1 Slutsky Matrix og Negative definiteness

This is exercise **2.F.10** from the book. Given the demand function x(p,w) from the book page 23, where $\beta = 1$ and $w = 1$, we shall :

1. Calculate the Slutsky matrix $S = D_p x(p,w) + D_w x(p,w) x(p,w)^T$
   - evaluate S at p = (1,1,1)
2. Show that x(p,w) does not fullfill the weak axiom.

Since we calculate the Slutsky Matrix and therefore changes in consumer income, we shall wait by inserting $w = 1$ assumption.

1.1 Solution

We start out by calculating $D_p x(p,w)$:

$$D_p x(p,w) = \begin{pmatrix} -\frac{p_2 w}{p_1^2 P} - \frac{p_2 w}{p_1 P^2} & \frac{w}{p_1 P} - \frac{p_2 w}{p_1 P^2} & -\frac{p_2 w}{p_1 P^2} \\ -\frac{p_3 w}{p_2 P^2} & -\frac{p_3 w}{p_2^2 P} - \frac{p_3 w}{p_2 P^2} & \frac{w}{p_2 P} - \frac{p_3 w}{p_2 P^2} \\ \frac{w}{p_3 P} - \frac{p_1 w}{p_3 P^2} & -\frac{p_1 w}{p_3 P^2} & -\frac{p_1 w}{p_3^2 P} - \frac{p_1 w}{p_3 P^2} \end{pmatrix} \tag{1}$$

Where $P \equiv (p_1 + p_2 + p_3)$. Now we have the raw effect on demand from changing prices. This effect includes income effects, and since we only want to consider substitution effects we have to compensate the consumer. Firstly we calculate the raw effects from changing income $D_w x(p,w)$:

$$D_w x(p,w) = \begin{pmatrix} \frac{p_2}{p_1 P} & \frac{p_3}{p_2 P} & \frac{p_1}{p_3 P} \end{pmatrix} \tag{2}$$

Then we calculate the full income effect as $D_w x(p,w) x(p,w)^T$, which is a 3×3 matrix.

$$D_w x(p,w) x(p,w)^T = \begin{pmatrix} \frac{p_2}{p_1 P} \\ \frac{p_3}{p_2 P} \\ \frac{p_1}{p_3 P} \end{pmatrix} \cdot \begin{pmatrix} \frac{p_2 w}{p_1 P} & \frac{p_3 w}{p_2 P} & \frac{p_1 w}{p_3 P} \end{pmatrix} \tag{3}$$

$$D_w x(p,w) x(p,w)^T = \begin{pmatrix} \frac{p_2^2 w}{p_1^2 P^2} & \frac{p_3 w}{p_1 P^2} & \frac{p_2 w}{p_3 P^2} \\ \frac{p_3 w}{p_1 P^2} & \frac{p_3^2 w}{p_2^2 P^2} & \frac{p_1 w}{p_2 P^2} \\ \frac{p_2 w}{p_3 P^2} & \frac{p_1 w}{p_2 P^2} & \frac{p_1^2 w}{p_3^2 P^2} \end{pmatrix} \tag{4}$$

So the isolated substitution effects is:

$$S = D_p x(p,w) + D_w x(p,w) x(p,w)^T = \begin{pmatrix} -\frac{p_2 w(2p_1+p3)}{p_1^2 P^2} & \frac{w(p_1+2p_3)}{p_1 P^2} & \frac{p_2 w(p_1-p_3)}{p_1 p_3 P^2} \\ \frac{p_3 w(p_2-p_1)}{p_1 p_2 P^2} & -\frac{p_3 w(p_1+2p_2)}{p_2^2 P^2} & \frac{w(2p_1+p_2)}{p_2 P^2} \\ \frac{w(2p_2+p_3)}{p_3 P^2} & -\frac{p_1 w(p2-p3)}{p_2 p_3 P^2} & -\frac{p_1 w(p_2+2p_3)}{p_3^2 P^2} \end{pmatrix} \tag{5}$$

When evaluating the Slutsky matrix in $p = (1,1,1)$ and $w = 1$, one gets:

$$S(1,1,1) = \frac{1}{3} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \tag{6}$$

This Matrix does not have full rank. Take as an example (- column vector 1 - column vector 3) = column vector 2. Also $p \cdot Sp = 0$, when p = (1,1,1). But this actually applies for all p.

We now examine if S is negative semidefinite $\forall v, v \cdot Sv \leq 0$. This is a necessary condition for x(p,w) fullfilling the weak axiom. Let $p = (1,1,\epsilon), \epsilon > 0$, and insert this subset of price vectors in S:

$$S(1,1,\epsilon) = \begin{pmatrix} -\frac{1}{2+\epsilon} & \frac{1+2\epsilon}{(2+\epsilon)^2} & \frac{1-\epsilon}{(2+\epsilon)^2\epsilon} \\ 0 & -\frac{3\epsilon}{(2+\epsilon)^2} & \frac{3}{(2+\epsilon)^2\epsilon} \\ \frac{1}{(2+\epsilon)\epsilon} & \frac{\epsilon-1}{(2+\epsilon)^2\epsilon} & -\frac{1+2\epsilon}{(2+\epsilon)^2\epsilon^2} \end{pmatrix} \tag{7}$$

We search for ϵ´s and vectors v where $S(1,1,\epsilon)$ is not negativ semidefinite (that is positive definite). We now apply **2.F.3**, which says that if x(p,w) fullfills WL and is homogenous of degree of 0, then $p \cdot S(p) = 0$ and $S(p)p = 0 \ \forall \ (p,w)$. From exercise 1, set 2 we know that x(p,w) fullfills WL and is homogenous of degree of 0. Therefore we can apply **M.D.4** page 939.

Theorem M.D.4: If $Sp = 0 \wedge p \cdot S = 0$ and a reduced matrix $\tilde{S}$ is negative definit, then S is negative definit for all vectors in the subspace $T_z = \{z | z \cdot p = 0\}$. We therefore examine the reduced matrix $\tilde{S}$ given by removing one row and one column:

$$\tilde{S}(1,1,\epsilon) = \begin{pmatrix} -\frac{1}{2+\epsilon} & \frac{1+2\epsilon}{(2+\epsilon)^2} \\ 0 & -\frac{3\epsilon}{(2+\epsilon)^2} \end{pmatrix} \tag{8}$$

We choose an arbitrary vector $\tilde{v} \in \mathbb{R}^2$ and calculate:

$$\tilde{v} \cdot \tilde{S}\tilde{v} = \frac{1+2\epsilon}{(2+\epsilon)^2}\tilde{v}_1\tilde{v}_2 - \frac{(2+\epsilon)}{(2+\epsilon)^2}\tilde{v}_1^2 - \frac{(3\epsilon)}{(2+\epsilon)^2}\tilde{v}_2^2 \tag{9}$$

Especially we search for vectors which ensures $\tilde{v} \cdot \tilde{S}\tilde{v} > 0$. Since $\epsilon > 0$ we can remove $(2+\epsilon)^2$ and reduce the expression:

$$\tilde{v} \cdot \tilde{S}\tilde{v} > 0 \Rightarrow (\tilde{v}_1\tilde{v}_2 - 2\tilde{v}_1^2) > \epsilon(\tilde{v}_1^2 + 3\tilde{v}_2^2 - 2\tilde{v}_1\tilde{v}_2) \tag{10}$$

The expression on the right hand side next to ϵ is allways positive, since:

$$(\tilde{v}_1^2 + 3\tilde{v}_2^2 - 2\tilde{v}_1\tilde{v}_2) = (\tilde{v}_1 - \tilde{v}_2)^2 + 2\tilde{v}_2^2 > 0 \tag{11}$$

Therefore we can convert the expression to:

$$\frac{(\tilde{v}_1\tilde{v}_2 - 2\tilde{v}_1^2)}{(\tilde{v}_1^2 + 3\tilde{v}_2^2 - 2\tilde{v}_1\tilde{v}_2)} > \epsilon > 0 \tag{12}$$

Basically we achieve our goal if $\tilde{v}_2 > 2 \cdot \tilde{v}_1$. Thus if we choose $\tilde{v} = (1,4)$ we will ensure that $\tilde{v} \cdot \tilde{S}\tilde{v} > 0$ for some ϵ. By insertion of $\tilde{v}$ we get:

$$\frac{(1 \cdot 4 - 2 \cdot 1^2)}{(1^2 + 3 \cdot 4^2 - 2 \cdot 1 \cdot 4)} = \frac{2}{41} > \epsilon > 0 \tag{13}$$

Thus $\tilde{S}$ is not negative definite for all vectors, thus S cannot be either. Renaming our v vector to Δp, we have found that:

$$\exists\ \Delta p \in \mathbb{R}^3;\ \ \Delta p = (1,4,0) \Rightarrow \Delta p \cdot S\Delta p > 0, \quad \text{when } p = (1,1,2/41 - \rho) \tag{14}$$

Thus there exists price changes Δp and prices p which makes S postive definite, and x(p,w) cannot fullfill the weak axiom. Q.E.D.

2 Strange demand changes

In execise 2.F.16 from the book we are given the following demand function:

$$x(p,w) = \begin{pmatrix} \frac{p_2}{p_3} \\ -\frac{p_1}{p_3} \\ \frac{w}{p_3} \end{pmatrix} \tag{15}$$

That is, the consumer demands positively good 1 and 3, but delivers good 2. Also the demand for the first two good does not vary with its own price or the consumers overall income w. We could think of good 1 as special consumption good which our consumer basically demands according to his real wage salery, namely p_2 p_3. Good 2 is then labour supply which is supplied according to consumer price p_1. Only good 3 seems to be a normal good, and could be thought of as some kind of investment good here bought from savings w.

The demand could be explained like this: Assume the worker know, at which prices he can buy good 1 for in the next period, but not what his working salary is. It could be that the payoff from work is uncertain. However, he chooses his labour supply, so that if prices on good 1 are high, then he will work more. When the work is completed, the project pays of at some price p_2, and all work income is spend. Basically this story involve prices being a signal to the consumer, thus changing the prices, changes the the signal and the behaivior even if the consumer are compensated through w. In the exercise we are asked to:

1. show that $x(p,w)$ is homogenous of degree of 0. and fullfills Walras Law.
2. show that $x(p,w)$ does not fullfill the weak axiom
3. show that the Slutsky matrix S fullfills $v \cdot Sv = 0 \;\; \forall \;\; v \in \mathbb{R}^3$.

2.1 Proof - Homogenity of degree 0. and Walras Law.

From insertion of the real number λ one gets:

$$x(\lambda p, \lambda w) = \begin{pmatrix} \frac{\lambda p_2}{\lambda p_3} \\ -\frac{\lambda p_1}{\lambda p_3} \\ \frac{\lambda w}{\lambda p_3} \end{pmatrix} = x(p,w) \tag{16}$$

Thus x(p,w) is homogenous of degree 0. Also x(p,w) satisfies Walras low since:

$$p \cdot x(p,w) = \frac{p_1 p_2}{p_3} - \frac{p_2 p_1}{p_3} + \frac{p_3 w}{p_3} = w \tag{17}$$

Both properties follows directly. Q.E.D.

2.2 Proof - Violation of the weak axiom

Next we show that x(p,w) does not satisfy the weak axiom. This is due to fact that w is not used in good 1 and 2. Let:

$$p = (1,1,1) \;\; \wedge \;\; w = 1 \;\; \Rightarrow \;\; x = (1,-1,1) \tag{18}$$

Now lets change price of good 1, and compensate our consumer, so he still can afford x in this new situation:

$$\tilde{p} = (2,1,1) \;\; \wedge \;\; \tilde{w} = \tilde{p} \cdot x = 2 \;\; \Rightarrow \;\; \tilde{x} = (1,-2,2) \tag{19}$$

So $\tilde{x}$ is revealed preferred to x, and x should not be revealed preferred $\tilde{x}$, which is basically the same as $\tilde{x}$ is not affordable in situation 1. Unfortunately that is indeed the case:

$$p = (1, 1, 1) \;\; \Rightarrow \;\; p \cdot \tilde{x} = 1 = w \tag{20}$$

Thus the weak axiom is not fullfilled. The problem is, that we use savings for compensating price changes, which cannot be remedied through increased savings. Q.E.D.

2.3 The Slutsky Matrix

Finally we deduct the Slutsky matrix. First demand changes from price changes.

$$D_p x(p, w) = \begin{pmatrix} 0 & \frac{1}{p_3} & -\frac{p_2}{p_3^2} \\ -\frac{1}{p3} & 0 & \frac{p1}{p_3^2} \\ 0 & 0 & -\frac{w}{p_3^2} \end{pmatrix} \tag{21}$$

Next the compensated income effect

$$D_w x(p, w) \cdot x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{p_2}{p_3^2} & -\frac{p_1}{p_3^2} & \frac{w}{p_3^2} \end{pmatrix} \tag{22}$$

Note that only good 3 is affected by our income changes. Finally we calculate the substitution effects shown in the Slutsky matrix S:

$$S = D_p x(p, w) + D_w x(p, w) x(x, p)^T = \begin{pmatrix} 0 & \frac{1}{p_3} & -\frac{p_2}{p_3^2} \\ -\frac{1}{p3} & 0 & \frac{p1}{p_3^2} \\ \frac{p_2}{p_3^2} & -\frac{p_1}{p_3^2} & 0 \end{pmatrix} \tag{23}$$

Choose a vector $v \in \mathbb{R}^3$ and calculat:

$$v \cdot Sv = \frac{v_1 v_2}{p3} - \frac{p2}{p3}\frac{v_3 v_1}{p_3} + \frac{p1}{p3}\frac{v_3 v_2}{p_3} - \frac{v_1 v_2}{p3} + \frac{p2}{p3}\frac{v_3 v_1}{p_3} - \frac{p1}{p3}\frac{v_3 v_2}{p_3} = 0 \tag{24}$$

So each pair in the sum negates each other. So 1. term negates 4. term and so on. So $v \cdot Sv = 0$ for arbitrary vectors v. Q.E.D.

Both exercises demonstrates an application of the negative definiteness of S. Firstly, S being negative semidefinite, is an necessary condition for the weak axiom. On the other hand, S can be negative semidefinite, even if the demand does not fullfill the weak axiom. Thus there is no biimplication in **proposition 2.F.2**.