# Price Reaction to Information with Heterogeneous Beliefs and Wealth Effects: Underreaction, Momentum, and Reversal*

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**Abstract**

This paper analyzes how asset prices in a binary market react to information when traders have heterogeneous prior beliefs. We show that the competitive equilibrium price underreacts to information when there is a bound to the amount of money traders are allowed to invest. Underreaction is more pronounced when prior beliefs are more heterogeneous. Even in the absence of exogenous bounds on the amount traders can invest, prices underreact to information provided that traders become less risk averse as their wealth increases. In a dynamic setting, underreaction results in initial momentum and then reversal in the long run.

*Keywords:* Aggregation of heterogeneous beliefs, Price reaction to information, Wealth effects.

*JEL Classification:* D82 (Asymmetric and Private Information), D83 (Search; Learning; Information and Knowledge), D84 (Expectations; Speculations).

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This paper investigates how asset prices relate to the beliefs of traders in financial markets. Our analysis uncovers a novel theoretical mechanism through which prices initially underreact to information under the realistic assumption that traders have heterogeneous beliefs and are subject to wealth effects. This result provides a simple explanation of pricing patterns that are widely documented in asset markets. Underreaction to information is consistent with post-earning announcement drift and stock price momentum in the short run. In addition, the same mechanism that leads to initial underreaction and momentum also explains reversal in the long run.

We formulate our results in a trading model for a binary event. Traders can take positions in two Arrow-Debreu contingent assets, each paying one dollar if the corresponding outcome occurs. Our underreaction result hinges on three characteristics of asset markets:

- Traders have heterogeneous *prior beliefs*, given their limited experience with the underlying event contingent on which the asset pays.[1] These initial opinions are subjective and thus are uncorrelated with the realization of the outcome.[2] Having different prior beliefs, traders gain from trading actively.

- Traders have access to public *information* (such as an earnings announcement) about the eventual realization of the outcome on which the market is liquidated. Information has an objective nature because it is correlated with the outcome.[3]

- Traders exhibit *wealth effects*, which can take one of two forms. Initially, we develop the intuition for underreaction in a simple setting in which traders are risk neutral but are exogenously bounded by their limited wealth. We then turn to a more standard setting with risk averse traders who endogenously limit their positions on the risky assets, and show that underreaction results when wealthier traders are willing to take on more risk.

[1] We thus depart from the common prior assumption associated to the so-called Harsanyi doctrine. We refer to Morris (1995b) for a discussion of the assumption of heterogeneous priors. See also Blume and Easley's (1998) survey of rational learning for well-behaved examples where there is no convergence to common beliefs.

[2] For the purpose of our analysis, traders' subjective prior beliefs play the role of exogenous parameters, akin to the role played by preferences. As in most work on heterogeneous priors, prior beliefs are given exogenously in our model. We refer to Brunnermeier and Parker (2005) for a model in which heterogeneous prior beliefs arise endogenously.

[3] This conceptual distinction between prior beliefs and information is standard—as Aumann (1976) notes, "reconciling subjective probabilities makes sense if it is a question of implicitly exchanging information, but not if we are talking about 'innate' differences in priors."

To sharpen our result, we assume that all traders interpret information in the same way, so that beliefs are *concordant* in Milgrom and Stokey's (1982) terminology. The heterogeneity of traders' posterior beliefs is thus uniquely due to the fixed amount of heterogeneity in their prior beliefs. How does the market price aggregate the traders' posterior beliefs? How does the equilibrium price react to information that becomes publicly available to all traders? We address these questions through a comparative statics analysis of how the market price depends on changes in information.

Our main contribution is the observation that the market price systematically underreacts to information, rather than behaving like a posterior belief. Initially, we focus on the case in which each trader's endowment is constant with respect to the outcome realization, so that trade is only motivated by differences in prior beliefs.

To understand the mechanism driving underreaction in a static setting, consider a hypothetical market based on which team, Italy or Denmark, will win a soccer game. Suppose that those traders who are subjectively more optimistic about Italy winning live further south. We begin in Section 1 by presenting the first incarnation of the result in a model with risk neutral traders and bounded wealth. In equilibrium, traders living south of a certain threshold latitude invest all their wealth in the asset that pays if Italy wins; likewise, traders north of the threshold latitude invest all their wealth in the Denmark asset (Proposition 1).

Now, what happens when traders observe information (such as a player injury) more in favor of Italy winning? This information causes the price of the Italy asset to be higher, while contemporaneously reducing the price of the Denmark asset, compared to the case with less favorable information. As a result, the southern traders (who are optimistic about Italy) are able to buy fewer Italy assets, which are now more expensive.[4] Similarly, the northern traders can afford, and thus demand, more Denmark assets, now cheaper. Hence, the market would have an excess supply of the Italy asset and excess demand for the Denmark asset. For the market to equilibrate, some northern traders must turn to the Italian side. In summary, when information more favorable to an outcome is available, the marginal trader who determines the price has a prior belief that is *less* favorable to that outcome. Through this countervailing adjustment, the heterogeneity in priors dampens

[4]This wealth effect is the equivalent loss in income suffered by an individual when a change in prices implies that the desired trade becomes more expensive.

the effect of information on the price.

This underreaction result (Proposition 2) amends the common interpretation that the price of an Arrow-Debreu asset represents the belief held by the market about the probability of the event. The reason why the price does not behave like a posterior belief is that there is no constant "market prior" belief for which the equilibrium price is the Bayesian posterior update that incorporates the available information. Instead, the marginal trader's prior changes in the direction opposite to information, and the more so the more heterogeneous beliefs are (Proposition 3). Underreaction is consistent with evidence from asset markets, as well as with the widespread observation of the favorite-longshot bias in betting and prediction markets, whereby prices of favorites underestimate the corresponding empirical probabilities, while prices of longshots overestimate them (Section 1.2 and Corollary 1).[5]

For the second step of our analysis, in Section 2 we turn to a more traditional asset market model with risk averse traders. We initially focus on the special case with homogeneous endowments across events. After characterizing the unique equilibrium in Proposition 4, Proposition 5 verifies that equilibrium prices react one-for-one to information, like posterior beliefs, if traders have Constant Absolute Risk Aversion (CARA) preferences. Proposition 6 establishes that underreaction holds under the empirically plausible assumption that traders have Decreasing Absolute Risk Aversion (DARA), *even* when no exogenous bound is imposed on the traders' wealth. The logic is the same as in our baseline model. When favorable information is revealed, traders who take long positions on the asset that now becomes more expensive suffer a negative wealth effect. Hence these traders become more risk averse and cut back their positions.

Our analysis combines elements of the "average investor" view with the "marginal investor" view à la Ali (1977) and Miller (1977), a view with a lineage that Mayshar (1983) traces back to John Maynard Keynes, John Burr Williams, and James Tobin. The average investor view prevails in the absence of wealth effect, given that heterogeneous beliefs can be aggregated under CARA, as shown by Wilson (1968) and Lintner (1969).[6] The marginal investor view prevails when heterogeneous beliefs are combined with wealth

[5] In addition, our testable prediction that underreaction is more pronounced when trader beliefs are more heterogeneous seems to be borne out by the data; see Section 1.3.

[6] The case with CARA preferences and heterogeneous priors is also analyzed by Varian (1989) in a generalization of Grossman (1976). (In their models, the price is also a vehicle through which information becomes public to all traders.)

effects. Under DARA, we show that wealth effects not only inhibit aggregation, but systematically generate underreaction to information because the price assigns an increased weight to traders with beliefs that are contrary to the realized information.

Heterogeneity in beliefs is essential to obtain underreaction and cannot merely be replaced by heterogeneity in endowments across traders. When beliefs are common, heterogeneity in endowments permits demand aggregation for a class of preferences with wealth effects, Hyperbolic Absolute Risk Aversion (HARA) with common cautiousness parameter (Gorman, 1953, and Rubinstein, 1974). In this case, more extreme information induces all traders, buyers as well as sellers, to take more extreme positions; under the HARA condition positions adjust in a balanced way, and the price reacts to information as a Bayesian posterior belief. However, this knife-edge result is again upset in the direction of underreaction in the more general (and relevant) case which combines heterogeneous priors with heterogeneous endowments. Proposition 7 establishes that underreaction holds if traders exhibit DARA as well as HARA with common positive cautiousness parameter, and if subjective prior beliefs are independent of individual endowment and preference parameters.[7]

For our third step, in Section 3 we turn to the correlation pattern of price changes over time in a dynamic extension of the model with new information arriving each period, as in Milgrom and Stokey (1982). After characterizing the equilibrium (Proposition 8), we find that underreaction entails two dynamic price patterns when our setting is ex ante symmetric with respect to the two events:[8]

- The first-round underreaction is immediately followed by price momentum (Proposition 9). Intuitively, the arrival of additional information over time partly undoes the initial underreaction. This first result is consistent with the observation of momentum—a long-standing puzzle documented by a large empirical literature in finance (for example see Jagadeesh and Titman, 1993, Bernard and Thomas, 1989, and Moskowitz, Ooi, and Pedersen, 2012).

- The initial underreaction implies a subsequent reversal (Proposition 10), given that

[7] On the optimal allocation of risk with heterogeneous prior beliefs and risk preferences, see also Gollier (2007) and references therein. To this literature we add the consideration of how information affects belief aggregation.

[8] Symmetry is a sufficient but not necessary assumption for our model to be consistent with underreaction and reversal.

the marginal trader has contrarian beliefs. Thus, long-term price changes are negatively correlated with medium-horizon price changes. This reversal is also consistent with empirical evidence (see DeBondt and Thaler, 1985, Fama and French, 1992, Lakonishok, Shleifer, and Vishny, 1994, and Asness, Moskowitz, and Pedersen, 2013).

Like Milgrom and Stokey (1982), our model allows traders to have arbitrary risk preferences, heterogeneous endowments, heterogeneous priors, and concordant information. To their well-known characterization of equilibrium, we add a comparative statics analysis of the first-round equilibrium price with respect to information as well as a characterization of the correlation of price changes over time. Our restriction to two events makes the analysis particularly tractable; we return to this point in Section 4.

While we maintain that all traders are rational and symmetrically informed, an alternative approach in the theoretical literature emphasizes the role (and pattern) of noise trading for obtaining deviations of market prices from fundamental values. To the extent that noise trade cannot be distinguished from informed trade, overreaction arises when risk-averse traders require a risk premium for absorbing noise trade.[9] In Serrano-Padial (2012), rational traders constrained by an auction mechanism can be unwilling to correct mispricing induced by naive traders, if overpricing occurs at lower values and underpricing at higher values. In a dynamic setting, Cespa and Vives (2012) obtain underreaction or overreaction depending on the opaqueness surrounding liquidation value and the predictability of noise traders. Instead, our pricing patterns are not driven by the exogenous process governing the dynamic arrival of noise traders.

Another strand of the literature allows traders to interpret the information incorrectly or differently, thus relaxing concordant beliefs. For example, Harris and Raviv (1993) assume that traders with common prior update beliefs to different extents in response to information, and obtain underreaction to information which contradicts earlier information. Barberis, Shleifer, and Vishny (1998) derive momentum by assuming that traders are mistaken about the correct information model, while Hong and Stein (1999) posit that information diffuses gradually and is initially understood only by some traders. Allen,

[9]Intuitively, a lower price in a noisy REE suggests the realization of lower demand by noise traders (or greater aggregate supply). Rational risk-averse traders can only be willing to take a larger position (which is necessary for the market to clear when the aggregate supply is high) if they expect the price to increase on average in the future—hence, the price must overreact to information in this noisy REE setting. See Vives (2008, page 121) for an analytical explanation along these lines.

Morris, and Shin (2006) consider short-lived traders with private information who forecast the next period average forecasts and so end up overweighting the common public information. Banerjee, Kaniel, and Kremer (2009) obtain momentum by assuming that traders do not recognize the information of other traders and thus do not react to the information contained in the equilibrium price.[10] In contrast, we obtain both short-term momentum and long-term reversal, even when all long-lived traders agree about the correct interpretation of information. Our results are driven by differential wealth effects across traders with different beliefs, an aspect that the previous literature seems to have disregarded.[11]

We collect the proofs of the main results in the Appendix. The relatively standard proof of Proposition 8 is in the Online Appendix.

# 1 Bounded Wealth Model

**Events.** Traders take positions on whether or not a binary event, $A$, is realized (e.g., the Democratic candidate wins the 2016 presidential election). There are two Arrow-Debreu assets corresponding to the two possible realizations: one asset pays out 1 unit of cash if event $A$ is realized and 0 otherwise, while the other asset pays out 1 cash unit if the complementary event $A^c$ is realized and 0 otherwise.[12]

**Wealth.** We assume that there is a continuum $I$ of competitive, risk-neutral traders.[13] Trader wealth in this market is bounded, as each trader $i$ initially holds a given safe endowment, the amount $w_{i0}$ of each asset. Traders exchange their assets with other traders in a competitive market. Traders are not allowed to hold a negative quantity of either asset. Thus, there is an endogenous upper bound on the number of asset units that each individual trader can purchase and eventually hold. Risk-neutral traders would gain from relaxing this exogenous bound.

[10] While we consider the arrival of information, they assume that dynamic price changes are driven by noise. They find that momentum is impossible with commonly known heterogeneous prior beliefs.

[11] A complementary approach in the literature seeks to explain asset pricing anomalies through agency problems in delegated portfolio management (see Shleifer and Vishny, 1997, and Vayanos and Woolley, 2013).

[12] Traders cannot affect the exogenously given event outcome. For an analysis of traders' incentives to manipulate the outcome see Ottaviani and Sørensen (2007), who disregard the wealth effect. Lieli and Nieto-Barthaburu (2009) extend the analysis to allow for the possibility of feedback, whereby a decision maker acts on the basis of the information revealed by the market.

[13] The results derived in this section immediately extend to the case of risk-loving traders, whose behavior is also to adopt an extreme asset position. We turn to risk-averse traders in Section 2.

**Priors.** Initially, trader $i$ has subjective prior belief $q_i$. For convenience, we normalize the aggregate endowment of each asset to 1. The initial distribution of assets over individuals is described by the cumulative distribution function $G$. Thus $G(q) \in [0,1]$ denotes the share of all assets initially held by individuals with subjective prior belief less than or equal to $q$. We assume that $G$ is continuous, and that $G$ is strictly increasing on the interval where $G \notin \{0,1\}$.[14]

**Information.** Before trading, all traders observe the realization of a public signal with likelihood ratio $L \in (0,\infty)$ for event $A$. By Bayes' rule the subjective posterior belief $\pi_i$ satisfies

$$\frac{\pi_i}{1-\pi_i} = \frac{q_i}{1-q_i} L. \tag{1}$$

Posterior beliefs are concordant, as Bayes' rule uses the same $L$ for every trader $i$. This setting amounts to assuming that the arrival of the public signal triggers the simultaneous dispersion of prior beliefs, and that this dispersion is stochastically independent of the realization of the public signal.[15]

**Equilibrium.** Competitive traders take asset prices as given. We normalize the sum of the two asset prices to one, and focus on the price $p$ of the asset paying in event $A$. Trader $i$ chooses a feasible asset position $(w_i(A), w_i(A^c))$ to maximize subjective expected value $\pi_i w_i(A) + (1-\pi_i) w_i(A^c)$. With Arrow-Debreu assets, $w_i(A), w_i(A^c)$ also denote the event-dependent cash payout. Markets clear when the aggregate demand for each asset precisely equals the aggregate endowment.[16]

## 1.1 Competitive Equilibrium

Solving the competitive demand problem of the risk-neutral traders is straightforward. Let the public information be realized with likelihood ratio $L$, and consider trader $i$ with posterior belief $\pi_i$ resulting via (1). Given market price $p$, the subjective expected return

[14] The assumption that the priors are continuously distributed is made to simplify the analysis, but is not essential for our underreaction result.

[15] Formally, let $\omega$ denote the payoff-relevant state and $s$ a payoff-irrelevant signal. The subjective belief of individual $i$ assigns joint density $f_i(s,\omega) = f(s|\omega) q_i(\omega)$. With binary $\omega \in \{A, A^c\}$, we have let $L(s) = f(s|A)/f(s|A^c)$.

[16] Note that the informational requirements for competitive equilibrium are very weak; e.g., see Morris (1995a). Submitting the individual demand in response to a price is a dominant strategy for each trader and does not require any knowledge about other traders.

on the asset that pays out in event $A$ is $\pi_i - p$, while the other asset's expected return is $(1-\pi_i) - (1-p) = p - \pi_i$. With the given bound on trades, risk-neutral demand thus satisfies the following: if $\pi_i > p$, trader $i$ exchanges the entire endowment of the $A^c$ asset into $(1-p)\, w_{i0}/p$ units of the $A$ asset. The final portfolio is then $(w_i(A), w_i(A^c)) = (w_{i0}/p, 0)$. Conversely, when $\pi_i < p$, the trader's final portfolio is $(w_i(A), w_i(A^c)) = (0, w_{i0}/(1-p))$. Finally, when $\pi_i = p$, the trader is indifferent over all feasible trades.

Aggregate demand for the $A$ asset is then given by $1/p$ times the cumulated wealth of traders with posterior belief above $p$. Markets clear when this equals the aggregate endowment, 1.

**Proposition 1** *The competitive equilibrium price, $p$, is the unique solution to the equation*

$$p = 1 - G\left(\frac{p}{(1-p)\, L + p}\right) \tag{2}$$

*and is a strictly increasing function of the information realization $L$.*

## 1.2 Underreaction to Information

Inverting Bayes' rule (1) after public information realization $L$, we can always interpret the price $p$ as the posterior belief of a hypothetical individual with initial belief $p/[(1-p)\, L + p]$. According to (2), this hypothetical individual is the marginal trader, and this initial belief might be interpreted as an aggregate of the heterogeneous subjective prior beliefs of the individual traders. However, this way of aggregating subjective priors cannot be separated from the realization of information. Our main result states that this initial belief of the marginal trader moves systematically against the public information available to traders.

This systematic change in the market prior against the information implies that the market price underreacts to information. Consider the inference of any outside observer with a fixed prior belief $q$. The observer's posterior probability, $\pi(L)$, for the event $A$ satisfies (1), or

$$\log \frac{\pi(L)}{1-\pi(L)} = \log \frac{q}{1-q} + \log L. \tag{3}$$

The expression on the left-hand side is the posterior log-likelihood ratio for event $A$, which clearly moves one-to-one with changes in $\log L$. Part (ii) of the following Proposition notes that the corresponding expression for the market price, $\log(p(L)/(1-p(L)))$ does not

possess this property, but rather moves less than one-for-one with the publicly observable $\log L$.

**Proposition 2** *Suppose that beliefs are truly heterogeneous, i.e., the distribution $G$ is non-degenerate. (i) The marginal trader moves opposite to the information, i.e., the implied ex ante market belief $p/\left[(1-p)L+p\right]$ is strictly decreasing in $L$. (ii) The market price underreacts to initial information: for any pair $L' > L$ we have*

$$\log L' - \log L > \log \frac{p(L')}{1-p(L')} - \log \frac{p(L)}{1-p(L)} > 0. \tag{4}$$

To understand the intuition for part (i), consider what happens when public information is more favorable to event $A$ (corresponding, say, to the Democratic candidate winning the election over the Republican candidate). Naturally, by (2) the price $p$ for asset $A$ is higher when $L$ is higher. The trading bound forces optimists (with high prior $q_i$) to purchase fewer units of asset $A$: the amount of $A$ assets which can be obtained through selling all the $A^c$ endowment is $(1-p)\,w_{i0}/p$, decreasing in $p$. If the marginal trader were unchanged at the higher price that results with higher $L$, there would be insufficient demand for the $A$ assets sold out by pessimists. To balance the market it is necessary that some traders who were buying the Republican asset before now change sides and put their money on the Democratic candidate. In the new equilibrium, the price must thus move traders from the pessimistic to the optimistic side. Hence, although the price, $p$, rises with the information, $L$, it rises more slowly than a posterior belief, because of this negative effect on the prior belief of the marginal trader.

The underreaction result hinges on the fact that the endogenous upper bound (equal to $w_{i0}/p$) on the individual position in asset $A$ is inversely related to its price.[17]

**Application to Prediction Markets.** Our assumptions of bounded wealth at risk and equal endowments across event realizations are particularly descriptive in the context of prediction markets. Prediction markets are trading mechanisms that target unique events, such as the outcome of a presidential election or the identity of the winner in a sport

[17] Underreaction would not appear if instead there were a price-independent cap on the number of assets that each trader can buy. Then a constant set of optimists (or pessimists) would buy the full allowance of the $A$ (or $A^c$) asset. The marginal trader would then be constant and there would be no underreaction.

contest.[18] Because the realized outcomes are observed, these simple markets are useful laboratories for testing asset pricing theories.

According to an institutional feature of prediction markets, individuals are typically allowed to allocate a bounded budget to the market, as in our model.[19] According to the following corollary of Proposition 2, underreaction implies that $\pi(L) > p(L)$ when $p(L)$ is high (so that event $A$ is a favorite) and $\pi(L) < p(L)$ when $p(L)$ is low (longshot).

**Corollary 1** *The market price exhibits a favorite-longshot bias, as there exists a price* $p^* \in [0,1]$ *such that* $p(L) > p^*$ *implies* $\pi(L) > p(L)$*, and* $p(L) < p^*$ *implies* $\pi(L) < p(L)$*.*

Thus, the favorite-longshot bias results, with longshot outcomes occurring less often than indicated by the price, while the opposite is true for favorites. The favorite-longshot bias is widely documented in the empirical literature on betting and prediction markets when comparing winning frequencies with market prices (see Thaler and Ziemba, 1988, Jullien and Salanié, 2008, and Snowberg and Wolfers, 2010).

Rearranging (4) with (3), we have that $\log(\pi/(1-\pi)) - \log(p/(1-p))$ is a strictly increasing function of $p$. Thus, when running the following regression

$$\log \frac{\pi_j}{1-\pi_j} = a + b \log \frac{p_j}{1-p_j} + \varepsilon_j, \tag{5}$$

Proposition 2 predicts that $b > 1$. Once we identify the posterior $\pi_j$ chance for an event with the empirical winning frequency corresponding to market price $p_j$, our model thus offers a new informational explanation of the favorite-longshot bias. Outcomes favored by the market occur more often than if the price is interpreted as a probability—and, conversely, longshots win less frequently than the price indicates.

Before proceeding, it is worth pausing to discuss the relation with the alternative explanation for the favorite-longshot proposed by Ali (1977) in a pioneering paper and recently revived by Manski (2006), Gjerstad (2005), and Wolfers and Zitzewitz (2005) in the fledgling literature on prediction markets. In a model of equilibrium betting with

[18] Partly thanks to their track record as forecasting tools, as documented, for instance, by Forsythe et al. (1992) and Berg et al. (2008), prediction markets have attracted some recent interest as mechanisms to collect information and improving decision making in business and public policy contexts. See Hanson (1999), Wolfers and Zitzewitz (2004), and Hahn and Tetlock (2005).

[19] For example, in the Iowa Electronic Markets each trader cannot invest more than \$500. Exemption from anti-gambling legislation is granted for such small stakes given the educational purpose of these markets. Naturally, traders have no endowment risk and are given an equal number of the two assets when they enter the market.

heterogeneous prior beliefs, Ali (1977) notes that if the median bettor thinks that one outcome (defined to be the favorite) is more likely than the other, then the equilibrium fraction of parimutuel bets on this favorite outcome is lower than the belief of the median bettor. Ali (1977, Theorem 2) explains the bias by making the auxiliary assumption that the median (or average) belief corresponds to the empirical probability. But this assumption is contentious. If the traders' beliefs really have information content, their positions should depend on the information about these beliefs that is contained in the market price. This tension underlies the modern information economics critique of the Walrasian approach to price formation with heterogeneous beliefs (see the discussion in Chapter 1 of Grossman, 1989). To the prediction markets literature, we contribute the observation that the favorite-longshot bias results *without* making any assumption on how the beliefs of the median member of the population relate to the empirical probability. Even if we remain agnostic about the relation between (the distribution of prior) beliefs and the empirical chance of the outcome, we show that underreaction results as a comparative statics result with respect to information.[20]

## 1.3 Comparative Statics in Prior Beliefs and Wealth

Our equilibrium price $p(L)$ is determined by (2) which depends on the primitive distribution $G$ of wealth across traders with different prior beliefs. Changes in this wealth distribution can affect the equilibrium and hence the extent of underreaction. We show that underreaction is more pronounced if this distribution is wider. Note that a wider distribution arises in a population where traders simply have greater belief heterogeneity. A wider distribution of wealth over beliefs also arises when more opinionated traders attract more resources, or when less opinionated traders stay away from the market.

In analogy with Rothschild and Stiglitz's (1970) definition of mean preserving spread, define distribution $G'$ to be a *median-preserving spread* of distribution $G$ if $G$ and $G'$ have the same median $m$ and satisfy $G'(q) \geq G(q)$ for all $q \leq m$ and $G'(q) \leq G(q)$ for all $q \geq m$.

**Proposition 3** *Suppose that $G'$ is a median-preserving spread of $G$, denoting the common*

[20] Ottaviani and Sørensen (2009) and (2010) offer a different explanation for the favorite-longshot bias in the context of a game-theoretic model of parimutuel betting where traders have a common prior but are unable to condition their behavior on the information that is contained in the equilibrium price.

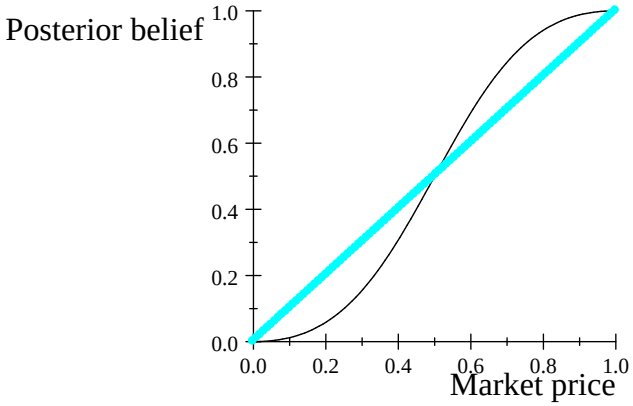


Figure 1: This plot shows the posterior probability for event $A$ as a function of the market price $p$ for the $A$ asset, when the prior beliefs of the risk-neutral traders are uniformly distributed ($\beta = 1$ in the example). The market price is represented by the dotted diagonal.

*median by* $m$*. Then, more underreaction results under* $G'$ *than under* $G$*:* $L > (1-m)/m$ *implies* $\pi(L) > p(L) > p'(L) > 1/2$*, and* $L < (1-m)/m$ *implies* $\pi(L) < p(L) < p'(L) < 1/2$.

This result is consistent with the observation of more pronounced favorite-longshot bias in political prediction markets, which are naturally characterized by a wider dispersion of beliefs; see Page and Clemen (2013) for corroborating evidence. Our baseline model with bounded wealth is also applicable to financial markets where traders typically have a finite wealth and/or can borrow a finite amount of money due to imperfections in the credit market. Empirical evidence by Verardo (2009) confirms that momentum profits are significantly larger for portfolios characterized by higher heterogeneity of beliefs.

**Example.** For illustration suppose that the distribution of subjective prior beliefs over the interval $[0,1]$ is $G(q) = q^\gamma / [q^\gamma + (1-q)^\gamma]$, where $\gamma > 0$ is a parameter that measures the concentration of beliefs. The greater $\gamma$ is, the less spread is this symmetric belief distribution around the average belief $q = 1/2$. For $\gamma = 1$ beliefs are uniformly distributed, as $\gamma \to \infty$ beliefs become concentrated near $1/2$, and as $\gamma \to 0$ beliefs are maximally dispersed around the extremes of $[0,1]$. The equilibrium market price $p(L)$ satisfies the linear relation

$$\log \frac{p(L)}{1-p(L)} = \frac{\gamma}{1+\gamma} \log L.$$

Hence, $\gamma/(1+\gamma) \in (0,1)$ measures the extent to which the price reacts to information. Price underreaction is minimal when $\gamma$ is very large, corresponding to the case with nearly homogeneous beliefs. Conversely, there is an arbitrarily large degree of underreaction when beliefs are maximally heterogeneous, corresponding to $\gamma$ close to zero.

Assume that a market observer's prior is $q = 1/2$ for event $A$, consistent with a symmetric market price of $p(L=1) = 1/2$ in the absence of additional information. The posterior belief associated with price $p$ then satisfies

$$\log \frac{\pi(L)}{1-\pi(L)} = \log L = \frac{1+\gamma}{\gamma} \log \frac{p(L)}{1-p(L)}.$$

This provides a particularly strong foundation for the linear regression (5). As illustrated in Figure 1 for the case with uniform beliefs ($\gamma = 1$), the market price overstates the winning chance of a longshot and understates the winning chance of a favorite by a factor of two.

# 2 Risk Aversion Model

So far we have assumed that each individual trader is risk neutral, and thus ends up taking as extreme a position as possible on either side of the market. Now, we show that our main result extends nicely to risk-averse traders, under the empirically plausible assumption that their absolute risk aversion is decreasing with wealth. This result does not rely on imposing exogenous constraints on trades.

## 2.1 Homogeneous Endowments

We first retain the assumption that traders are initially endowed with the same number of each asset, $w_{i0}(A) = w_{i0}(A^c) = w_{i0}$. Public information $L$ and the wealth distribution $G$ are as in the baseline model.

The first difference to the former model is that trader $i$ now maximizes subjective expected utility, $\pi_i u_i(w_i(A)) + (1-\pi_i) u_i(w_i(A^c))$, where the utility function $u_i$ is twice differentiable with $u_i' > 0$ and $u_i'' < 0$. We assume that $u_i$ satisfies the DARA assumption that the de Finetti-Arrow-Pratt coefficient of absolute risk aversion, $-u_i''/u_i'$, is weakly decreasing in its argument $w_i$.

The second change is that we allow traders to adopt negative positions in the assets. There is no longer any exogenous bound on portfolios — traders can exchange as many

units as they like of one asset into the other. Risk aversion implies that they prefer not to go to extremes.

The combination of DARA with homogeneous endowments implies that aggregate demand is well behaved, in analogy with Proposition 1:

**Proposition 4** *There exists a unique competitive equilibrium. The price, $p$, is a strictly increasing function of the information realization $L$.*

**Belief Aggregation with CARA Preferences.** Our contention is that underreaction results once we relax simultaneously two assumptions that are commonly made in asset pricing models with information: no wealth effects and common prior. With heterogeneous priors but without wealth effects there is no underreaction. To see this, suppose here that all traders have constant absolute risk aversion (CARA) utility functions, with possibly heterogeneous degrees of risk aversion, such that $u_i(w) = -\exp(-w/t_i)$, where $t_i > 0$ is constant. Denoting the relative risk tolerance of trader $i$ in the population by $\tau_i = t_i / \int_0^1 t_j dG(q_j)$, we have:

**Proposition 5** *Suppose traders have CARA preferences and heterogeneous beliefs. If we define an average prior belief $q$ by*

$$\log \frac{q}{1-q} = \int_0^1 \tau_i \log \frac{q_i}{1-q_i} dG(q_i), \tag{6}$$

*then the equilibrium price satisfies Bayes' rule with market prior $q$.*

Under CARA, wealth effects vanish and heterogeneous beliefs can be aggregated, according to formula (6), consistent with the classic result of Wilson (1968), Lintner (1969), and Rubinstein (1974); a similar result has also been obtained by Varian (1989). The market price thus behaves as a posterior belief and there is no underreaction.

**Underreaction with DARA Preferences.** We have seen that CARA preferences lead to an unbiased price reaction to information in equilibrium. Now we verify that, for strict DARA preferences, a bias arises in the price. When $L$ rises, the rising equilibrium price yields a negative wealth effect on any optimistic individual (with $\pi_i > p$) who is a net demander ($w_i(A) > w_i(A^c)$). Conversely, pessimistic traders benefit from the price increase. With DARA preferences, the wealth effect implies that optimists become more

risk averse while pessimists become less risk averse. Although the price rises with $L$, it is less reactive than a posterior belief, because pessimists trade more heavily in the market when information is more favorable.[21]

**Proposition 6** *Suppose that beliefs are truly heterogeneous and that all individuals have strict DARA preferences. The market price underreacts to information, satisfying (4) for any pair, $L' > L$.*

The asset pricing literature often assumes that traders have a common prior belief (Grossman, 1976). Under the common prior assumption, the price reacts one-for-one to information, regardless of risk attitudes. Our underreaction result thus holds once we allow for *both* heterogeneous priors *and* wealth effects. The intuition for this result is the same as in the baseline model with limited wealth. As $L$ increases, optimists suffer a negative wealth effect, become more risk averse, and thus optimally reduce their demand of the $A$ assets. The converse holds for pessimists. Thus, the equilibrium price adjusts by increasing the weight assigned to traders with prior beliefs less favorable to $A$.

Traders constrained by risk aversion choose an asset bundle that satisfies a familiar first-order condition for optimality,

$$\frac{\pi_i}{1-\pi_i}\frac{u_i'\left(w_i\left(A\right)\right)}{u_i'\left(w_i\left(A^c\right)\right)} = \frac{p}{1-p}. \tag{7}$$

According to this consumption-based asset pricing relation, the price is proportional to the subjective expected marginal utility of payoffs. Since subjective beliefs are updated according to Bayes' rule, our price underreaction result can be alternatively interpreted as a systematic change in marginal utilities with DARA preferences. Our proposition proves that due to the wealth effect, as $L$ rises, $u_i'\left(w_i\left(A\right)\right)/u_i'\left(w_i\left(A^c\right)\right)$ falls for all traders.

**Example with Logarithmic Preferences.** Suppose traders have logarithmic preferences, $u_i\left(w\right) = \log w$, satisfying DARA. In order to highlight the difference between Propositions 2 and 6, namely the inclusion of unconstrained traders, we remove completely the trading constraint. The well-known solution to this individual demand problem with Cobb-Douglas preferences gives $w_i\left(A\right) = \pi_i\left(W_i + w_0\right)/p$. The market-clearing price is then a

[21] Given that CARA is the knife-edge case, by reversing the logic of Proposition 6 it can be shown that overreaction results when risk aversion is increasing but not too much (so that demand monotonicity is preserved).

wealth-weighted average of the posterior beliefs,[22]

$$p(L) = \int_0^1 \pi(L)\, dG(q) = \int_0^1 \frac{qL}{qL + (1-q)}\, dG(q). \tag{8}$$

When $G$ is uniform, integration by parts of (8) yields $p(L) = L(L - 1 - \log L)/(L-1)^2$ for all $L \neq 1$. If $p(1) = \int_0^1 q\, dq = 1/2$ is the prior belief of an outside observer, the favorite-longshot bias can be illustrated in a graph similar to Figure 1.

## 2.2 Heterogeneous Endowments

We now allow trader $i$'s initial asset endowment to vary across events, $w_{i0}(A) \neq w_{i0}(A^c)$, as is natural in financial markets.[23] In order to derive results in this more general case, we restrict the class of individual preferences. Suppose that there exist constants $\alpha_i$ and $\beta$ such that trader $i$ has Hyperbolic Absolute Risk Aversion (HARA), $-u_i''(w)/u_i'(w) = 1/(\alpha_i + \beta w)$. The fact that $\beta$ is constant across traders means that traders are equally cautious.[24] We will focus attention on the case where cautiousness satisfies the DARA assumption that $\beta > 0$.

The individual characteristics, namely the endowment vector $w_{i0}$, the preference parameter $\alpha_i$, and the prior $q_i$, are jointly distributed on $\mathbb{R}^4$ with probability measure $H$. We assume that the aggregate endowments $w_0(A) = \int w_{i0}(A)\, dH$ and $w_0(A^c) = \int w_{i0}(A^c)\, dH$ as well as the average preference parameter $\alpha = \int \alpha_i dH$ are well-defined finite numbers. We likewise assume that $\int (q_i/(1-q_i))^\beta\, dH$ and $\int ((1-q_i)/q_i)^\beta\, dH$ are both finite—this technical condition helps in our proofs, and is satisfied when individual prior beliefs near the extremes 0 and 1 are not too common. We finally assume that $(w_{i0}(A), w_{i0}(A^c), \alpha_{i0})$ are stochastically independent of $q_i$.

In the special case of common prior belief $q$, the HARA assumption guarantees that there exists a representative trader; see Rubinstein (1974). This means that the aggregate demand is invariant to redistribution of the initial endowment and can be expressed as the individual demand function derived from the representative trader's utility function. In

[22] Cobb-Douglas preferences are homothetic, so that wealth expansion paths are linear. With more general utility functions, this property fails, and the extent of underreaction can be affected by a proportional resizing of wealth across the population of traders.

[23] See also Musto and Yilmaz (2003) for a model in which traders are subject to wealth risk, because they are differentially affected by the redistribution associated with different electoral outcomes.

[24] In the special case with $\beta > 0$, the absolute risk aversion is decreasing in $w$, so that these preferences are a special case of DARA preferences. CARA results when $\beta = 0$.

equilibrium, this representative trader must demand the constant aggregate endowment $(w_0(A), w_0(A^c))$, and the equilibrium price $p(L)$ must satisfy the first-order condition (7). Hence, there is no underreaction in this setting, since $\log[p(L)/(1-p(L))] - \log(L)$ is constant in $L$.

We can show that the introduction of prior belief heterogeneity results again in price underreaction to information. As before, traders with higher prior beliefs tend to take larger positions in asset $A$, and react relatively more when news favors event $A$. The extra complication is that individual trade heterogeneity depends not only on beliefs but just as much on endowments and preferences. Thus, some optimists for $A$ actually trade against $A$ in the market because they initially hold even more $A$ assets than they would like to keep, and the size of traders' reaction to news depends on preference parameter $\alpha_i$. Intuitively, however, the underreaction effect appears once we average over endowments and preferences. Technically, such averaging is feasible since the HARA demand function is multiplicatively separable in beliefs and other individual characteristics.

**Proposition 7** *Assume that all traders have HARA preferences with common cautiousness parameter $\beta > 0$. If prior beliefs are truly heterogeneous and independent of other individual characteristics, then the market price underreacts to information.*

**Edgeworth Box Illustration.** The Edgeworth box in Figure 2 graphically illustrates our logic for a market with two types of traders (with prior beliefs $q_1 < q_2$). Traders have convex indifference curves, which are not drawn to avoid cluttering the picture. Given that the slope of the indifference curves at any safe allocation is $-\pi_i/(1-\pi_i) = -q_i L/(1-q_i)$, trader 2 (optimist) has steeper indifference curves than trader 1 (pessimist) along the diagonal. In equilibrium, the marginal rates of substitution are equalized. In the figure, there is aggregate risk as $w_0(A) > w_0(A^c)$, but in the limit where endowments are homogeneous, the Edgeworth box would be a square with the initial endowment, $e$, lying on the common diagonal. We denote the equilibrium allocation by $w^*$. In the picture, the less optimistic trader 1 sells on net asset $A$, as is always the case with homogeneous endowments.

How is the equilibrium affected by an exogenous change in information from $L$ to $L' > L$? Marginal rates of substitution are affected such that all indifference curves become steeper by a factor of $L'/L$. For the sake of argument, imagine that the price were to change

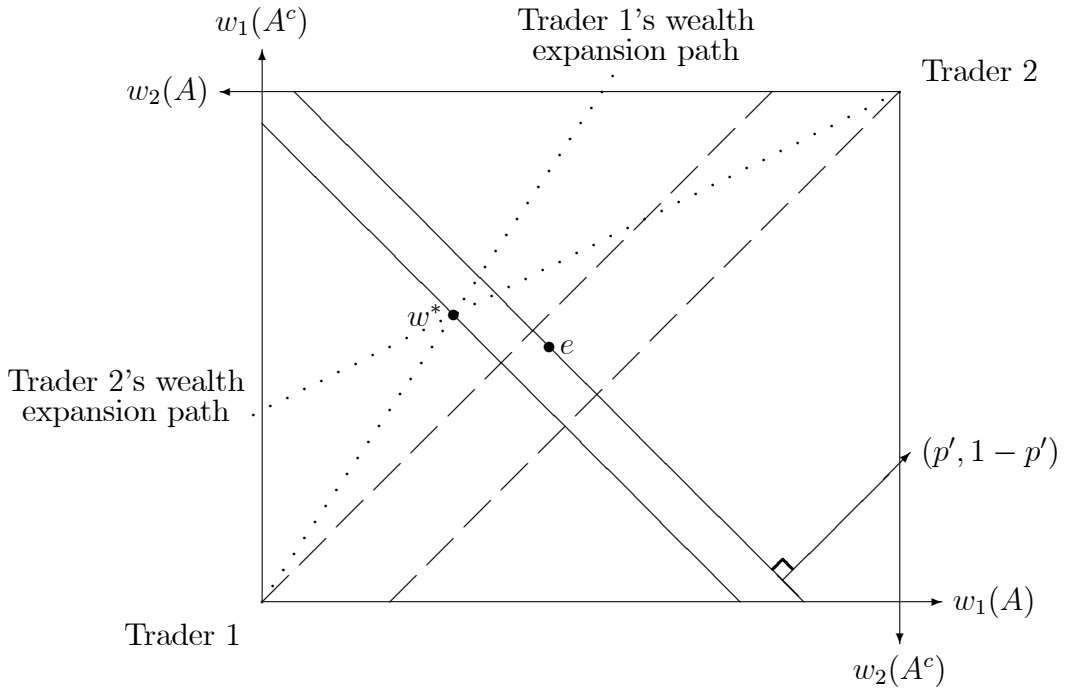


Figure 2: Edgeworth box representation of the underreaction result. Logarithmic preferences result in linear wealth-expansion paths.

as a Bayesian update of market belief $p(L)$ to $p' = p(L) L' / [p(L) L' + (1 - p(L)) L]$. Since $p' > p(L)$, the new budget line through $e$ passes above $w^*$, illustrating the wealth effect which is positive for the pessimistic trader 1. Now, as it has been well known since Arrow (1965), DARA implies that the wealth expansion paths diverge from the diagonal. The richer trader 1 thus demands a riskier bundle further away from the diagonal than at $w^*$, whereas the poorer trader 2 demands a safer bundle closer to the diagonal. To reach a new equilibrium in our picture, the price must adjust so as to eliminate the excess demand for asset $A^c$. This is achieved by a relative reduction in the relative price for asset $A$, so that $p(L') < p'$. Thus prices must underreact to information when endowments are homogeneous.

With heterogeneous endowments and preferences outside the HARA class, neither under nor overreaction need result. For instance, an equilibrium may exist where the two risk-averse traders hold a bundle on the same side of their respective diagonal in the Edgeworth box (i.e., $w_1(A) > w_1(A^c)$ and $w_2(A) > w_2(A^c)$). The DARA wealth expansion

paths no longer force the price to underreact in response to information, as a rising price of asset 1 consistently takes the net buyer of asset 1 closer to the diagonal, and the other trader further from the diagonal.

# 3 Dynamic Price Effects

In this section we extend our model to a dynamic setting in which information arrives to the market sequentially after the initial round of trade. To set the stage, we verify that there exists an equilibrium where the initial round of trade is captured by our baseline model, and where there is no trade in subsequent periods (Proposition 8); this result is consistent with Milgrom and Stokey's (1982) no trade theorem. We then obtain our two substantive results about the price path. First, we show that the initial underreaction of the price to information implies momentum of the price process in subsequent periods—if the initial price movement is upward, prices subsequently move up on average (Proposition 9, part a). Under a sufficient symmetry assumption, this implies positive autocorrelation in price changes in the short run (Proposition 9, part b). Second, symmetry also suffices to obtain a later price reversal after the initial momentum (Proposition 10). These momentum and reversal effects are aftershocks of the initial underreaction, and thus appear in our model even though there is no trading after the first period.

## 3.1 Model

Consider a constant set of traders $I$ who are initially in the same situation as in either of Sections 1, 2.1, or 2.2. In the latter two cases, we assume that all traders' utility functions exhibit strictly DARA. Each trader is allowed to trade at every time date $t \in \{1, \ldots, T\}$ at price $p_t$ that is determined competitively. The joint information publicly revealed to traders up until period $t$ has likelihood ratio $L_t$, so that $L_t$ encompasses $L_{t-1}$ and the new information observed in period $t$. The asset position of trader $i$ after trade at period $t$ is denoted by $\Delta x_{it}$. At time $T+1$ the true event is revealed, and the asset pays out. Each trader aims to maximize the subjective expected utility of period $T+1$ wealth.

A dynamic competitive equilibrium is defined as follows. First, for every $t = 1, \ldots, T$, there is a price function $p_t(L_t)$. By convention, $p_{T+1} = 1$ when $A$ is true, and $p_{T+1} = 0$ when $A^c$ is true. Second, given these price functions, every trader $i$ chooses a contingent

strategy of asset trades in order to maximize expected utility of final wealth. If wealth is constrained as in Section 1, the trader's wealth must always stay non-negative. Finally, in every period $t$ at any information $L_t$ realization, the market clears.

**Proposition 8** *There exists a dynamic competitive equilibrium with the following properties. In the first round of trade, the price $p_1(L_1)$ is the static equilibrium price $p(L_1)$ from either of Propositions 1, 4, or 7. In all subsequent periods there is no trade, and the price satisfies Bayes' updating rule,*

$$\frac{p_t(L_t)}{1-p_t(L_t)} = \frac{L_t}{L_1}\frac{p_1(L_1)}{1-p_1(L_1)}. \tag{9}$$

As in Milgrom and Stokey (1982), when beliefs are concordant there will be no trade after the first period. After one round of trade, the marginal rate of substitution for every trader is equal to the ratio of the competitive prices of the two assets. When beliefs are concordant, information changes this marginal rate of substitution for every trader in the same way. Thus, the allocation resulting at the end of the first period remains an equilibrium allocation, voiding future trade.

In the remainder of this section, we write $p_t$ for the equilibrium price at date $t$, thus suppressing its dependence on $L_t$.

The marginal trader, who holds posterior belief $\pi_i(L_1) = p_1$ after the first round of trading, remains the marginal trader at future dates. The market price $p_t$ is the Bayesian update of this trader's prior belief $p_1/[(1-p_1)L_1 + p_1]$ with information available at time $t$. From this trader's point of view prices follow a martingale, i.e., $E[p_{t_2}|L_{t_1}] = p_{t_1}$ for all $t_2 > t_1 \geq 1$.

Every trader who is initially more optimistic than this marginal trader, and hence has first-round posterior $\pi_i(L_1) > p_1$ and has chosen $\Delta x_i > 0$, believes that the price process is a sub-martingale (trending upwards). Despite this belief, the no-trade theorem establishes that such a trader does not wish to alter the position away from the initial $\Delta x_i$. The position already reflects a wealth- or risk-constrained position on the asset eventually rising in price, and there is no desire to further speculate on the upward trend in future asset prices.

## 3.2 Early Momentum

By Propositions 2, 6, and 7, an observer with neutral prior belief $q = p_1(1)$ sees initial price underreaction, disagreeing with the marginal trader of posterior belief $\pi_i(L_1) = p_1$. As more information arrives over time, both the price and the observer's belief are updated with Bayes' rule. From the observer's point of view, how are asset prices expected to develop over time? How does the initial disagreement change over time?

As a benchmark, recall that the marginal trader believes prices satisfy the martingale property, $E[p_{t_2} - p_{t_1}|L_{t_1}] = 0$ for all $t_2 > t_1 \geq 0$, when we let $p_0$ denote this trader's prior belief. The martingale property implies that $\text{Cov}(p_{t_3} - p_{t_2}, p_{t_2} - p_{t_1}|L_{t_1}) = E[(p_{t_3} - p_{t_2})(p_{t_2} - p_{t_1})|L_{t_1}] = 0$ for all $t_3 > t_2 > t_1 \geq 0$.[25]

We show that the outside observer with prior $q$ sees a different relation between initial and future price changes. Following the initial price reaction, prices exhibit early momentum, consistent with the empirical findings of Jagadeesh and Titman (1993) and subsequent literature. More precisely, the early price change $E[p_t - p_1|L_1]$ is no longer zero, but has the same sign as the initial price movement $p_1 - q$. Intuitively, a fixed prior disagreement matters less for the posterior beliefs, when the observer and marginal trader update their beliefs in concordance with subsequent information. Since $p_t$ is the marginal trader's posterior belief, the observer is expecting the asset price $p_t$ to gradually shed its initial underreaction. Underreaction is followed by a correcting outward price movement.

We further find that this momentum effect shows up as positive correlation between $p_t - p_1$ and $p_1 - q$ if $E[p_1 - q] = 0$.[26] The latter unbiased initial disagreement holds under the following *first-period symmetry* assumption: The distribution of priors satisfies $G(1-q) = 1 - G(q)$ for all $q \in [0,1]$, and first-period signals satisfy that $L_1$ has the same distribution as $1/L_1$.[27]

**Proposition 9** *Suppose that beliefs are truly heterogeneous. Fix the observer's prior at*

[25] Under this belief we have $E[(p_{t_3} - p_{t_2})(p_{t_2} - p_{t_1})|L_{t_1}] = E[E[p_{t_3} - p_{t_2}|L_{t_2}](p_{t_2} - p_{t_1})|L_{t_1}]$ and $E[p_{t_3} - p_{t_2}|L_{t_2}] = 0$ for all $L_{t_2}$.

[26] In price data where events $A$ and $A^c$ are arbitrarily labeled there should be no average direction to the disagreement among observer and marginal trader.

[27] Suppose that events $A$ and $A^c$ are equally likely, and let $f_1(s_1|A)$ and $f_1(s_1|A^c)$ denote conditional densities for the publicly observed signal $s_1$. Then $L_1 = f_1(s_1|A)/f_1(s_1|A^c)$ is a transformation of $s_1$, with conditional distributions that can be derived from the distributions of $s_1$. If $L_1$ is distributed in event $A$ as $1/L_1$ is distributed in event $A^c$, then first-period symmetry is satisfied.

*the neutral level* $q = p_1(1)$. *(a) Prices exhibit early momentum, i.e., for any date* $t > 1$,

$$E\left[(p_t - p_1)(p_1 - q) | L_1\right] \geq 0, \tag{10}$$

*with strict inequality when* $L_1 \neq 1$ *and the distribution of* $L_t/L_1$ *is non-degenerate. (b) If also first-period symmetry holds, then there is positive autocovariance in price changes, i.e., for any date* $t > 1$,

$$\text{Cov}\left(p_t - p_1, p_1 - p_0\right) \geq 0, \tag{11}$$

*with strict inequality when the distributions of* $L_1$ *and* $L_t/L_1$ *are non-degenerate.*

Part (b) predicts that in a regression of subsequent price changes $p_t - p_1$ on initial price reactions $p_1 - p_0$ there should be a positive coefficient.[28] In addition, the symmetry condition for part (b) also implies that the observer's momentum return $p_{T+1} - p_1$ is negatively skewed. Under symmetry, initial underreaction means that the observer considers the asset price too high if and only if the observer's posterior $L_1/(1 + L_1)$ exceeds $p_0 = 1/2$. The conditional expected return $p_{T+1} - p_1$ follows a binomial distribution (since the asset's payout $p_{T+1}$ is either 1 or 0). The binomial distribution is negatively skewed precisely when the probability of the high outcome exceeds 1/2. Negative skewness in momentum returns is consistent with empirical evidence, e.g., Amin, Coval and Seyhun (2004).

Proposition 9 is also consistent with the seemingly conflicting findings on price drift recently documented by Gil and Levitt (2007) and Croxson and Reade (2014) in the context of sport betting markets. On the one hand, Gil and Levitt (2007) find that the immediate price reaction to goals scored in the 2002 World Cup games is sizeable but incomplete and that price changes tend to be positively correlated, as predicted by our model. On the other hand, Croxson and Reade (2014) find no drift during the half-time break, thus challenging the view that the positive correlation of price changes during play time indicates slow incorporation of information. Consistent with this second bit of evidence, our model predicts the absence of drift when no new information arrives to the market, as it is realistic to assume during the break when the game is not played. These results follow when $L_t = L_{t'} = L_{t''}$ for all periods $t$ in the break $\{t', ..., t''\}$.

[28] Although the present analysis focuses on the periods that follow an initial period in which trade opens, our results apply more broadly to trading environments in which the arrival of new information coincides with trade—either because of added liquidity reasons or differential interpretation of information, from which the present analysis abstracts away.

### 3.3 Late Reversal

Momentum suggests a tendency for a correction of the initial price underreaction over time. However, there is an additional effect if we extend the symmetry assumption to later periods. *Later-period symmetry* holds when, for every $t > 1$, the distribution of $L_t/L_1$ conditional on $(A, L_1)$ is the same as the distribution of $L_1/L_t$ conditional on $(A^c, L_1)$.[29] From the outside observer's perspective at date 1, for every interior date $t$ strictly between the opening date 1 and payout date $T+1$, the later price change $p_{T+1} - p_t$ can be expected to reverse the earlier price change $p_t - p_1$. The two changes have negative covariance.

**Proposition 10** *Suppose that prior beliefs are truly heterogeneous, and that first-period and later-period symmetry hold. Fix the observer's prior at $q = 1/2$. Prices exhibit late reversal, i.e., for any date $t > 1$,*

$$E\left[\left(p_{T+1} - p_t\right)\left(p_t - p_1\right)|L_1\right] \leq 0 \tag{12}$$

*and*

$$\mathrm{Cov}\left(p_{T+1} - p_t, p_t - p_1\right) \leq 0, \tag{13}$$

*with strict inequality provided $L_t$ is not perfectly revealing and the distribution of $L_t/L_1$ is non-degenerate.*

This reversal effect is consistent with empirical findings in the asset pricing literature. Thus, in our model reversal is a necessary counterpart to initial momentum. Both effects are driven by the same initial disagreement between a market observer and the marginal trader.

# 4 Conclusion

To recap, our analysis combines three key ingredients—heterogeneous beliefs, information, and wealth effects—all hallmarks of financial markets (see Hong and Stein 2007). While previous literature has considered the effect of these ingredients either in isolation or in

[29] Continuing from footnote 27, let public signal $s_t$ be observed at time $t$ and added to the history of previously observed signals. Suppose that conditional signal distributions are independent, with densities $f_t\left(s_t|A\right)$ and $f_t\left(s_t|A\right)$. If the likelihood ratio $f_t\left(s_t|A\right)/f_t\left(s_t|A^c\right)$ is distributed in $A$ as $f_t\left(s_t|A^c\right)/f_t\left(s_t|A\right)$ is distributed in $A^c$, then $L_t/L_1 = \prod_{t'=2}^{t}\left(f_{t'}\left(s_{t'}|A\right)/f_{t'}\left(s_{t'}|A^c\right)\right)$ satisfies the assumptions.

partial combination, the simultaneous presence of all three ingredients delivers realistic pricing patterns. Information results in a redistribution of wealth across traders with different beliefs, so that prices tend to underreact to information when traders are subject to wealth effects, either because they have limited wealth to invest or because their absolute risk aversion decreases with wealth. This initial underreaction is followed by momentum in the short run and reversal in the long run. Our mechanism provides a single explanation for pricing patterns in financial markets that have typically been explained through separate channels.

We see our analysis as a first step toward understanding price reaction to information in the presence of heterogeneous beliefs and wealth effects. From a methodological perspective, we contribute a tractable model of asset pricing with two events. The model specification with bounded wealth at risk is particularly simple and thus could be a useful tool for analyzing asset pricing with wealth effects. The specification with risk aversion allows for a characterization of equilibrium asset prices with general risk preferences without making additional parametric or distributional assumptions.

The working paper version of this article analyzes a more general model that encompasses these two specifications by allowing traders to have general risk-averse preferences while at the same time constraining the wealth they can invest (for example because of limited borrowing capacity).[30] All our results continue to hold for this combination, even if all traders have CARA preferences as long as our exogenous wealth constraint binds for some traders. Underreaction does not result when our wealth-constraint is replaced by a short-selling constraint (as in Chen, Hong, and Stein, 2001, page 176); like DARA preferences, wealth constraints implicitly bound the position that traders can take on either side of the market.[31]

While our model is more general in some dimension than the traditional CARA-normal paradigm, the restriction to two events is special. It is natural to wonder whether underreaction also holds in settings with more than two events. Suppose the outcome space is partitioned into $K > 2$ mutually exclusive evens, and focus attention on a change of public information in favor of event $k$. To fix ideas, consider a version of the model with risk-neutral traders and bounded wealth. A trader with posterior belief vector $\pi$, facing

[30] Future work could add explicit consideration of borrowing that differentially relaxes the wealth constraints of traders; see Fostel and Geanakoplos (2008) for an initial investigation in this direction.

[31] See also Baker, Coval, and Stein (2007, page 270).

price vector $p$, invests all money on asset $k$ where the ratio $\pi_k/p_k$ is maximal. It is not hard to verify that there is a convex set of prior beliefs, $Q_k$, for which it is optimal to take position on asset $k$. Market clearing implies that $p_k$ equals the relative wealth endowment held by traders in set $Q_k$. The information favoring event $k$ implies that $p_k$ rises; hence the wealth endowment of the set $Q_k$ grows. This wider interest is our central underreaction mechanism. Our insight that an information change in favor of $k$ must attract traders with prior beliefs less favorable to $k$ is thus robust. Complicating the picture, however, the equilibrium reduction of wealth allocated to other assets is not necessarily proportional to their prices. In general the set $Q_k$ may grow in certain directions while shrinking in other directions, as the relative prices of other assets change, so the overall effect on the price of asset $k$ is unclear in general. Contagion adjusts the relative prices of other assets and may feed back to the demand for the focus asset in such a way that its price actually overreacts to the information.

The article focuses on the reaction to public information. What if instead information is initially privately held by the traders? The working paper extends our results to the case where the information is privately held by the traders rather than being public. The extension works because in our setting all private information held by traders is revealed in a fully revealing rational expectations equilibrium (REE). However, there is a tension between the assumption of heterogeneous priors and the equilibrium notion. The learning that is necessary for strategic (rather than competitive) equilibrium play to become sensible could also eliminate heterogeneity in prior beliefs; see Dekel, Fudenberg, and Levine (2004).

Our results are particularly striking given our focus on concordant beliefs—even though all traders agree on how to interpret information, underreaction results in the aggregate through a composition effect. A similar composition effect is at play when traders interpret information in a non concordant way. The key is that the wealth effect moves the marginal trader against the information. In an extension of Section 1's static model allowing for non-concordant beliefs as in Harris and Raviv (1995) and Kandel and Pearson (1996), we have verified that underreaction also results when some traders assign too great (reflecting overconfidence) or too small (underconfidence) a weight on information relative to Bayesian updating, provided the distribution of weights in the population is symmetric around Bayesian updating.[32] The occurrence of speculative retrade à la Harrison and Kreps (1978)

[32]See also Palfrey and Wang (2012) who modify Harris and Raviv's (1993) price formation process and

in dynamic extensions poses serious analytical challenges.[33] Banerjee and Kremer (2010) make progress in a two-period setting. Analyses with non-concordant beliefs along these lines promise to deliver realistic predictions regarding the amount of trade.

## Appendix

**Proof of Proposition 1.** For a given likelihood ratio $L$, the prior of an individual with posterior belief $\pi_i$ is, using (1), $q_i = \pi_i / [(1-\pi_i) L + \pi_i]$. The $A^c$ asset is demanded in amount $w_{i0}/(1-p)$ by every individual with $\pi_i < p$, or equivalently $q_i < p/[(1-p) L + p]$. The aggregate demand for this asset is then $G(p/[(1-p) L + p])/(1-p)$. In equilibrium, aggregate demand is equal to aggregate supply, equal to 1, resulting in equation (2).

Next, we establish that the price defined by (2) is a strictly increasing function of $L$. The left-hand side of (2) is a strictly increasing continuous function of $p$, which is 0 at $p = 0$ and 1 at $p = 1$. For any $L \in (0, \infty)$, the right-hand side is a weakly decreasing continuous function of $p$, for the cumulative distribution function $G$ is non-decreasing. The right-hand side is equal to 1 at $p = 0$, while it is 0 at $p = 1$. Thus there exists a unique solution, such that $G \notin \{0, 1\}$. When $L$ rises, the left-hand side is unaffected, while the right-hand side rises for any $p$, strictly so near the solution to (2) by the assumptions on $G$. Hence, the solution $p$ must be increasing with $L$.

**Proof of Proposition 2.** (i) When $L$ increases, so does $p(L)$. By equation (2), when $p(L)$ increases, $p(L)/[(1-p(L)) L + p(L)]$ must fall, because the cumulative distribution function $G$ is non-decreasing. (ii) By Proposition 1, $p(L') > p(L)$. Note that (4) is equivalent to

$$\frac{p(L')}{1-p(L')} \frac{1}{L'} < \frac{p(L)}{1-p(L)} \frac{1}{L}.$$

Using the strictly increasing transformation $z \to z/(1+z)$ on both sides of this inequality, it is equivalent to

$$\frac{p(L')}{[1-p(L')] L' + p(L')} < \frac{p(L)}{[1-p(L)] L + p(L)},$$

which is true by part (i).

obtain overreaction to good news and underreaction to bad news. However, they do not address potential momentum or reversal.

[33] See also Morris (1996) and the survey by Scheinkman and Xiong (2004).

**Proof of Corollary 1.** Combining (4) with (3), we see from (ii) that the function

$$\Psi(L) = \log\left(\frac{\pi(L)}{1-\pi(L)}\right) - \log\left(\frac{p(L)}{1-p(L)}\right)$$

is strictly increasing in $L$. Hence, one of the following three cases will hold. In the first case, there exists an $L^* \in (0,\infty)$ such that $\Psi(L)$ is negative for $L < L^*$ and positive for $L > L^*$—in this case, the result follows with $p^* = p(L^*)$. In the second case, $\Psi(L)$ is negative for all $L$, and the result holds for $p^* = 1$. In the third case, $\Psi(L)$ is positive for all $L$, and the result is true with $p^* = 0$.

**Proof of Proposition 3.** Note first that $p((1-m)/m) = 1/2$ by (2). Consider now $L > (1-m)/m$ such that the equilibrium prices satisfy $\pi(L) > p(L), p'(L) > 1/2$. If, contrary to the claim, $p(L) < p'(L)$, then (2) implies that

$$G\left(\frac{p(L)}{(1-p(L))L + p(L)}\right) = 1 - p(L) > 1 - p'(L) = G'\left(\frac{p'(L)}{(1-p'(L))L + p'(L)}\right).$$

Further,

$$\frac{p'(L)}{(1-p'(L))L + p'(L)} > \frac{p(L)}{(1-p(L))L + p(L)},$$

while $p'(L) > 1/2$ in equilibrium implies

$$\frac{p'(L)}{(1-p'(L))L + p'(L)} < m.$$

Thus the median preserving spread property implies the contradiction,

$$G\left(\frac{p(L)}{(1-p(L))L + p(L)}\right) < G'\left(\frac{p'(L)}{(1-p'(L))L + p'(L)}\right).$$

A similar argument applies when $L < (1-m)/m$.

**Proof of Proposition 4.** Let $\Delta x_i$ denote the choice variable of trader $i$, such that $p\Delta x_i$ units of the $A^c$ asset are exchanged for $(1-p)\Delta x_i$ units of the $A$ asset. Note that this is a zero net value trade, since the asset sale generates $(1-p)p|\Delta x_i|$ of cash that is spent on buying the other asset. The final wealth levels in the two events are:

$$w_i(A) = w_{i0} + (1-p)\Delta x_i, \tag{14}$$

$$w_i(A^c) = w_{i0} - p\Delta x_i. \tag{15}$$

The individual trader maximizes $\pi_i u_i(w_i(A)) + (1-\pi_i) u_i(w_i(A^c))$ over $\Delta x_i$. The first derivative with respect to $\Delta x_i$ is

$$\pi_i (1-p) u_i'(w_i(A)) - (1-\pi_i) p u_i'(w_i(A^c)).$$

Strict concavity of $u_i$ ensures that the maximand $\Delta x_i$ is the unique solution to the first-order condition (7). By the Theorem of the Maximum, $\Delta x_i$ is a continuous function of $\pi_i$ and $p$. We first show that the optimizer $\Delta x_i$ is strictly decreasing in $p$ and strictly increasing in $\pi_i$.

Since $u_i' > 0$, the cross-partial of the objective with respect to the choice variable $\Delta x_i$ and the exogenous $\pi_i$ is strictly positive, and hence $\Delta x_i$ is strictly increasing in $\pi_i$. A sufficient condition for a strictly negative cross-partial with respect to $\Delta x_i$ and $p$ is

$$\Delta x_i [\pi_i (1-p) u_i''(w_i(A)) - (1-\pi_i) p u_i''(w_i(A^c))] > 0. \tag{16}$$

Using the first-order condition for interior optimality, the second factor of (16) is positive if and only if

$$-\frac{u_i''(w_i(A^c))}{u_i'(w_i(A^c))} > -\frac{u_i''(w_i(A))}{u_i'(w_i(A))}.$$

By the DARA assumption, this inequality holds if and only if $w_i(A) > w_i(A^c)$, i.e., $\Delta x_i > 0$. Thus the cross-partial is strictly negative for all $\Delta x_i \neq 0$, so that $\Delta x_i$ is strictly decreasing in $p$.

Equilibrium is characterized by the requirement that the aggregate purchase of asset $A$ must be zero, i.e., $\int_0^1 \Delta x_i(p, q_i, L)\, dG = 0$. When $p = 0$, every trader has $\pi_i > p$ and hence $\Delta x_i > 0$, while the opposite relation holds when $p = 1$. Individual demands are continuous and strictly decreasing in $p$, so there exists a unique equilibrium price in $(0,1)$. When $L$ is increased, $\pi_i(L)$ rises, and hence $\Delta x_i$ rises for every trader. In order to restore equilibrium, the price must then strictly increase.

**Proof of Proposition 5.** The necessary and sufficient first-order condition (7) for the individual optimum is solved by

$$\Delta x_i = t_i \log\left(\frac{1-p(L)}{p(L)} \frac{\pi_i(L)}{1-\pi_i(L)}\right). \tag{17}$$

Market clearing occurs when $\int_0^1 \Delta x_i dG = 0$. By (17) and using $\pi_i(L)/(1-\pi_i(L)) = q_i L/(1-q_i)$ this is solved by $p(L) = qL/(qL+1-q)$.

**Proof of Proposition 6.** The result follows as in the proof of Proposition 2, once we establish that $\log\left[p\left(L\right)/\left(1-p\left(L\right)\right)\right]-\log\left(L\right)$ is strictly decreasing in $L$. Suppose, for a contradiction, that $\log\left[p\left(L\right)/\left(1-p\left(L\right)\right)\right]-\log\left(L\right)$ is non-decreasing near some $L$. We will argue in the next paragraph that individual demand satisfies $d\Delta x_i/dL<0$. Since market clearing $\int_0^1 \Delta x_i\left(p\left(L\right),q_i,L\right)dG=0$ implies $\int_0^1\left[d\Delta x_i\left(p,q_i,L\right)/dL\right]dG=0$, we will then obtain a contradiction establishing the claim.

Since $\log[\pi_i(L)/(1-\pi_i\left(L\right))]-\log(L)$ is constant, (7) implies that $u_i'(w_i(A))/u_i'(w_i(A^c))$ is non-decreasing in $L$. Using the expressions for the final wealth levels (14) and (15), non-negativity of the derivative of $u_i'\left(w_i\left(A\right)\right)/u_i'\left(w_i\left(A^c\right)\right)$ implies that

$$u_i''\left(w_i(A)\right)u_i'\left(w_i(A^c)\right)\left[(1-p)\,\tfrac{d\Delta x_i}{dL}-\Delta x_i\tfrac{dp}{dL}\right]\geq -u_i''\left(w_i(A^c)\right)u_i'\left(w_i(A)\right)\left[p\tfrac{d\Delta x_i}{dL}+\Delta x_i\tfrac{dp}{dL}\right].$$

The second derivative of the utility function is negative, so this implies

$$\frac{d\Delta x_i}{dL}\leq \Delta x_i\frac{dp}{dL}\frac{u_i''\left(w_i\left(A\right)\right)u_i'\left(w_i\left(A^c\right)\right)-u_i''\left(w_i\left(A^c\right)\right)u_i'\left(w_i\left(A\right)\right)}{\left(1-p\right)u_i''\left(w_i\left(A\right)\right)u_i'\left(w_i\left(A^c\right)\right)+pu_i''\left(w_i\left(A^c\right)\right)u_i'\left(w_i\left(A\right)\right)}. \tag{18}$$

On the right-hand side of (18), $dp/dL>0$ by Proposition 4, and the denominator is negative. Recall that $\Delta x_i>0$ if and only if $w_i\left(A\right)>w_i\left(A^c\right)$. By DARA, this implies that

$$-\frac{u_i''\left(w_i\left(A\right)\right)}{u_i'\left(w_i\left(A\right)\right)}<-\frac{u_i''\left(w_i\left(A^c\right)\right)}{u_i'\left(w_i\left(A^c\right)\right)}$$

or that the numerator is positive. Likewise, when $\Delta x_i<0$, the numerator is negative. In either case, the right-hand side of (18) is strictly negative. Hence, $d\Delta x_i/dL<0$ for every trader.

**Proof of Proposition 7.** The proof proceeds in five steps, first deriving individual demand functions, deriving an equation to characterize equilibrium, establishing two helpful technical results, proving existence and uniqueness of the equilibrium, and finally proving underreaction.

**Step 1.** We first consider the individual asset demand problem of trader $i$, given market price $p$ and information realization $L$. It follows from integration of the HARA property that

$$\frac{u_i'\left(w_i\left(A\right)\right)}{u_i'\left(w_i\left(A^C\right)\right)}=\left(\frac{\alpha_i+\beta w_i\left(A^C\right)}{\alpha_i+\beta w_i\left(A\right)}\right)^{1/\beta}.$$

The necessary first order condition (7) then reduces to

$$q_i^{\beta}L^{\beta}\left(1-p\right)^{\beta}\left(\alpha_i+\beta w_i\left(A^C\right)\right)=\left(1-q_i\right)^{\beta}p^{\beta}\left(\alpha_i+\beta w_i\left(A\right)\right). \tag{19}$$

The budget constraint is

$$pw_i(A) + (1-p)\, w_i(A^c) = pw_{i0}(A) + (1-p)\, w_{i0}(A^c).$$

Solving these two linear equations for the pair $(w_i(A), w_i(A^c))$, we find

$$\begin{aligned} w_i(A) &= \frac{(1-p)^\beta q_i^\beta L^\beta \beta[pw_{i0}(A)+(1-p)w_{i0}(A^c)]-(1-p)\left[p^\beta(1-q_i)^\beta-(1-p)^\beta q_i^\beta L^\beta\right]\alpha_i}{p(1-p)^\beta q_i^\beta L^\beta\beta+(1-p)p^\beta(1-q_i)^\beta\beta}, \\ w_i(A^c) &= \frac{p^\beta(1-q_i)^\beta \beta[pw_{i0}(A)+(1-p)w_{i0}(A^c)]+p\left[p^\beta(1-q_i)^\beta-(1-p)^\beta q_i^\beta L^\beta\right]\alpha_i}{p(1-p)^\beta q_i^\beta L^\beta\beta+(1-p)p^\beta(1-q_i)^\beta\beta}. \end{aligned}$$

It follows that asset demand is multiplicatively separable,

$$\begin{aligned} \alpha_i+\beta w_i(A) &= \left[\alpha_i+\beta\left[pw_{i0}(A)+(1-p)\, w_{i0}(A^c)\right]\right]\frac{(1-p)^\beta q_i^\beta L^\beta}{p(1-p)^\beta q_i^\beta L^\beta+(1-p)p^\beta(1-q_i)^\beta}, \\ \alpha_i+\beta w_i(A^c) &= \left[\alpha_i+\beta\left[pw_{i0}(A)+(1-p)\, w_{i0}(A^c)\right]\right]\frac{p^\beta(1-q_i)^\beta}{p(1-p)^\beta q_i^\beta L^\beta+(1-p)p^\beta(1-q_i)^\beta}. \end{aligned}$$

**Step 2.** We next turn to the aggregate property of the equilibrium. Averaging across the four individual characteristics, keeping in mind that the markets must clear, and employing the assumption that $q_i$ is independent of the other individual characteristics,

$$\begin{aligned} \alpha+\beta w_0(A) &= \left[\alpha+\beta\left[pw_0(A)+(1-p)\, w_0(A^c)\right]\right]E\left[\frac{(1-p)^\beta q_i^\beta L^\beta}{p(1-p)^\beta q_i^\beta L^\beta+(1-p)p^\beta(1-q_i)^\beta}\right], \\ \alpha+\beta w_0(A^c) &= \left[\alpha+\beta\left[pw_0(A)+(1-p)\, w_0(A^c)\right]\right]E\left[\frac{p^\beta(1-q_i)^\beta}{p(1-p)^\beta q_i^\beta L^\beta+(1-p)p^\beta(1-q_i)^\beta}\right]. \end{aligned}$$

The expectations operator here denotes averaging over the distribution of the prior $q_i$. Thus, there exists a positive constant $K = [\alpha+\beta w_0(A)]/[\alpha+\beta w_0(A^c)]$ such that, for every information realization $L$, the equilibrium price $p(L)$ satisfies

$$E\left[\frac{(1-p(L))^\beta q_i^\beta L^\beta}{p(L)(1-p(L))^\beta q_i^\beta L^\beta+(1-p(L))p(L)^\beta(1-q_i)^\beta}\right] = KE\left[\frac{p(L)^\beta(1-q_i)^\beta}{p(L)(1-p(L))^\beta q_i^\beta L^\beta+(1-p(L))p(L)^\beta(1-q_i)^\beta}\right],$$

i.e.,

$$0 = E\left[\frac{\left(\frac{1-p(L)}{p(L)}L\right)^\beta - K\left(\frac{1-q_i}{q_i}\right)^\beta}{\left(\frac{1-p(L)}{p(L)}L\right)^\beta + \frac{1-p(L)}{p(L)}K\left(\frac{1-q_i}{q_i}\right)^\beta}\right], \tag{20}$$

where we removed a non-zero factor $p(L)$ before arriving at the last line. This equation characterizes the equilibrium price $p(L) \in (0,1)$.

**Step 3.** We derive two convenient results. (i) If $a, b, c, d, x$ are positive reals, then $(ax-b)/(d+cx) \leq (ax-b)/(d+bc/a)$. To prove this, note that both denominators are positive. The desired inequality is then equivalent to $(ax-b)(d+bc/a) \leq (ax-b)(d+cx)$,

i.e., $(ax-b)\,bc/a \leq (ax-b)\,cx$, equivalent to the true $0 \leq ac\,(x-b/a)^2$. (ii) Suppose that $Y$ is a non-degenerate random variable, $f$ is a positive, strictly increasing function, and $g$ is a strictly increasing function, such that $E[f(Y)], E[g(Y)]$ and $E[f(Y)\,g(Y)]$ are well-defined finite numbers. Then $E[f(Y)\,g(Y)] > E[f(Y)]\,E[g(Y)]$. To see this, let $P$ denote the probability distribution for $Y$, and define the probability distribution $P'$ by $dP'/dP = f/E[f(Y)]$. Since the likelihood ratio $f/E[f(Y)]$ is strictly increasing, $P'$ first-order stochastically dominates $P$. Since $g$ is increasing, it follows that

$$E[f(Y)\,g(Y)] = \int fg dP = E[f(Y)] \int g dP' > E[f(Y)] \int g dP = E[f(Y)]\,E[g(Y)].$$

**Step 4.** We verify that there exists a uniquely defined strictly increasing equilibrium price function $p(L)$ solving (20). Write $x = (1-p)/p$. We aim to show that $x$ is a decreasing function of $L$ when

$$0 = E\left[\frac{(xL)^\beta - K\left(\frac{1-q_i}{q_i}\right)^\beta}{(xL)^\beta + xK\left(\frac{1-q_i}{q_i}\right)^\beta}\right]. \tag{21}$$

First, note that the expectation is a continuous function of $x > 0$. From Step 3 part (i), we have

$$E\left[\frac{(xL)^\beta - K\left(\frac{1-q_i}{q_i}\right)^\beta}{(xL)^\beta + xK\left(\frac{1-q_i}{q_i}\right)^\beta}\right] \leq E\left[\frac{(xL)^\beta - K\left(\frac{1-q_i}{q_i}\right)^\beta}{(xL)^\beta + x\,(xL)^\beta}\right] = \frac{(xL)^\beta - KE\left[\left(\frac{1-q_i}{q_i}\right)^\beta\right]}{(xL)^\beta + x\,(xL)^\beta}.$$

The latter expression is negative when $x$ is sufficiently close to zero, so the right-hand side of (21) is negative in that region. Likewise,

$$E\left[\frac{(xL)^\beta - K\left(\frac{1-q_i}{q_i}\right)^\beta}{(xL)^\beta + xK\left(\frac{1-q_i}{q_i}\right)^\beta}\right] = E\left[\frac{(xL)^\beta\left(\frac{q_i}{1-q_i}\right)^\beta - K}{(xL)^\beta\left(\frac{q_i}{1-q_i}\right)^\beta + xK}\right] \geq \frac{(xL)^\beta\,E\left[\left(\frac{q_i}{1-q_i}\right)^\beta\right] - K}{K + xK}$$

which is positive once $x$ is sufficiently large. Hence, a solution exists. We now show that every solution to (21) is an up-crossing, which proves uniqueness, and helps toward proving monotonicity. The partial derivative with respect to $x$ is

$$\begin{aligned} & E\left[\frac{\beta x^\beta L^\beta K\left(\frac{1-q_i}{q_i}\right)^\beta + \beta x^{\beta-1}L^\beta K\left(\frac{1-q_i}{q_i}\right)^\beta - (xL)^\beta K\left(\frac{1-q_i}{q_i}\right)^\beta + K^2\left(\frac{1-q_i}{q_i}\right)^{2\beta}}{\left[(xL)^\beta + xK\left(\frac{1-q_i}{q_i}\right)^\beta\right]^2}\right] \\ = \; & E\left[\frac{\beta x^\beta L^\beta K\left(\frac{1-q_i}{q_i}\right)^\beta + \beta x^{\beta-1}L^\beta K\left(\frac{1-q_i}{q_i}\right)^\beta}{\left[(xL)^\beta + xK\left(\frac{1-q_i}{q_i}\right)^\beta\right]^2}\right] - E\left[\frac{(xL)^\beta - K\left(\frac{1-q_i}{q_i}\right)^\beta}{\left[(xL)^\beta + xK\left(\frac{1-q_i}{q_i}\right)^\beta\right]}\frac{K\left(\frac{1-q_i}{q_i}\right)^\beta}{\left[(xL)^\beta + xK\left(\frac{1-q_i}{q_i}\right)^\beta\right]}\right]. \end{aligned}$$

Here, the second expectation is negative at any crossing by Step 3 part (ii) with $f(y) = Ky^\beta / \left((xL)^\beta + xKy^\beta\right)$ and $g(y) = \left(Ky^\beta - (xL)^\beta\right) / \left((xL)^\beta + xKy^\beta\right)$ where $E[g(Y)] = 0$ by (21). The first expectation is positive, so the partial derivative with respect to $x$ is positive. Finally, we note that the expectation in (21) is increasing in $L$ since the partial derivative is

$$E\left[\frac{\beta x^{\beta+1}L^{\beta-1}K\left(\frac{1-q_i}{q_i}\right)^\beta + \beta x^\beta L^{\beta-1}K\left(\frac{1-q_i}{q_i}\right)^\beta}{\left[(xL)^\beta + xK\left(\frac{1-q_i}{q_i}\right)^\beta\right]^2}\right].$$

**Step 5.** For our main comparative statics result, define $z = (1-p(L))L/p(L)$. To show underreaction, we aim to establish that $z$ is an increasing function of $L$, or equivalently that $z$ is a decreasing function of $x = (1-p(L))/p(L)$. The equilibrium relationship in (20) can be rewritten as

$$0 = E\left[\frac{z^\beta - K\left(\frac{1-q_i}{q_i}\right)^\beta}{z^\beta + xK\left(\frac{1-q_i}{q_i}\right)^\beta}\right]. \tag{22}$$

The partial derivative with respect to $z$ is

$$E\left[\frac{\beta z^{\beta-1}xK\left(\frac{1-q_i}{q_i}\right)^\beta + \beta z^{\beta-1}K\left(\frac{1-q_i}{q_i}\right)^\beta}{\left[z^\beta + xK\left(\frac{1-q_i}{q_i}\right)^\beta\right]^2}\right] > 0.$$

The partial derivative with respect to $x$ is

$$-E\left[\frac{z^\beta - K\left(\frac{1-q_i}{q_i}\right)^\beta}{\left[z^\beta + xK\left(\frac{1-q_i}{q_i}\right)^\beta\right]}\frac{K\left(\frac{1-q_i}{q_i}\right)^\beta}{\left[z^\beta + xK\left(\frac{1-q_i}{q_i}\right)^\beta\right]}\right] > 0,$$

where the sign follows from Step 3 part (ii), using $f(y) = Ky^\beta / \left(z^\beta + xKy^\beta\right)$ and $g(y) = \left(Ky^\beta - z^\beta\right) / \left(z^\beta + xKy^\beta\right)$ with $E[g(Y)] = 0$ by (22).

**Proof of Proposition 8.** See the Online Appendix for this proof.

**Proof of Proposition 9.** (a) The price at $t$ satisfies Bayes' rule,

$$p_t(L_t) = \frac{p_1(L_1)L_t}{p_1(L_1)L_t + (1-p_1(L_1))L_1} = p_1(L_1) + \frac{(1-p_1(L_1))p_1(L_1)(L_t - L_1)}{p_1(L_1)L_t + (1-p_1(L_1))L_1}.$$

The observer's posterior at time 1 is $\pi(L_1) = qL_1/(qL_1 + 1 - q)$. For the observer,

$$\begin{aligned} & E[p_t(L_t) - p_1(L_1)\,|L_1] \\ = \;& \pi(L_1)\, E[p_t(L_t) - p_1(L_1)\,|A, L_1] + (1 - \pi(L_1))\, E[p_t(L_t) - p_1(L_1)\,|A^c, L_1] \\ = \;& (\pi(L_1) - p_1(L_1))\,\{E[p_t(L_t) - p_1(L_1)\,|A, L_1] - E[p_t(L_t) - p_1(L_1)\,|A^c, L_1]\}, \end{aligned}$$

using the martingale property of Bayes updated prices.

At time 1, there is uncertainty about the realization of the future $L_t$. Bayes' rule implies $L_t/L_1 = f(L_t|A, L_1)/f(L_t|A^c, L_1)$ where $f$ denotes the p.d.f. for $L_t$. For any realization of $L_1$, we write $p_1$ for the known $p_1(L_1)$. Then

$$\begin{aligned} & E[(p_t(L_t) - p_1(L_1))(p_1(L_1) - q)\,|L_1] \\ = \;& (p_1 - q)(\pi(L_1) - p_1)\int_0^\infty (p_t(L_t) - p_1)(L_t - L_1)\, f(L_t|A^c, L_1)\, dL_t \\ = \;& (p_1 - p_1(1))(\pi(L_1) - p_1)\int_0^\infty \frac{(1 - p_1)\, p_1 (L_t - L_1)^2}{p_1 L_t + (1 - p_1) L_1} f(L_t|A, L_1)\, dL_t. \end{aligned}$$

All terms inside the integral are positive, and the entire integral is positive when $L_t/L_1$ has a non-degenerate distribution. By underreaction, $(p_1(L_1) - p_1(1))(\pi(L_1) - p_1(L_1)) > 0$ for all $L_1 \neq 1$. Hence, $E[(p_t(L_t) - p_1(L_1))(p_1(L_1) - q)\,|L_1] > 0$ for all $L_1 \neq 1$.

(b) By equation (2), symmetry of $G$ implies that $p_1(L_1) = 1 - p_1(1/L_1)$. In particular our observer has a fair prior, $q = p_1(1) = 1/2$. The assumption on the distribution of $L_1$ implies that $p_1$ has the same distribution as $1 - p_1$, so $E[p_1] = 1/2$. Thus, $E[p_1 - q] = 0$. Averaging (10) over $L_1$, $E[(p_t - p_1)(p_1 - q)] > 0$. Now, $\mathrm{Cov}(p_t - p_1, p_1 - q) = E[(p_t - p_1)(p_1 - q)] - E[p_t - p_1]\, E[p_1 - q] = E[(p_t - p_1)(p_1 - q)] > 0$.

**Proof of Proposition 10.** Symmetry implies that the distribution of $L_t/(L_t + L_1)$ conditional on $(A, L_1)$ is identical to the distribution of $L_1/(L_1 + L_t)$ conditional on $(A^c, L_1)$. It follows that $f(L_1^2/L_t|A, L_1) = (L_t/L_1)^2 f(L_t|A^c, L_1) = (L_t/L_1) f(L_t|A, L_1)$, when again $f$ denotes the p.d.f. for $L_t$.[34] Recall from the text that the expectation of $(p_{T+1} - p_t(L_t))(p_t(L_t) - p_1(L_1))$ is zero under the marginal trader's belief implied by $L_1$.

[34]Event invariance implies that $\Pr\left(L_1^2/L \leq L_t|A, L_1\right) = \Pr(L_1/(L_1 + L) \leq L_t/(L_t + L_1)\,|A, L_1) = \Pr(L_1/(L_1 + L) \leq L_1/(L_1 + L_t)\,|A^c, L_1) = \Pr(L_t \leq L|A^c, L_1)$. The expression for the conditional densities follows from differentiation with respect to $L$ on both sides.

For the observer, recalling expressions from the proof of Proposition 9, then

$$\begin{aligned}
& E\left[\left(p_{T+1}-p_t\left(L_t\right)\right)\left(p_t\left(L_t\right)-p_1\left(L_1\right)\right)|L_1\right] \\
= & \pi\left(L_1\right)\int_0^\infty\left(1-p_t\left(L_t\right)\right)\left(p_t\left(L_t\right)-p_1\left(L_1\right)\right)f\left(L_t|A,L_1\right)dL_t \\
& -\left(1-\pi\left(L_1\right)\right)\int_0^\infty p_t\left(L_t\right)\left(p_t\left(L_t\right)-p_1\left(L_1\right)\right)f\left(L_t|A^c,L_1\right)dL_t \\
= & \left(\pi\left(L_1\right)-p_1\left(L_1\right)\right)\int_0^\infty\left(1-p_t\left(L_t\right)\right)\left(p_t\left(L_t\right)-p_1\left(L_1\right)\right)f\left(L_t|A,L_1\right)dL_t \\
& +\left(\pi\left(L_1\right)-p_1\left(L_1\right)\right)\int_0^\infty p_t\left(L_t\right)\left(p_t\left(L_t\right)-p_1\left(L_1\right)\right)f\left(L_t|A^c,L_1\right)dL_t \\
= & \left(\pi\left(L_1\right)-p_1\left(L_1\right)\right)\int_0^\infty\frac{p_t\left(L_t\right)}{p_1\left(L_1\right)}\left(p_t\left(L_t\right)-p_1\left(L_1\right)\right)f\left(L_t|A^c,L_1\right)dL_t,
\end{aligned}$$

where we employed Bayes' rule

$$\frac{p_t\left(L_t\right)}{1-p_t\left(L_t\right)}=\frac{p_1\left(L_1\right)}{1-p_1\left(L_1\right)}\frac{L_t}{L_1}=\frac{p_1\left(L_1\right)}{1-p_1\left(L_1\right)}\frac{f\left(L_t|A,L_1\right)}{f\left(L_t|A^c,L_1\right)}.$$

Recalling the expressions for $p_t\left(L_t\right)$ and $p_t\left(L_t\right)-p_1\left(L_1\right)$,

$$\int_0^\infty\frac{p_t\left(L_t\right)}{p_1}\left(p_t\left(L_t\right)-p_1\right)f\left(L_t|A^c,L_1\right)dL_t=\int_0^\infty\frac{L_1\left(1-p_1\right)p_1\left(L_t-L_1\right)}{\left[p_1L_t+\left(1-p_1\right)L_1\right]^2}f\left(L_t|A,L_1\right)dL_t.$$

We will employ the symmetry property to prove that this integral has the same sign as $1-2p_1\left(L_1\right)$. Thus, over the range from 0 to $L_1$, change variables to $L=L_1^2/L_t$. The integral over this range becomes

$$\begin{aligned}
& \int_{L_1}^\infty\frac{L_1\left(1-p_1\right)p_1\left(\left(L_1^2/L_t\right)-L_1\right)}{\left[p_1\left(L_1^2/L_t\right)+\left(1-p_1\right)L_1\right]^2}f\left(\left(L_1^2/L_t\right)|A,L_1\right)\left(\frac{L_1}{L}\right)^2dL \\
= & \int_{L_1}^\infty\frac{L_1\left(1-p_1\right)p_1\left(L_1-L\right)}{\left[p_1L_1+\left(1-p_1\right)L\right]^2}f\left(L|A,L_1\right)dL.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \int_0^\infty\frac{L_1\left(1-p_1\right)p_1\left(L_t-L_1\right)}{\left[p_1L_t+\left(1-p_1\right)L_1\right]^2}f\left(L_t|A,L_1\right)dL_t \\
= & L_1\left(1-p_1\right)p_1\int_{L_1}^\infty\left(\frac{L_t-L_1}{\left[p_1L_t+\left(1-p_1\right)L_1\right]^2}-\frac{L_t-L_1}{\left[p_1L_1+\left(1-p_1\right)L_t\right]^2}\right)f\left(L_t|A,L_1\right)dL_t.
\end{aligned}$$

Observe that

$$\frac{1}{\left[p_1L_t+\left(1-p_1\right)L_1\right]^2}>\frac{1}{\left[p_1L_1+\left(1-p_1\right)L_t\right]^2}$$

if and only if $\left(2p_1-1\right)\left(L_t-L_1\right)<0$. This holds over the entire range where $L_t>L_1$ if and only if $p_1<1/2$. Hence, the entire integral has the same sign as $1-2p_1\left(L_1\right)$, as

desired. Its product with $\pi(L_1) - p_1(L_1)$ is negative for all $L_1 \neq 1$: when $L_1 < 1$ we have $\pi(L_1) < p_1(L_1) < 1/2$, while when $L_1 > 1$ we have $\pi(L_1) > p_1(L_1) > 1/2$. The desired inequality (12) follows. Finally, averaging (12) over $L_1$ and using the symmetry property $E[p_t] = E[p_1] = 1/2$, it follows that $\text{Cov}(p_{T+1} - p_t, p_t - p_1) = E[(p_{T+1} - p_t)(p_t - p_1)] - E[p_{T+1} - p_t] E[p_t - p_1] < 0$.

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## Online Appendix
## Proof of Proposition 8

We verify that the described outcome is an equilibrium. For the final equilibrium condition, note that the market clears because trader positions are the same as in the static equilibrium. The remainder of the proof verifies that this constant position is indeed optimal in the individual dynamic optimization problem.

Let $\Delta x_{it}(L_t)$ denote the contingent net position of trader $i$ in period $t$ after information realization $L_t$. By convention, $\Delta x_{i0} = 0$. The trader's wealth evolves randomly over time as $w_{it}(L_t) = w_{it-1}(L_{t-1}) + (p_t(L_t) - p_{t-1}(L_{t-1}))\Delta x_{it-1}(L_{t-1})$ for $t = 1, \ldots, T+1$, with $w_{i0} > 0$ given as before. If constrained, the trader's net position choice at $t-1$ must satisfy $\Delta x_{it-1}(L_{t-1}) \in [-w_{it-1}(L_{t-1})/(1 - p_{t-1}(L_{t-1})), w_{it-1}(L_{t-1})/p_{t-1}(L_{t-1})]$.

Suppose at period $t$, information $L_t$ has been realized. To save notation, write $p_t$ for the realization of $p_t(L_t)$ and $w_{it}$ for $w_{it}(L_t)$. Two observations are essential. First, $\Delta x_{it}$ is at the upper bound (interior, lower bound) of the constraint set $[-w_{it}/(1-p_t), w_{it}/p_t]$ if and only if, for all $L_{t+1}$, $\Delta x_{it+1}$ is on the upper bound (interior, lower bound) of constraint set $[-w_{it+1}(L_{t+1})/(1 - p_{it+1}(L_{t+1})), w_{it+1}(L_{t+1})/p_{it+1}(L_{t+1})]$. Second, for all realizations of the string $(L_{t+1}, \ldots, L_T)$, the feasible choice $\Delta x_{iT}(L_T) = \ldots = \Delta x_{it+1}(L_{t+1}) = \Delta x_{it}$ implies

$$\frac{u_i'(w_{iT}(A))}{u_i'(w_{iT}(A^c))} = \frac{u_i'(w_{it} + (1-p_t)\Delta x_{it})}{u_i'(w_{it} - p_t \Delta x_{it})}.$$

Both observations follow from the wealth evolution equation $w_{i\tau}(L_t) = w_{i\tau-1}(L_{\tau-1}) + (p_\tau(L_\tau) - p_{\tau-1}(L_{\tau-1}))\Delta x_{i\tau-1}(L_{\tau-1})$ for periods $\tau = t+1, \ldots, T$.

To prove our claim that the trader in every period selects the same position $\Delta x_{it} = \Delta x_{i1}(L_1)$ as in the static model given price $p_1(L_1)$, we proceed by backwards induction. The induction hypothesis $t$ states that the agent in period $t$ given price $p_t(L_t)$ (i) chooses $\Delta x_{it}$ to satisfy the static first-order condition

$$\frac{p_t(L_t)}{1 - p_t(L_t)} = \frac{\pi_i(L_t)}{1 - \pi_i(L_t)} \frac{u_i'(w_{it}(L_t) + (1 - p_t(L_t))\Delta x_{it})}{u_i'(w_{it}(L_t) - p_t(L_t)\Delta x_{it})}$$

if feasible, or (ii) chooses $\Delta x_{it} = w_{it}(L_t)/p_t(L_t)$ if the left-hand side of this static condition is below the right-hand side at this choice, and (iii) chooses $\Delta x_{it} = -w_{it}(L_t)/(1 - p_t(L_t))$ if the left-hand side of this static condition exceeds the right-hand side at this choice. Note from the previous two essential observations, that once we have proved the induction

hypothesis for all $t$, we have $\Delta x_{iT}(L_T) = \ldots = \Delta x_{i1}(L_1)$, and $\Delta x_{i1}(L_1)$ is the solution to the individual problem in Proposition 4.

The induction hypothesis $T$ is satisfied because the static first-order condition characterizes the solution to the remaining one-period problem. We now assume that the induction hypothesis is true at $t+1, \ldots, T$, and will prove that induction hypothesis $t < T$ is true. Suppose at period $t$, information $L_t$ is realized. Final wealth levels are then

$$w_{iT}(A) = w_{it} + (p_{t+1}(L_{t+1}) - p_t)\Delta x_{it} + (1 - p_{t+1}(L_{t+1}))\Delta x_{it+1}(L_{t+1})$$

and

$$w_{iT}(A^c) = w_{it} + (p_{t+1}(L_{t+1}) - p_t)\Delta x_{it} - p_{t+1}(L_{t+1})\Delta x_{it+1}(L_{t+1})$$

where $\Delta x_{it+1}(L_{t+1})$ is the reaction prescribed by induction hypothesis $t+1$. The time $t$ problem is

$$\max_{\Delta x_{it} \in [-w_{it}/(1-p_t), w_{it}/p_t]} \pi_i(L_t) E[u_i(w_{iT}(A))|A] + (1 - \pi_i(L_t)) E[u_i(w_{iT}(A^c))|A^c]$$

where the expectations are taken over the realization of $L_{t+1}$. In case (i), the static first-order condition can be satisfied with an interior choice of $\Delta x_{it}$. Evaluated at this choice, the derivative of the time $t$ objective function is, by the envelope theorem,

$$\begin{aligned} & \pi_i(L_t) E[(p_{t+1}(L_{t+1}) - p_t) u_i'(w_{iT}(A))|A] \\ & + (1 - \pi_i(L_t)) E[(p_{t+1}(L_{t+1}) - p_t) u_i'(w_{iT}(A^c))|A^c] \\ = \; & p_t E\left[\tfrac{\pi_i(L_t) u_i'(w_{iT}(A))}{p_t}(p_{t+1}(L_{t+1}) - p_t)|A\right] \\ & + (1 - p_t) E\left[\tfrac{(1-\pi_i(L_t)) u_i'(w_{iT}(A^c))}{1-p_t}(p_{t+1}(L_{t+1}) - p_t)|A^c\right]. \end{aligned}$$

Here $w_{iT}(A)$ and $w_{iT}(A^c)$ are constant across realizations of $L_{t+1}$. The static first-order condition then allows us to rewrite the derivative with respect to the control variable as

$$\frac{\pi_i(L_t) u_i'(w_{iT}(A))}{p_t} \{p_t E[p_{t+1}(L_{t+1}) - p_t|A] + (1 - p_t) E[(p_{t+1}(L_{t+1}) - p_t)|A^c]\}.$$

By the martingale property of Bayes-updated prices at market belief $p_t$, we have

$$p_t E[p_{t+1}(L_{t+1}) - p_t|A] + (1 - p_t) E[(p_{t+1}(L_{t+1}) - p_t)|A^c] = 0.$$

Thus the first-order condition for optimality of $\Delta x_{it}$ is satisfied at the choice resulting from the static model. The other two cases (with constrained choices) follow likewise.