slope of the $\dot{z} = 0$ line is positive (though less than 1) and ε is sufficiently above α so that the $\dot{x} = 0$ line is steeper than the $\dot{z} = 0$ line (which, when $\alpha > 1$, is required for stability of the steady state). The case shown in Fig. 2 is one of an initially ‘resource-poor’ economy since $B(0)z^{*1/\gamma} > u^*$. (In the opposite case, that of a ‘resource-rich’ economy, the stippled curve would cross the saddle path to the right of the steady-state point.)

In view of the boundary value condition arising from (49), we shall define the original three-dimensional dynamical system as ‘saddle-point stable’ if and only if its Jacobian has one negative eigenvalue and two positive eigenvalues (or two eigenvalues with a positive real part). On the other hand, we shall call a dynamical system totally unstable if all the eigenvalues are positive or have positive real parts.

Proposition 4 The dynamics of the model (1)–(5) and (14) is described by the differential equations (42), (43), and (45). A steady state (z^*, x^*, u^*) is saddle-point stable if and only if $D(\equiv 1 - \alpha + (\varepsilon - 1)\gamma) > 0$, and it is totally unstable if $D < 0$.

Proof See Appendix. □

Corollary A saddle-point stable steady state has $g_c^* > 0$ if and only if

$$\alpha + \beta > 1 \quad \text{and} \quad n > \frac{\gamma\rho}{\alpha + \beta + \gamma - 1} \tag{50}$$

Proof By the proposition, a saddle-point stable steady state has $D > 0$. Now use the corollary to Proposition 3. \square

Together, the Propositions 2 and 4 imply the existence of a stable steady state even outside the range of parameter values examined in Stiglitz (1974).²⁰ Indeed, Figures 1 and 2 illustrate that the existence of this stable steady state is compatible with values of α above unity, so that there can be increasing returns to scale with respect to capital. Further, the implication of the corollary to Proposition 4 is that stable optimal *per capita* growth calls for two features to be simultaneously present: increasing returns to capital and labour taken together and a sufficient amount of population growth.²¹ On the other hand, if $D < 0$ there exist parameter constellations such that steady *per capita* growth is both technically feasible and satisfies the first order and transversality conditions (arising from utility discounting and consumption smoothing), but nevertheless the steady state is not stable: After a disturbance in one of the predetermined variables, K , N , or S , it is not optimal for (z, x, u) to move back to (z^*, x^*, u^*) . The next section gives an example of this.

Before considering this example, we may relate the above results to the ongoing discussion on endogenous growth. In the introduction we defined (strictly) endogenous growth as growth in *per capita* consumption in the long run at a positive constant rate, even in the absence of *any* exogenously growing factor. Likewise, weakly endogenous or semi-endogenous growth (as introduced by Groth, 1992, and Jones, 1995) was defined as growth in *per capita* consumption in the long run at a positive rate, even without any exogenously given technology growth. With these definitions, Proposition 4 and its corollary have the following interesting implications:

Main result For a Cobb-Douglas one-sector optimal growth model with non-renewable natural resources, the following holds: (a) a stable steady state with (strictly) endogenous growth does not exist, not even as a knife-edge case; (b) a

²⁰ As mentioned in footnote 13, Stiglitz has in his analysis of technically feasible growth paths a short remark on the case $\alpha + \beta + \gamma > 1$, but the CRS assumption ($\alpha + \beta + \gamma = 1$) is maintained throughout his analysis of existence and stability of optimal growth.

²¹ One of the models examined in Aghion and Howitt (1998, pp.162–3) is an AK model with a non-renewable resource ($\alpha = 1, \gamma > 0$). In that model, however, labour does not appear in the production function ($\beta = 0$). Because of this specification, positive long-run growth is not possible. By introducing population growth and an explicit productive role for labour, the present model reverses this result (without violating stability).

Aghion and Howitt carry on with another model based on what they call a Schumpeterian approach to non-renewable resources (Aghion and Howitt 1998, pp.163–4). Here the economy has two sectors, a manufacturing sector and an R&D sector, both with constant returns to scale to producible inputs. The authors find that now growth is possible (without population growth) and they ascribe this to the Schumpeterian approach as distinct from the AK approach. In our view, the key is rather that the ‘growth engine’ (the R&D sector) is modelled without any dependence on the resource (not even indirectly since the R&D sector does not use capital).

stable steady state²² with semi-endogenous growth exists if and only if (i) $\alpha + \gamma < 1 + \varepsilon\gamma$ (i.e. $D > 0$); and (ii) there is simultaneously increasing returns to capital and labour taken together *and* enough population growth. Numerical example: $\alpha = 0.90, \beta = 0.25, \gamma = 0.02, \rho = 0.01, n = 0.01, \varepsilon = 2.00$, and δ arbitrary; then the steady state is stable and has $g_c^* = 0.013$; and (c) semi-endogenous growth features the property (similar to that of strictly endogenous growth) that the long-run *per capita* growth rate depends on technology as well as preference parameters.²³

This last observation, which follows from (36), allows a potential for tax and subsidy policies to influence not only the ‘level’ along which growth occurs, but also the long-run growth rate in a market economy.²⁴ This is in contrast to conventional semi-endogenous growth models (e.g. those surveyed in Eicher and Turnovsky 1999) where policy has only level effects. This difference derives from the fact that in these models non-renewable resources do not enter the growth engine.

4. The Case of no population growth

To get a more concrete picture of the character of steady states that are unstable, let us study the special case of no population growth. Unless otherwise indicated, when speaking of a ‘steady state’ we still understand a steady state satisfying the first order and transversality conditions.

Proposition 5 Assume $n = 0$. Then: (a) in view of $\rho > 0$, there can be no steady state with $g_c^* = 0$. When $D(\equiv 1 - \alpha + (\varepsilon - 1)\gamma) = 0$, there exists no steady state at all; (b) when $D \neq 0$, there exists a steady state with $g_c^* > 0$ if and only if $\alpha > 1$, $D < 0$, and $\delta < -(1 - \gamma)\rho/D$; and (c) a steady state with $g_c^* > 0$ is totally unstable.

Proof See Appendix.

We already know from Proposition 1 that when $n = 0$, the mere technical feasibility of sustained growth requires the strong assumption of increasing returns to capital ($\alpha > 1$). But as Proposition 5 reminds us, the realization of this potential growth depends on preferences. The aversion towards consumption variation (that is the elasticity of marginal substitution, ε) must be not too large (since $D < 0$ requires $\varepsilon < 1 + (\alpha - 1)/\gamma$). And, given the positive rate of time preference, ρ , the drain of investment represented by capital depreciation must not be too large.

²² As to optimality of the steady state fulfilling the first order and transversality conditions, the proviso mentioned in Remark 2 to Proposition 2 should be kept in mind.

²³ Stiglitz (1974) shows this feature to hold in the case where $\alpha + \beta + \gamma = 1$ with exogenous technical progress.

²⁴ In Groth and Schou (2001) the potential for economic policy are explored in detail.

The requirement of increasing returns to capital is due to the fact that $\alpha > 1$ is needed to make the Keynes-Ramsey Rule support sustained growth when $n = 0$. If, on the contrary, $\alpha \leq 1$, then the marginal product of capital tends to zero as capital accumulates, labour input stays constant, and resource use diminishes.

In any case, due to its lack of stability, the steady state with growth is not very appealing. Its realization requires appropriate initial conditions to be satisfied. Indeed, for any given K_0 and N_0 , S_0 should satisfy the boundary value condition

$$K_0^{\alpha-1} N_0^\beta S_0^\gamma = z^* u^{*\gamma} \tag{51}$$

from (49). Consider the following thought experiment where, for ease of exposition, we let $\delta = 0$ (in addition to $n = 0$). Suppose, that by some exceptional coincidence the economy has been in the steady state (z^*, x^*, u^*) until time t_1 , that is, in (z, u) space, the point (z^*, u^*) in Fig. 3. In the corresponding phase diagram for z and x (which we don't show), the steady state point (z^*, x^*) is a source (totally unstable). Figure 3 presupposes $z = z^*$ and $x = x^*$ for all $t < t_1$. Then an unforeseen shift upwards in the rate of time preference ρ occurs, implying that z^* , x^* , and u^* shift upwards in the same proportion, as shown by (33), (34), and (35) with $n = \delta = 0$, i.e. the shift would be along a line like OA in Fig. 3. Imagine for a moment that the jump variables z and x immediately shift to their new steady state values $z^{*'}$ and $x^{*'}$. The arrows in Fig. 3 show the direction of movement of u after t_1 according to the differential equation $\dot{u} = (-\alpha x^{*'} / (1 - \gamma) + u)u$ which is the same as (45) when $n = 0 = \delta$ and $(z, x) = (z^{*'}, x^{*'})$. This implies a vertical movement upwards starting at the point B in Fig. 3 (B is the point where the stippled curve representing the boundary value condition (51), with z^* and x^* replaced by z and x , respectively, crosses the vertical line $z = z^{*'}$). The increasing extraction rate

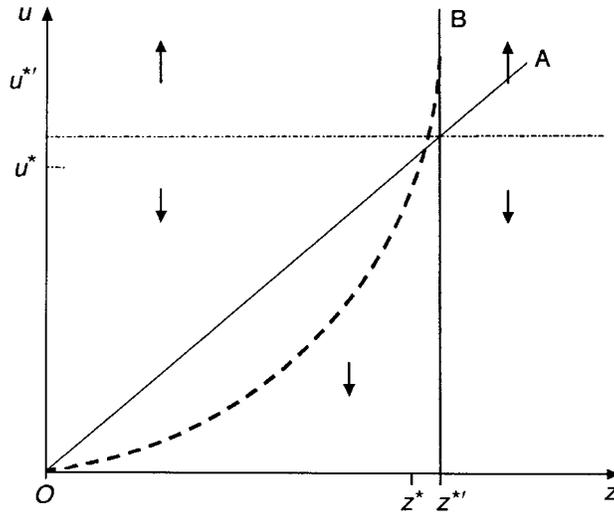


Fig. 3. Dynamics of u in the unstable case $n = 0$ (and $\delta = 0$).

u will cause the resource stock to be exhausted in finite time which cannot be optimal. In the opposite case where at time t_1 the rate of time preference ρ shifts downwards, the point B would be below the new steady state point $(z^{*'}, u^{*'})$ so that the resulting movement of u will be downwards leaving some of the resource stock totally unused forever. Therefore, the provisional assumption that at time t_1 , z , and x shift to their new steady state values is refused; after a disturbance the economy does not return to the (new) steady state.

The lesson to be learnt from this is that not only does a desire for consumption smoothing combined with utility discounting restrict the viability of sustained growth; it also excludes stability of such paths when $n = 0$.

We conclude that when non-renewable resources enter the growth engine in an essential way strictly endogenous stable growth does not exist, not even as a knife-edge case. One might alternatively say that a knife-edge reappears in a different and even more precarious form: Steady growth without the exogenous stimulus of population growth requires not only the strong assumption of $\alpha > 1$, but also that the initial conditions exactly 'match' the steady state conditions as shown by (51).²⁵

A further problem is the well-known fact that when there is instability, comparative statics do not give fruitful results. Indeed, in the present case we get paradoxical results such as: More impatience leads to higher consumption growth, a higher elasticity of marginal utility leads to larger differences in the consumption level over time, etc.²⁶ Due to the instability of the steady state with growth, such comparative statics are not useful.

5. Conclusion

We have studied a Cobb-Douglas one-sector optimal growth model with non-renewable resources and no exogenous technical progress. A stable steady state exists for a larger range of parameter values than examined in Stiglitz (1974). Indeed, the existence and stability of a steady state is shown to be compatible with increasing returns to the reproducible factor, capital. A high output elasticity of capital need not lead to explosive growth paths because the increasing need to use the natural resource sparingly has a counter-balancing effect. However, population

²⁵ The implication we want to draw is not that absence of population growth necessarily makes an economy unstable, but rather that one should not expect continuing exponential growth if the real world in the long run is characterized by (i) an elasticity of substitution between non-renewable resources and other inputs not above one; (ii) a strictly positive rate of time preference, and (iii) a non-increasing population and no exogenous technical progress.

²⁶ The same kind of 'pathological' behaviour is implied by the Chiarella model (Chiarella, 1980), mentioned in the introduction. The paper is an early endogenous growth contribution with an R&D sector which indirectly depends on inputs of a non-renewable resource. Having no population growth, the model can be seen as an extension of the case considered here. The dynamics of the model is, however, complex and its analysis not complete. Seemingly, the steady state is unstable though the author claims the contrary.

growth is a necessary condition for stable sustained *per capita* growth. This means that the extended Stiglitz optimal growth model presented in this paper does not underpin endogenous growth in the strict sense of the word, i.e. the model cannot generate positive and stable *per capita* growth without relying on population growth.

Therefore, the challenge of generating a generic model with strictly endogenous growth is not met by simply including non-renewable resources as necessary inputs in production. On the contrary, in this setting the knife-edge problem of endogenous growth is turned into a stability problem unless there is population growth. Hence, to the extent that the presence of such indispensable natural resources in a macro production function is considered realistic, the analysis has identified a new problem for the concept of strictly endogenous growth: It does not seem possible to have stable, strictly endogenous growth when non-renewable resources enter the growth engine in an essential way. This also applies, then, to market economies if they are considered to be replica, through tax and subsidy interventions, of social optima.

It is a possibility (but we think an unlikely one) that this negative conclusion could change in a multi-sector model where the reproducible asset(s) generating growth would depend on the resource (possibly indirectly). (The multi-sector endogenous growth models with non-renewable resources mentioned in the introduction all share the feature that the growth engine is not dependent on the resource).²⁷ This problem is left for future research. Similarly, it would be interesting to extend the analysis to cover the case where capital is required in order to exploit the natural resource.

The more positive aspect of our analysis is its calling attention to *semi-endogenous* growth. The semi-endogenous framework requires a less extreme value of the share of producible inputs in production. We showed that with non-renewable resources entering the growth engine, existence and stability of optimal sustained per capita growth requires only increasing returns to capital and labour taken together—albeit combined with population growth above some minimum. In this setting semi-endogenous growth features the property that the long-run *per capita* growth rate depends on technology as well as preference parameters, a property which has often been seen as belonging exclusively to models of strictly endogenous growth.

Supplementary material

Supplementary material can be found at <http://www.oep.oupjournals.org/cgi/data/54/3/386/DC1/1>

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²⁷ Here we exclude Chiarella (1980), commented on in footnote 26 above.

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Appendix

Proof of Proposition 2

By definition of a steady state, z and x are constant; hence, $g_C = g_Y = g_K = g$, a constant. In addition, being a special case of a BGP, a steady state has g_S constant and $g_S = g_R < 0$, from Lemma 1. (b) Therefore, the conditions of Lemma 5 are satisfied by an optimal steady state, $(g^*, g_R^*, z^*, x^*, u^*)$, if one exists, and given $D \neq 0$, it equals $(\bar{g}, \bar{g}_R, \bar{z}, \bar{x}, \bar{u})$, hence it is unique. (a) By definition, a steady state requires \bar{u}, \bar{z} , and $\bar{x} > 0$. Hence, the ‘only if’ part of (a) follows from (29) (since $\bar{g}_S = \bar{g}_R < 0$) and (28) of Lemma 5, (28) giving $\bar{x} > 0$ only if (a.iii) holds. On the other hand, suppose $\alpha, \beta, \gamma, \varepsilon, \rho, n$, and δ are such that (a.i), (a.ii), and (a.iii) hold. Then, by the corollary to Lemma 5, $\bar{x} > 0$. We can now construct a BGP satisfying the first order conditions by determining \bar{g} from (27) and putting $g_C = g_Y = g_K = \bar{g}$ and $g_S = -\bar{u} = \bar{g}_R < 0$. By (b) and (c) of Lemma 5, the two transversality conditions also hold. This proves the ‘if’-part. Finally, to see that the set of allowed parameter values has a non-empty interior, let α, β , and γ be such that $\alpha + \beta + \gamma = 1$; then $\bar{z} > 0$, $D > 0$; in addition, let $0 < n < (1 - \alpha)\rho / [\varepsilon(1 - \alpha - \beta) + \beta]$; then $\bar{u} = -\bar{g}_R > 0$, and $\bar{x} > 0$ by the corollary to Lemma 5, since $\alpha < 1$. \square

Proof of Proposition 3

We already know from Proposition 2 that a steady state, fulfilling the first order and transversality conditions, exists if and only if (a.i), (a.ii), and (a.iii) of Proposition 2 are

satisfied. In such a steady state, by (36), $g_c^* > 0$ if and only if (37) holds. Finally, from (a) of Lemma 6 we see that (a.ii) of Proposition 2 is automatically satisfied when $g_c^* > 0$ since this implies $\bar{g} = g_c^* + n > 0$. \square

Proof of Proposition 4

The Jacobian of the dynamical system for $z, x,$ and $u,$ evaluated at the steady state, is

$$\bar{J} = \begin{bmatrix} \frac{\partial \dot{z}}{\partial z} & \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial u} \\ \frac{\partial \dot{x}}{\partial z} & \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial u} \\ \frac{\partial \dot{u}}{\partial z} & \frac{\partial \dot{u}}{\partial x} & \frac{\partial \dot{u}}{\partial u} \end{bmatrix} = \begin{bmatrix} (\alpha - 1)z^* & \frac{1 - \alpha - \gamma}{1 - \gamma} z^* & 0 \\ \left(\frac{\alpha}{\varepsilon} - 1\right)x^* & x^* & 0 \\ 0 & \frac{-\alpha}{1 - \gamma} u^* & u^* \end{bmatrix} \tag{A1}$$

The trace is

$$(\alpha - 1)z^* + x^* + u^* = 2u^* > 0 \tag{A2}$$

by (9), (27), and Proposition 2. The determinant, $\bar{\Delta}$, is

$$\bar{\Delta} = \alpha \frac{1 - \alpha + (\varepsilon - 1)\gamma}{(\gamma - 1)\varepsilon} z^* x^* u^* \equiv \alpha \frac{D}{(\gamma - 1)\varepsilon} z^* x^* u^* \tag{A3}$$

From (A1) we see that u^* is an eigenvalue, i.e. at least one eigenvalue is real and positive. If and only if $D > 0$, then $\bar{\Delta} < 0$. Hence, if and only if $D > 0$, there will be one negative eigenvalue and two positive real eigenvalues. In view of the boundary value condition (49) this implies saddle-point stability.

Assume, on the contrary, $D < 0$. Then $\bar{\Delta} > 0$, and on the face of it there are two possible cases: either all three eigenvalues are positive (or have positive real part) or there is one positive eigenvalue and two eigenvalues with non-positive real part. But this last-mentioned case can be excluded since the trace of the upper left 2×2 submatrix of \bar{J} equals $-g_R^* = u^* > 0$ as shown in Section 3.2. Hence, the system is totally unstable. \square

Proof of Proposition 5 (the case $n = 0$)

Assume $n = 0$. (a) If $D \neq 0$, then $g_c^* \neq 0$, from (36). If $D = 0$, then (25) and (26) imply non-existence of a steady state when $n = 0$. (b) The statement is a special case of Proposition 3. Indeed, (37) shows: $g_c^* > 0 \Leftrightarrow -\gamma\rho/D > 0 \Leftrightarrow D < 0$. When $D < 0$, by (32), $g_R^* < 0 \Leftrightarrow \alpha > 1$. Furthermore, for $D < 0$ and $\alpha > 1$, by (34), $x^* > 0 \Leftrightarrow \delta < -(1 - \gamma)g_R^*/(\alpha - 1) \Leftrightarrow \delta < (1 - \gamma)\rho/(-D)$ from (32). (c) From (b), a steady state with $g_c^* > 0$ presupposes $D < 0$. Then, by Proposition 4, the steady state is completely unstable. \square